Complex singularities in 2-d Euler flows

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with thanks to Uriel Frisch, Jeremie Bec, Takeshi Matsumoto.
Complex singularities must precede real ones

- take a flow evolving under the Euler equation, e.g.,
  \[ \partial_t \omega = J_x(\psi, \omega), \quad -\omega = \nabla_x^2 \psi, \quad u = (\partial_{x_2} \psi, -\partial_{x_1} \psi). \]
  \[ \psi|_{t=0} = \cos x_1 + \cos 2x_2 \]
- extend as a function of complex coordinates
  \[ x_1 \rightarrow z_1 = x_1 + iy_1, \quad x_2 \rightarrow z_2 = x_2 + iy_2, \]
- Bardos/Benachour (1976): width of analyticity strip \( \delta(t) \) cannot decrease discontinuously in time
- singularities can be tracked in the complex plane and their nature understood
- only progress achieved in 2-d so far (singularities will not reach real axis)
High precision numerical simulations (Matsumoto, Bec, Frisch 2005)

- measure width from Fourier spectra
- clean logarithmic decrease in $\delta(t)$
- also Taylor-Green vortex in 3-d (Brachet et al.1983)
Extend to complex plane
(Bec, Frisch, Pauls, Matsumoto 2003-2010)

- Complexify \( x_1 \rightarrow z_1 = x_1 + iy_1, \ x_2 \rightarrow z_2 = x_2 + iy_2, \)

- and FIX the real coordinates, \( x = (\pi, 0) \) varying the imaginary parts ONLY

\[
\partial_{x_1} = -i\partial_{y_1}, \quad \partial_{x_2} = -i\partial_{y_2}, \quad J_x = -J_y, \quad \nabla^2_x = -\nabla^2_y, \\
\partial_t \omega = -J_y(\psi, \omega), \quad \omega = \nabla^2_y \psi.
\]

- retaining only the modes that grow with increasing imaginary parts

\[
\psi|_{t=0} = ae^{p \cdot y} + be^{q \cdot y}, \quad p \equiv p(\cos \phi_p, \sin \phi_p), \quad q \equiv q(\cos \phi_q, \sin \phi_q),
\]

- singularities are a distance \( O(\log t) \) away for \( t<<1 \)

\[
\tilde{y} = y - \lambda \log t^{-1},
\]

- rescale to obtain pseudo-hydrodynamic flow \( \psi_{ps} = \lambda_2 \tilde{y}_1 - \lambda_1 \tilde{y}_2 + t\psi, \quad \omega_{ps} = -t\omega, \)
Pseudo-flow (PMFB06)

• obeys steady Euler equation
  \[ \omega_{ps} = J \bar{y}(\psi_{ps}, \omega_{ps}), \quad \omega_{ps} = -\nabla_{\bar{y}}^2 \psi_{ps}. \]
• with Rayleigh friction

• and matching to exponentials
  \[ \psi_{ps} \sim -\frac{1}{2} e^{y_1} + \frac{1}{2} e^{2y_2} \]
• curved singular manifold

• measure scaling as approached

• at the manifold vorticity scales as
  \[ \omega_{ps} \sim s^{-\beta}, \quad \beta \approx 5/6, \quad \phi = \pi/2 \]

• exponent depends on angle between initial modes
  \[ \phi = 0, \quad \beta \approx 1, \quad \phi = \pi/4, \quad \beta \approx 0.96. \]
Our aim: pin down scalings by formal asymptotics

