MAS4010 ADVANCED TOPICS IN ALGEBRA

SOLUTIONS TO EXERCISE SHEET 4

Question 1

Let G be the alternating group A_6. For each prime p dividing $|G|$, give one example of a Sylow p-subgroup of G, and find the number of such subgroups.

Solution:

$A_6 = \frac{1}{2} \times 6! = 360 = 2^3 \cdot 3^2 \cdot 5$. So we need to consider the primes $p = 2, 3, 5$.

$p = 2$: One Sylow 2-subgroup is $\langle (1234)(56), (13)(56) \rangle \cong D_4$. (Notice we need the (56) contribution to get even permutations). Every conjugate of this contains two elements of order 4, with cycle type $t_4 t_2$, and every element of this type occurs in exactly one Sylow 2-subgroup. So the number of Sylow 2-subgroups is

$$\frac{1}{2} \times \text{number of elements of cycle type } t_4 t_2 = \frac{1}{2} \times \frac{6 \cdot 5 \cdot 4 \cdot 3}{4} = 45.$$

$p = 3$: One Sylow 3-subgroup is $\langle (123), (456) \rangle \cong C_3 \times C_3$. Any Sylow 3-subgroup contains 4 elements of cycle type t_3^2, each element with this cycle type occurring in exactly one Sylow 3-subgroup. So the number of Sylow 3-subgroups is

$$\frac{1}{4} \times \text{number of elements of cycle type } t_3^2 = \frac{1}{4} \times \frac{(6 \cdot 5 \cdot 4)(3 \cdot 2 \cdot 1)}{3 \cdot 3 \cdot 2} = 10.$$

$p = 5$: One Sylow 5-subgroup is $\langle 12345 \rangle \cong C_5$. Each Sylow 5-subgroup contains 4 5-cycles, and every 5-cycle is contained in exactly one Sylow 5-subgroup. So the number of Sylow 5-subgroups is

$$\frac{1}{4} \times \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{5} = 36.$$

[5 marks]

Question 2

Let G be a group, and, for any prime p, let n_p be the number of Sylow p-subgroups of G. Find the values of n_p permitted by Sylow's Theorems, for each p dividing $|G|$, in the following cases. Are there any conclusions about G that you can easily draw? (You do not have to show that groups with these values of n_p actually exist).

(i) $|G| = 35$;

(ii) $|G| = 155$;

(iii) $|G| = 240$;

(iv) $|G| = 72$;

(v) $|G| = 105$.

Solution:

(i) $|G| = 35$: $n_7 \equiv 1 \pmod 7$ and n_7 divides 35, so $n_7 = 1$. Similarly $n_5 = 1$. So G has only 6 elements of order 7 and only 4 elements of order 5. Thus there are some elements of order 35 (in fact 24 of them), so G must be cyclic. **[3 marks]**

(ii) $|G| = 155 = 5 \times 31$: $n_{31} \equiv 1 \pmod{31}$ and n_{31} divides 155, so $n_{31} = 1$. On the other hand n_5 divides 155 and satisfies $n_5 \equiv 1 \pmod 5$, which gives two possibilities $n_5 = 1$ or $n_5 = 31$. Thus G always has a normal subgroup of order 31, but might or might not be cyclic. **[3 marks]**

(iii) $|G| = 240 = 2^4 \cdot 3 \cdot 5$. Here we have $n_2 = 1$ or 3 or 5 or 15; $n_3 = 1$, 4, 10, 16 or 40 and $n_5 = 1$, 6 or 16. **[4 marks]**

(iv) $|G| = 72 = 2^3 \cdot 3^2$. Here $n_2 = 1$, 3 or 9 and $n_3 = 1$ or 4. We can't easily calculate the number of elements of 2-power or 3-power order, since we do not know the orders of the intersections of two Sylow 2-subgroups (or 3-subgroups). [**marks**]

(v) $|G| = 105 = 3 \cdot 5 \cdot 7$. We have $n_3 = 1$ or 7; $n_5 = 1$ or 21; $n_7 = 1$ or 15.

If $n_3 = 7$ then G has 15 elements of order 3. If $n_5 = 21$ then G has 84 elements of order 5. If $n_7 = 15$ then G has 90 elements of order 7.

Thus if $n_7 = 15$ we must have $n_5 = n_3 = 1$. Thus we can conclude that either G has a normal subgroup of order 7, or G has both a normal subgroup of order 3 and a normal subgroup of order 5. **[4 marks]**

[Total: 17 marks]

Question 3

Let G be a group of order $351 = 27 \times 13$. By considering the number of Sylow p-subgroups of G for appropriate primes G, show that G must have a proper normal subgroup.

