Nonlinear Evolution of a Vortex Ring

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The vortex ring can be regarded as the simplest vortical structure which has curvature. Its stability is one of the important problems in vortex dynamics not only in its own right but also from a wider perspective of curved vortical structures. Recently we have discovered a new linear instability of a vortex ring, which is a direct effect of curvature, by normal-mode analysis [1] and local stability analysis [2] assuming vanishing viscosity. It is an $O(\epsilon)$ curvature effect, with $\epsilon$ being a ratio of the core radius to the ring radius of the vortex ring, in contrast to the well-known Widnall instability which is an $O(\epsilon^2)$ effect. The maximum growth rate of this curvature instability is larger than that of the Widnall instability in a large region of parameter space.

Our next step is to investigate when the curvature instability is observed and what role it takes in practical situations. Although the Widnall instability has been observed in the experiments there have been no results which show the curvature instability to our knowledge. One of the important differences between our linear stability analysis and most of the experiments is in the vorticity distribution. We chose Kelvin's vortex ring as base flow; its vorticity has a jump at the boundary of the core. It is most likely that the vortex rings created in the experiments have smooth vorticity distributions. The wave on a smooth vortex ring may decay and may be no longer neutral if there is a critical layer at which relative angular velocity vanishes. Bending waves, which are subjected to the Widnall instability, have no critical layer even for a smooth vortex ring. On the other hand the waves on Kelvin's vortex ring which are subjected to curvature instability possess critical layer when the vorticity distribution is smoothed.

If we consider a vortex ring confined in a torus, however, the waves do not have critical layers since they usually appear in the skirt of vorticity distribution and are outside of the torus. In this paper the nonlinear evolution of the vortex ring confined in a torus is investigated in order to seek a possible occurrence of the curvature instability. After establishing numerical evidence, the vortex ring in a torus would be a suitable model for investigating the interaction of waves on vortices and the route to turbulence.

Direct numerical simulation is performed by solving the three-dimensional Navier-Stokes equations expressed in the toroidal coordinate system $(r, \theta, s)$. Spectral method is used for spatial discretization: Chebyshev collocation method for $r$ and Fourier collocation method for $\theta$ and $s$. Solving the Poisson equation with high accuracy, which usually appears in the numerical simulation of incompressible flows, is not a simple task in the present case. This is because the Poisson equation has terms of which coefficients depend on $\theta$ so that it is not separable. Since high accuracy is required to resolve small amplitude waves, we have developed a new method for solving the Poisson equation in the toroidal coordinate system efficiently with high accuracy.

Figures 1 and 2 show the numerical evidence of curvature instability. The energies of the unstable waves are seen to grow exponentially (Fig. 1). They are close to the solid line which corresponds to the theoretical prediction $\propto e^{2\sigma t}$ [1]. Actually the theoretical line is a little steeper than the numerically obtained lines; this would be due to viscous effects. The structure of the destabilized vortex ring is shown in Fig. 2. More details will be reported in the talk.

Fig.1: Evolution of energy of unstable modes.

Fig.2: Vorticity distribution of growing disturbance. $t = 40$. Contours of the component normal to a cross section.