- start with Euler
  \[ \partial_t \omega = J_x(\psi, \omega), \quad -\omega = \nabla_x^2 \psi, \]
  \[ \psi|_{t=0} = ae^{p \cdot y} + be^{q \cdot y}, \quad p \equiv p(\cos \phi_p, \sin \phi_p), \quad q \equiv q(\cos \phi_q, \sin \phi_q), \quad \phi \equiv \phi_q - \phi_p, \]
- complexify
  \[ \partial_t \omega = -J_y(\psi, \omega), \quad \omega = \nabla_y^2 \psi. \]
- change variables
  \[ \psi(y_1, y_2, t) = t^{-1} \Psi(Y_1, Y_2), \quad \omega(y_1, y_2, t) = t^{-1} \Omega(Y_1, Y_2). \]
  \[ Y_1 = te^{p \cdot y}, \quad Y_2 = te^{q \cdot y}. \]
- rescale (including time) to obtain
  \[ (\frac{-1 + Y_1 \partial_1 + Y_2 \partial_2)}{Y_1 Y_2} \Omega = -Y_1 Y_2 [(\partial_1 \Psi)(\partial_2 \Omega - (\partial_2 \Psi)(\partial_1 \Omega)] \]
  \[ \Omega = \eta^{-1} Y_1 \partial_1 (Y_1 \partial_1 \Psi) + 2 \cos \phi Y_1 Y_2 \partial_1 \partial_2 \Psi + \eta Y_2 \partial_2 (Y_2 \partial_2 \Psi), \quad \Psi \sim -Y_1 + Y_2 \quad (Y_1, Y_2 \rightarrow 0) \]
- two parameters \( \{\eta, \phi\}. \quad \eta = q/p. \) Zero angle limit OK (hydrostatic limit: Brenier).
- expand in 2-d Taylor series corresponds precisely to PMFB06 (very high accuracy arithmetic)
Expansion schemes

- start with PDEs in \((Y_1, Y_2)\)
- PMFB06 expand as Taylor series in \(Y_1^m Y_2^n\)
- and measure scaling in rational directions
- radius of convergence gives singular manifold location
- next term gives vorticity blow-up scaling exponent
- we expand as ODEs in one direction

\[
f_0(Y_1) + f_1(Y_1) Y_2 + f_2(Y_1) Y_2^2 + \cdots\]
Hierarchy of ODEs

- we expand as a hierarchy of ODEs
  \[
  \Psi = f_0(Y_1) + f_1(Y_1)Y_2 + f_2(Y_1)Y_2^2 + \cdots, \quad \Omega = h_0(Y_1) + h_1(Y_1)Y_2 + h_2(Y_1)Y_2^2 + \cdots
  \]
- solutions blow up rapidly for large \( Y_1 \) so we define
  \[
  f_n(Y_1) = e^{nY_1}Y_1^{-n-1}F_n(Y_1), \quad h_n(Y_1) = e^{nY_1}Y_1^{-n+1}H_n(Y_1),
  \]

- functions \( \{F_n, H_n\} \) expand in inverse powers of \( Y_1 \) for large \( Y_1 \)

- collocation method used
Hierarchy of linear 3rd order ODE systems

- limiting ratios give radius of convergence in $Y_2$ and so location of singular manifold
- can also recover PMFB06 pseudo-flow

\begin{align*}
Y_1^2 H_n' + \eta^{-1} n F_n &= -S_n, \\
Y_1 G_n &= Y_1 F_n' + (nY_1 - n - 1) F_n, \\
Y_1 H_n &= \eta^{-1} [Y_1 G_n' + (nY_1 - n) G_n] + 2n \cos \phi G_n + \eta n^2 Y_1^{-1} F_n, \\
S_n &= \sum_{r=1}^{n-1} [(n - r)Y_1 F_r' H_{n-r} - rY_1 F_r H_{n-r}' - n F_r H_{n-r}],
\end{align*}

- but hard to measure scaling laws on the singular manifold from our method (PMFB06 superior accuracy).
Expansion

- try expansion as

\[ F_n = n^{-\alpha-2}e^{-n\delta(Y_1)}(A_0(Y_1) + \cdots), \quad H_n = n^{-\alpha}e^{-n\delta(Y_1)}(C_0(Y_1) + \cdots) \]