Solution:

The relevant primes are 3 and 13. (Note that 27 is not prime!) As $n_3 \equiv 1 \pmod 3$ and n_3 divides 13, we have $n_3 = 1$ or 13. Similarly $n_{13} = 1$ or 27. If $n_{13} = 27$ then G contains 27×12 elements of order 13, and hence only 27 other elements, so that $n_3 = 1$ in this case. (Any Sylow 3-subgroup will have order 13). Hence either G has a normal subgroup of order 13 (if $n_{13} = 1$) or a normal subgroup of order 27 (if $n_{13} = 27$, so that $n_3 = 1$). Either way, G has a proper normal subgroup. **[7 marks]**

Question 4

Let G be a group of order pq, where p and q are primes and $p > q$. Show that

(i) G has a proper normal subgroup;

(ii) if $p \not\equiv 1 \pmod q$ then G is cyclic.

Solution:

As n_p divides pq we have $n_p = 1$, p, q, or pq. But also $n_p \equiv 1 \pmod p$. Since $1 < q < p$, the only possibility for n_p is 1. Thus G has a proper normal subgroup of order p.
As n_q divides pq but is not divisible by q we have $n_q = 1$ or p. If $p \not\equiv 1 \pmod q$ then the second possibility is ruled out, so $n_q = n_p = 1$. Then G has only $p - 1$ elements of order p, only $q - 1$ elements of order q, and only one element (the identity) of order 1. This leaves $(p-1)(q-1) > 0$ elements which must have order pq, so that G is cyclic.

[12 marks]

Question 5

Show that any p-subgroup of a finite group G is contained in some Sylow p-subgroup of G. (*Hint:* Adapt the proof of the 2nd Sylow Theorem).

Solution:

Let P be a Sylow p-subgroup of G. (We know P exists by the 1st Sylow Theorem.) Let Q be any p-subgroup in G, and let $|G| = p^r m$ where p does not divide m.
G acts by left multiplication on the set $S = \{gP \mid g \in G\}$ of the m left cosets of P. Hence Q also acts on this set, and each Q-orbit has size a power of p (since it must divide $|Q|$). As these orbits partition S, and $|S| = m$ is not divisible by p, there must be at least one orbit of size $p^0 = 1$. Thus, for some $g \in G$ we have $QgP = gP$. Then $Qg \subseteq gP$, so that $Q \subseteq gPg^{-1}$. Now gPg^{-1} is a Sylow p-subgroup since it has the same order as P. So we have shown that every p-subgroup Q of G is contained in some Sylow p-subgroup.

[8 marks]

Question 6

Show that a normal p-subgroup of a finite group G is contained in every Sylow p-subgroup of G.

Solution:

Let Q be a normal p-subgroup of G. By Question 5, $Q \subseteq P$ for some Sylow p-subgroup of G. By Sylow's 2nd Theorem, every Sylow p-subgroup of G has the form gPg^{-1}, and since Q is normal in G we have $Q = gQg^{-1} \subseteq gPg^{-1}$. Hence Q is contained in every Sylow p-subgroup of G.

[4 marks]

Question 7

Let S be the subset of $G = \mathrm{GL}_n(\mathbb{F}_p)$ consisting of upper triangular matrices with 1's on the main diagonal. Verify that S is a subgroup of G, and deduce that it is a Sylow p-subgroup.

Solution

We have that $A = (a_{ij}) \in S$ if and only if $a_{ii} = 1$ for all i and $a_{ij} = 0$ whenever $i > j$.
Clearly $I \in S$. Given $A = (a_{ij})$ and $B = (b_{ij})$ in S, let $C = AB = (c_{ij})$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Now $a_{ik} = 0$ if $i > k$ and $b_{jk} = 0$ if $k > j$. Thus if $i > j$ then each term in the sum is 0, so $c_{ij} = 0$. If $i = j$ then $c_{ij} = c_{ii} = a_{ii} b_{ii} = 1$. Thus $C = AB \in S$, so S is closed under multiplication.
Now each $A \in S$ is an element of the finite group G, so it has some finite order n. Since S is closed under multiplication, $A^{-1} = A^{n-1} \in S$, so S is also closed under taking inverses. Hence S is a subgroup of G.

