MAS4010 ADVANCED TOPICS IN ALGEBRA

SUMMARY OF PROPERTIES OF CHARACTERS

Here is a summary of facts which are useful in finding the character table of a finite group G.

1. The number of irreducible characters is the same as the number of conjugacy classes. (Convention: χ_1 is the trivial character $\chi_1(g) = 1$ for all $g \in G$. The first conjugacy class representative is $g_1 = e$.) It is useful to record

$$\begin{aligned} n_i &= \text{size of conjugacy class of } g_i, \\ c_i &= \text{order of centraliser of } g_i, \end{aligned}$$

   for each conjugacy class representative g_i. (Thus $n_i c_i = |G|$).

2. (Character degrees). Let $d_i = \chi_i(e)$, the degree of the irreducible character χ_i. Then

$$\sum_{i=1}^{r} d_i^2 = |G|.$$

   (Also, each d_i divides $|G|$ – proof omitted.)

3. (Degree 1 characters). These are just homomorphisms $G \longrightarrow \mathbb{C}^\times$, and are necessarily irreducible. The product of any two degree 1 characters is a degree 1 character. For an abelian group G, all irreducible characters have degree 1, and there are $|G|$ of them.

4. Orthogonality relations:

$$\sum_{k=1}^{r} n_k \chi_i(g_k)\overline{\chi_j(g_k)} = |G|\delta_{ij} \qquad \text{(Row Orthogonality);}$$

$$\sum_{k=1}^{r} \chi_k(g_i)\overline{\chi_k(g_j)} = c_i\delta_{ij} \qquad \text{(Column Orthogonality).}$$

5. (Inflating characters). If N is a normal subgroup of G, we get an irreducible character χ of G by taking an irreducible character $\hat{\chi}$ of G/N and setting $\chi(g) = \hat{\chi}(gN)$.(This is particularly useful if G/N is abelian, since we can then write down $|G/N|$ degree 1 characters of G.)

6. $\chi(g^{-1}) = \overline{\chi(g)}$. In particular, $\chi_i(g) \in \mathbb{R}$ for all irreducible characters χ_i if and only if g is conjugate to g^{-1}.

7. The complex conjugate of any row (column) in the character table must occur as a row (column).

8. The product of any irreducible character with a degree 1 character is an irreducible character.

9. The product $\psi = \chi_i \chi_j$ of any two irreducible characters χ_i and χ_j is again a character, but in general will not be irreducible. However, it can be written

$$\chi_i \chi_j = \sum_{k=1}^{r} m_k \chi_k$$

for some non-negative integers m_k. Using inner products, we have $\sum_k m_k^2 = \langle \psi, \psi \rangle$, and, for those irreducible characters χ_k that we already know, we can find $m_k = \langle \psi, \chi_k \rangle$. Subtracting off the $m_k \chi_k$ for all such k, we obtain from ψ a linear combination of the remaining irreducible characters. (If we are lucky, just one of these occurs in ψ, so we can fill in another row of the character table.)

10. (Permutation characters). If G acts on some set S of size n, we get a permutation character χ_S for G, of degree n. For each $g \in G$ we have that $\chi_S(g)$ is the number of points of S fixed by g. Again, χ_S need not be irreducible, but we can investigate its irreducible constituents as in 9.

    We can obtain further permutation characters by considering actions of G on other sets, e.g. G will permute the collection of all 2-element subsets of S, and this gives a permutation character of degree $\binom{n}{2}$.

11. (The regular character). This is the special permutation character where G acts on itself by left multiplication. It has values

$$\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = e, \\ 0 & \text{if } g \neq e, \end{cases}$$

and its decomposition into irreducible characters is $\chi_{\text{reg}} = \sum_{i=1}^{r} d_i \chi_i$.

12. $\chi_i(g)$ is a sum of d_i mth roots of unity, where g has order m. In particular, character values are always algebraic integers.

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