- location of manifold is given by decrement \( \delta(Y_1) \)

- power law blow-up of vorticity controlled by \( \alpha = 1 - \beta \),

- rest is a series in inverse powers of \( n \)

- substitute into hierarchy and look at behaviour as \( n \to \infty \)

- only successfully sorted out for \( \alpha = 0 \) and so vorticity blow-up \( \beta = 1 \)

- start with this and see what we can deduce
Large-n expansion : sketch

• substitute expansion in equations:

\[ F_n = n^{-2} e^{-n\delta(Y_1)}(A_0(Y_1) + n^{-1} A_1(Y_1) + \cdots), \]
\[ H_n = e^{-n\delta(Y_1)}(C_0(Y_1) + n^{-1} C_1(Y_1) + \cdots) \]

\[ Y_1^2 H'_n + \eta^{-1} n F_n = -S_n, \]
\[ Y_1 G_n = Y_1 F'_n + (nY_1 - n - 1) F_n, \]
\[ Y_1 H_n = \eta^{-1} [Y_1 G'_n + (nY_1 - n) G_n] + 2n \cos \phi G_n + \eta n^2 Y_1^{-1} F_n, \]

\[ S_n = \sum_{r=1}^{n-1} [(n - r) Y_1 F'_r H_{n-r} - r Y_1 F_r H'_{n-r} - n F_r H_{n-r}], \]

• expansion cannot be used for head of series, \( n = O(1) \) only for tails
Large-n expansion 1: stream function / vorticity

- substitute expansion in equations:
  \[ F_n = n^{-2}e^{-n\delta(Y_1)}(A_0(Y_1) + n^{-1}A_1(Y_1) + \cdots), \]
  \[ H_n = e^{-n\delta(Y_1)}(C_0(Y_1) + n^{-1}C_1(Y_1) + \cdots) \]
  \[ Y_1^2 H_n' + \eta^{-1}n F_n = -S_n, \]
  \[ Y_1 G_n = Y_1 F_n' + (nY_1 - n - 1) F_n, \]
  \[ Y_1 H_n = \eta^{-1}[Y_1 G_n' + (nY_1 - n)G_n] + 2n \cos \phi G_n + \eta n^2 Y_1^{-1} F_n, \]
  \[ S_n = \sum_{r=1}^{n-1} [(n - r)Y_1^2 F_r H_{n-r} - rY_1 F_r H_{n-r}' - nF_r H_{n-r}], \]

- link between stream function and vorticity was second order ODE

- but now becomes a purely multiplicative relationship

  \[ C_0 = LA_0, \]
  \[ L(Y_1) = Y_1^{-2}[\eta^{-1}(-Y_1 \delta' + Y_1 - 1)^2 + 2 \cos \phi (-Y_1 \delta' + Y_1 - 1) + \eta] \geq 0. \]
Large-n expansion 2: log terms

- substitute expansion in equations:
  \[ F_n = n^{-2}e^{-n\delta(Y_1)}(A_0(Y_1) + n^{-1}A_1(Y_1) + \cdots), \]
  \[ H_n = e^{-n\delta(Y_1)}(C_0(Y_1) + n^{-1}C_1(Y_1) + \cdots) \]
  \[ Y_1^2H'_n + \eta^{-1}nF_n = -S_n, \]
  \[ Y_1G_n = Y_1F'_n + (nY_1 - n - 1)F_n, \]
  \[ Y_1H_n = \eta^{-1}[Y_1G'_n + (nY_1 - n)G_n] + 2n\cos \phi G_n + \eta n^2Y_1^{-1}F_n, \]
  \[ S_n = \sum_{r=1}^{n-1} [(n - r)Y_1F'_rH_{n-r} - rY_1F_rH'_{n-r} - nF_rH_{n-r}], \]