[Alternative proof that S is closed under inverses, which works over infinite fields as well:
Let $A = (a_{ij}) \in S$ and let $D = (d_{ij}) = A^{-1} \in G$. Then

$$\sum_{k=1}^{n} d_{ik} a_{kj} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

We show by induction on m, $1 \le m \le n$, that

$$d_{ii} = 1 \text{ if } i \le m; \qquad d_{ik} = 0 \text{ if } i > k \text{ and } k \le m.$$

The case $m = n$ shows $D \in S$.
Assuming this induction hypothesis holds for $m-1$ (no assumption for $m = 1$) we have

$$\delta_{im} = \sum_{k=1}^{n} d_{ik} a_{km} = \sum_{\{k \mid i \le k < m\}} d_{ik} a_{km} + d_{im}.$$

If $i \ge m$, the last sum is 0, so $d_{mm} = \delta_{mm} = 1$ and if $i > m$ then $d_{im} = 0$. This shows that the induction hypothesis also holds for m.]
Now we know

$$|G| = \prod_{j=0}^{n-1} (p^n - p^j) = p^{\binom{n}{2}} \prod_{j=0}^{n-1} (p^{n-j} - 1),$$

so any subgroup of G of order $p^{\binom{n}{2}}$ will be a Sylow p-subgroup, Now to specify an element of S, we must choose a_{ij} whenever $1 \le i < j \le n$. There are $\binom{n}{2}$ such matrix entries, each of which can be any of the p elements of $\mathbb{F}_p$. Hence $|S| = p^{\binom{n}{2}}$. Thus S is a Sylow p-subgroup of G. **[5 marks]**

Question 8

The normaliser of the group S in Question 7 is the group of all upper triangular invertible matrices. Using this fact (which you should have already proved in the case $n = 2$ in Question 5 on Sheet 3), determine the number of Sylow p-subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$.

Solution

The number of Sylow p-subgroups is the index of the normaliser of any one of them, so we just need to find the index of the group N of upper triangular invertible matrices in $G = \mathrm{GL}_n(\mathbb{F}_p)$. To specify an element of N we must first choose all the above-diagonal entries, which amounts to choosing an element of S, and then choose each of the n diagonal entries, which can be any element of $\mathbb{F}_p^\times$ (but can't be 0 since the matrix must be invertible). So

$$|N| = |S|(p-1)^n,$$

and the number of Sylow p-subgroups in G is

$$\frac{|G|}{|N|} = \prod_{j=0}^{n-1} \frac{p^{n-j}-1}{p-1} = \prod_{k=2}^{n} \frac{p^k-1}{p-1}.$$

[5 marks]

Question 9

The special linear group $SL_n(\mathbb{F}_p)$ is defined as the subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ consisting of matrices whose determinant (as an element of $\mathbb{F}_p$) is 1. Find a Sylow p-subgroup of $\mathrm{SL}_n(\mathbb{F}_p)$, and determine the number of such subgroups.

Solution

The determinant is a surjective homomorphism $\det\colon \mathrm{GL}_n(\mathbb{F}_p) \longrightarrow \mathbb{F}_p^\times$ with kernel $\mathrm{SL}_n(\mathbb{F}_p)$, so

$$|\mathrm{SL}_n(\mathbb{F}_p)| = \frac{|\mathrm{GL}_n(\mathbb{F}_p)|}{p-1}.$$

Thus a Sylow p-subgroup of $\mathrm{SL}_n(\mathbb{F}_p)$ will have the same order as one of $\mathrm{GL}_n(\mathbb{F}_p)$. Now the group S of upper triangular matrices with 1's on the main diagonal (Question 7) is a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$, and is clearly contained in $\mathrm{SL}_n(\mathbb{F}_p)$. Thus S is also a Sylow p-subgroup of $\mathrm{SL}_n(\mathbb{F}_p)$.
The normaliser N' of S in $\mathrm{SL}_n(\mathbb{F}_p)$ is just $\mathrm{SL}_n(\mathbb{F}_p) \cap N$, where N is the normaliser of S in $\mathrm{GL}_n(\mathbb{F}_p)$, as in Question 8. Thus the normaliser N' of S in $\mathrm{SL}_n(\mathbb{F}_p)$ is just the group of upper triangular matrices with determinant 1, and N' has index $p-1$ in N. This is the same as the index of $\mathrm{SL}_n(\mathbb{F}_p)$ in $\mathrm{GL}_n(\mathbb{F}_p)$. So the index of N' in $\mathrm{SL}_n(\mathbb{F}_p)$ is the same as that of S in $\mathrm{GL}_n(\mathbb{F}_p)$. Hence the number of Sylow p-subgroups in $\mathrm{SL}_n(\mathbb{F}_p)$ is the same as in $\mathrm{GL}_n(\mathbb{F}_p)$, namely

$$\prod_{k=2}^{n} \frac{p^k-1}{p-1}.$$

[5 marks]

Question 10

The quaternion group Q_8 of order 8 has presentation

$$Q_8 = \langle a, b \mid a^4 = e, b^2 = a^2, bab^{-1} = a^{-1} \rangle.$$

Show that every element of Q_8 can be written in the form $a^r b^s$ with $r \in \{0, 1, 2, 3\}$ and $s \in \{0, 1\}$, but that Q_8 is not isomorphic to D_4.
Let A and B be the matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Verify that Q_8 has a representation ρ of degree 2 given by $\rho(a) = A$ and $\rho(b) = B$ by checking that A and B satisfy $A^4 = I$, $A^2 = B^2$, $BAB^{-1} = A^{-1}$.