- evaluate quadratic terms \( \Sigma_{P,Q} = \sum_{r=1}^{n-1} r^{-P}(n - r)^{-Q} \)
  \[ = \begin{cases} 
  O(\log n) & (1, 0), (0, 1), \\
  O(n^{-1} \log n) & (1, 1), \\
  O(n^{-\min(P,Q)}) & \text{otherwise.} 
  \end{cases} \]

- gives log terms that cannot be balanced and so we set these to zero
  \[ 0 = Y_1(A_0C_0)', \]
  \[ A_0C_0 = K_0^2 = \text{const.} \]
  \[ A_0 = K_0L^{-1/2}, \quad C_0 = K_0L^{1/2}. \]
Large-n expansion 3: uniformity assumption

- substitute expansion in equations:
  \[ F_n = n^{-2} e^{-n \delta(Y_1)} (A_0(Y_1) + n^{-1} A_1(Y_1) + \cdots), \]
  \[ H_n = e^{-n \delta(Y_1)} (C_0(Y_1) + n^{-1} C_1(Y_1) + \cdots) \]
  \[ Y_1^2 H'_{n} + \eta^{-1} n F_n = -S_n, \]
  \[ Y_1 G_n = Y_1 F'_n + (nY_1 - n - 1) F_n, \]
  \[ Y_1 H_n = \eta^{-1} [Y_1 G'_n + (nY_1 - n) G_n] + 2n \cos \phi G_n + \eta n^2 Y_1^{-1} F_n, \]
  \[ S_n = \sum_{r=1}^{n-1} [(n-r) Y_1 F'_r H_{n-r} - r Y_1 F_r H'_{n-r} - n F_r H_{n-r}], \]

- leading non-log term gives ODE
  \[ Y_1^2 C'_0 = \gamma E Y_1 (A_0 C_0)' + \sum_{r=1}^\infty r^{-1} Y_1 [(\hat{F}_r - A_0) C_0]' + (Y_1 A'_0 - A_0) C_0, \]

- if we make a uniformity assumption that the limiting functions also expand in inverse powers of \( Y_1 \) for large \( Y_1 \), we are left with equating
  \[ Y_1^2 C'_0 \sim -C_{01} \quad -A_0 C_0 \sim -A_{00} C_{00} \quad A_0(Y_1) = A_{00} + A_{01} Y_1^{-1} + \cdots \]

- fixes some terms
  \[ C_{01} = A_{00} C_{00} = K_0^2 \quad K_0 = \eta^{1/2} (\cos \phi - \eta^{-1}) \]
Large-n expansion 4: tying the tails

- have expanded and focussed on right-hand tail of manifold

- could equally well expand and look at left-hand tail

- equating in an overlap region only compatible with parallel original modes

\[ K_0 = \eta^{1/2}(\cos \phi - \eta^{-1}) = -\bar{K}_0 = -\eta^{-1/2}(\cos \phi - \eta). \]

\[ \cos \phi = 1 \quad \phi = 0 \]
Results

• have shown that vorticity blow-up on singular manifold \( \omega_{ps} \sim s^{-\beta} \)

• with \( \beta = 1 \) corresponding to \( \alpha = 0 \) (as \( \alpha = 1 - \beta \), ) plus our asymptotic expansions and uniformity assumption

• corresponds only to the case of initially parallel modes \( \phi = 0 \),

• have analytical results about tails of the manifold,

• borne out by exact solutions (Maple) for \( \phi = 0 \),

\[
\begin{align*}
f_n &= \sum_{j=0}^{n\eta} \sum_{k=0}^{n} f_{njk} Y^{-j} e^{kY}, \\
h_n &= \sum_{j=0}^{n\eta} \sum_{k=0}^{n} h_{njk} Y^{-j} e^{kY}
\end{align*}
\]