[Note that the question should refer to D_4 not D_8.]

Solution

Any element of Q_8 can be written in the form $a^{r_1}b^{s_1} \dots a^{r_t}b^{s_t}$ for some t and some integers r_j, s_j. Using the properties $a^4 = e$ and $b^2 = a^2$, we can rewrite this so that $0 \le r_j \le 3$ and $0 \le s_j \le 1$ for each j, From the property $bab^{-1} = a^{-1}$ we have $ba = a^{-1}b = a^3 b$. Using this repeatedly, we can move every occurrence of b to the right of every occurrence of a, so that we end up with an expression $a^r b^s$ with $0 \le r \le 3$ and $0 \le s \le 1$. Thus every element of Q_8 can be written in this form.
Now in Q_8 we have $a^2 = b^2 \ne e$ so that a, a^3, b, b^3 are 4 distinct elements of order 4. (In fact ab and a^3b also have order 4). However in D_4 there are only two elements of order 4 (the rotations by $\pm 90^0$), all other elements except the identity having order 2. Thus the two nonabelian groups D_4, Q_8 of order 8 cannot be isomorphic.
To check that ρ is a representation, we just have to verify that the matrices A, B satisfy the same relations as the generators a, b of Q_8. Now

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I,$$

so that $A^4 = I$, and

$$B^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A^2.$$

Also

$$\begin{aligned} BAB^{-1} = -BAB &= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -A = A^{-1}. \end{aligned}$$

[8 marks]

Question 11

Let ρ_1 be the degree 2 representation of S_3 given by identifying S_3 with the group D_3 of symmetries of an equilateral triangle. (This identification is possible since all permutations of the vertices of the triangle are obtained by rotations or reflections.) Write out the 6 matrices $\rho_1(g)$ for $g \in S_3$. Check that $\rho_1((12))\rho_1((123)) = \rho_1((23))$.

[Note that the question should refer to D_3 not D_6.]

Solution

Number the vertices of the triangle 1, 2, 3 in anticlockwise order, with 1, 2 on the base. Then we have

$$\rho_1(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_1((123)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \rho_1((132)) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\rho_1((12) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_1((23)) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \rho_1((13)) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

We check

$$\rho_1((12))\rho_1((123)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \rho_1((23)).$$

[8 marks]

Question 12

Let ρ_2 be the permutation representation of S_3 (so ρ_2 has degree 3). Write out the 6 matrices $\rho_2(g)$ for $g \in S_3$. Check that $\rho_2((12))\rho_2((123)) = \rho_2((23))$.

Solution

We have

$$\rho_2(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_2((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\rho_2((132)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho_2((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho_2((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \rho_2((13)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We check

$$\begin{aligned} \rho_2((12))\rho_2((123)) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

[8 marks]

Question 13

Let ρ_1 and ρ_2 be the representations of S_3 in Questions 11 and 12 respectively. What are the degrees of the representations $\rho_3 = \rho_1 \oplus \rho_2$ and $\rho_4 = \rho_1 \otimes \rho_2$? Write out the 6 matrices $\rho_i(g)$ for $g \in S_3$, and check that $\rho_i((12))\rho_i((123)) = \rho_i((23))$, for $i = 3$ and for $i = 4$.

Solution

Since ρ_1 and ρ_2 have degrees 2, 3 respectively, $\rho_3 = \rho_1 \oplus \rho_2$ has degree $2+3 = 5$ and $\rho_4 = \rho_1 \otimes \rho_2$ has degree $2 \times 3 = 6$.
For ρ_3, the matrices are

$$\rho_3(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\rho_3((123)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\rho_3((132)) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\rho_3((12)) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\rho_3((23)) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\rho_3((13)) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We check

$$\begin{aligned} \rho_3((12))\rho_3((123)) &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ &= \rho_3((23)). \end{aligned}$$

For ρ_4, the matrices are

$$\rho_4(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\rho_4((123)) = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix};$$

$$\rho_4((132)) = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix};$$

$$\rho_4((12)) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\rho_4((23)) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix};$$

$$\rho_4((13)) = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix}.$$

We check

$$\rho_4((12))\rho_4((123)) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$= \rho_4((13)).$$

[8 marks]

Nigel Byott
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