\[
\begin{align*}
f_1 &= 2Y^{-2}e^Y - 2Y^{-1} - 2Y^{-2}, \\
h_1 &= (1 + Y^{-1})e^Y - Y^{-1}, \\
f_2 &= (Y^{-3} - 3Y^{-4})e^{2Y} + 4Y^{-3}e^Y + Y^{-3} + 3Y^{-4}, \\
h_2 &= (2Y^{-1} - 3Y^{-2} - \frac{5}{2}Y^{-3})e^{2Y} + (2Y^{-1} + 6Y^{-2} + 2Y^{-3})e^Y + \frac{1}{2}Y^{-3}, \\
f_3 &= \left(\frac{8}{9}Y^{-4} - \frac{122}{27}Y^{-5} + \frac{160}{27}Y^{-6}\right)e^{3Y} + \left(4Y^{-4} - 10Y^{-5}\right)e^{2Y} \\
&\quad + \left(4Y^{-4} - 2Y^{-5}\right)e^Y - \frac{34}{27}Y^{-5} - \frac{160}{27}Y^{-6}, \\
h_3 &= \left(4Y^{-2} - \frac{41}{3}Y^{-3} + \frac{73}{9}Y^{-4} + \frac{179}{27}Y^{-5}\right)e^{3Y} + \left(8Y^{-2} - 22Y^{-4} - 5Y^{-5}\right)e^{2Y} \\
&\quad + \left(2Y^{-2} + 9Y^{-3} + 5Y^{-4} - Y^{-5}\right)e^Y - \frac{17}{27}Y^{-5},
\end{align*}
\]
Discussion

• have good description of singular manifold for 2-d Euler flow in limiting case of parallel modes $\phi = 0$, and vorticity singularity $\beta = 1$, $\alpha = 0$; $\alpha = 1 - \beta$, still do not know everything though

• with parameters $\{\eta, \phi\}$ problem of general angle $\phi$ or general exponent $\alpha(\eta, \phi)$ remains elusive

• criticisms: `uncontrolled' expansions, no rigorous theorem

• unclear how to relax present framework to cope with general case

• need better numerics to motivate analysis, need better analysis to improve numerics

• 3-d has not yet been looked at
Ferromagnetic microswimmers

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Magnetically driven swimmers

• two unequal beads

• elastically coupled

• in an external field

• both beads are magnetic dipoles and so there is a force between them

• one bead is `hard': magnetic direction fixed

• one bead is `soft' (ferromagnetic) and direction tends to follow external field
Simplify

- non-dimensionalise and drop inertia
- internal dynamics of swimmer
  \[
  \begin{align*}
  \omega \dot{s} + s - 1 &= A_{\text{mag}} s^{-4} [\cos(\alpha_2 - \alpha_1) - 3 \cos \alpha_1 \cos \alpha_2], \\
  \omega s^2 \dot{\phi} &= A_{\text{ext}} b [\sigma \sin(\psi - \phi - \alpha_1) + \sigma^{-1} \sin(\psi - \phi - \alpha_2)]
  \end{align*}
  \]
- motion of Centre of Reaction
  \[
  (\rho + \rho^{-1}) X = -\frac{3\epsilon}{4} (\chi_2 - \chi_1) \int_0^t s^{-1} (2s \hat{r} + s \hat{\phi} \hat{\phi}) \, dt
  \]
- magnetic dipole angles
  \[
  \zeta \omega \alpha_j = (1 - \kappa_j) b \sin(\psi - \phi - \alpha_j) - \kappa_j \sin 2\alpha_j,
  \]
- take one hard magnet (fixed) and one completely soft (follows external field)
Simulation: soft-hard system

Swimming! NW direction
Synchronised swimming and braiding

- effect of one swimmer on another is approximately inverse square law velocity
- e.g. 2 swimmers rotate about each other
- 3 can give braiding
Streak lines and mixing

- release trail of particles from above one swimmer
- on 2 different scales
Discussion

- laboratory realised devices for experiment and biomedical application
- swimming and mixing
- mixing on small and larger scales