Ferromagnetic microswimmers

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Swimming

- Motion of a body through a fluid by means of change of shape

\[ \partial_t u + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u \]
\[ \nabla \cdot u = 0 \]

- Reynolds number \( \text{Re} = \text{Length} \times \text{Velocity} / \text{Viscosity} \)

- Familiar with swimming at large-ish Re (also ships, planes, fish ...): inertia important

- On small scales (or slow motions, viscous fluids) Re is small and inertia irrelevant

- Flows are dominated by viscosity
Purcell's scallop theorem

- Simple, time-reversible motions do not generate persistent motion
- e.g. scallop opening and closing, tail flapping back and forth
- but a more complex motion, for example rotating a spiral tail, can do
- movies ....
- much work on how microorganisms swim, but how can we fabricate a swimmer? How can we provide an energy source?
- Why? Drug delivery, microscale mixers, pumps, valves, lab-on-a-chip devices
Magnetically driven swimmers

- two unequal beads
- elastically coupled
- in an external field

- both beads are magnetic dipoles and so there is a force between them
- one bead is 'hard': magnetic direction fixed
- one bead is 'soft' (ferromagnetic) and direction tends to follow external field
Mechanisms

• torque from the external field on the beads
  • balanced by Stokes drag

• external field affects magnet dipole direction and so attraction/repulsion between beads
  • balanced by Stokes drag + elastic force
Lagrangian formalism

- generalised coordinates \( \mathbf{q} = (X, Y, s, \phi) \)
- positions \( \mathbf{r}_1 = X + \chi_1 s \hat{r}, \quad \mathbf{r}_2 = X - \chi_2 s \hat{r} \)
- velocities \( \mathbf{\dot{r}}_1 = \dot{X} + \chi_1 (s \dot{\hat{r}} + s \dot{\phi} \dot{\phi}), \quad \mathbf{\dot{r}}_2 = \dot{X} - \chi_2 (s \dot{\hat{r}} + s \dot{\phi} \dot{\phi}) \).
- choose \( \mathbf{X} = (X, Y) \) as the Centre of Reaction (cf. Centre of Mass)

\[ \chi_1 = \frac{R_2}{R_1 + R_2}, \quad \chi_2 = \frac{R_1}{R_1 + R_2}. \]
Lagrangian: conservative pieces

- Lagrangian
  \[ L = T - V_{\text{spring}} - V_{\text{mag}} - V_{\text{ext}} \]

- K.E.
  \[ T = \frac{1}{2}(\nu_1 + \nu_2) \dot{X}^2 + \frac{1}{2}(\nu_1 \chi_1 + \nu_2 \chi_2)(\dot{s}^2 + s^2 \dot{\phi}^2) + (\nu_1 \chi_1 - \nu_2 \chi_2)(\dot{s} \hat{r} + s \dot{\phi} \hat{\phi}) \cdot \dot{X} \]

- Spring energy
  \[ V_{\text{spring}} = \frac{1}{2} k(s - l_0)^2 \]

- Dipole interaction energy
  \[ V_{\text{mag}} = \frac{\mu_0}{4\pi s^3} \left[ m_1 \cdot m_2 - 3(m_1 \cdot \hat{r})(m_2 \cdot \hat{r}) \right] \]
  \[ = \frac{\mu_0 m_1 m_2}{4\pi s^3} \left[ \cos(\alpha_2 - \alpha_1) - 3 \cos \alpha_1 \cos \alpha_2 \right] \]

- External field energy
  \[ V_{\text{ext}} = -m_1 \cdot B_{\text{ext}} - m_2 \cdot B_{\text{ext}} \]
  \[ = -B_{\text{ext}} b [m_1 \cos(\phi + \alpha_1 - \psi) + m_2 \cos(\phi + \alpha_2 - \psi)] \]

- Angles \( \alpha_j(t) \) taken as prescribed (outside Lagrangian framework)

- External field
  \[ B_{\text{ext}} = B_{\text{ext}}(b_x, b_y) \quad b_x \equiv b \cos \psi = \alpha \cos t, \quad b_y \equiv b \sin \psi = \beta \sin t \]
Lagrangian: fluid drag and interaction

- Lagrangian \( L(q, \dot{q}, t) \)

- Euler-Lagrange equations
  \[
  p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad \dot{p}_i - \frac{\partial L}{\partial q_i} = Q_i, \quad Q_i = \sum_j F_j \cdot \frac{\partial r_j}{\partial q_i}
  \]

- Generalised forces \( Q = F_1 + F_2, \quad Q_s = (\chi_1 F_1 - \chi_2 F_2) \cdot \hat{r}, \quad Q_\phi = (\chi_1 F_1 - \chi_2 F_2) \cdot s\hat{\phi} \)

- Fluid drag and interaction terms
  \[
  F_1 = -6\pi \eta R_1 \dot{r}_1 + \frac{9\pi \eta R_1 R_2}{2s} (\hat{r} \hat{r} + \mathbf{I}) \cdot \dot{r}_2 + \cdots
  \]

- Expansion in a small parameter
  \[
  \varepsilon = \sqrt{R_1 R_2} / l_0 \ll 1
  \]
Magnetic dipole evolution

- Soft magnets adjust to external field on a nanosecond timescale $\tau$.

- Minimise energy
  \[ E_j = -\mathbf{M}_j \cdot \mathbf{H}_{\text{ext}} + K_j \sin^2 \alpha_j. \]

- Use gradient flow
  \[ \tau \dot{\alpha}_j = (1 - \kappa_j) b \sin(\psi - \phi - \alpha_j) - \kappa_j \sin 2\alpha_j. \]

- $\alpha_j(t)$ angle with `easy axis'.

- Entirely soft magnet: $\kappa_j = 0$, $\alpha_j = \psi - \phi$.

- Entirely hard magnet: $\kappa_j = 1$, $\alpha_j = 0$.

- Possible hysteretic effects: jumps in direction.
Simplify

• non-dimensionalise and drop inertia (Stokes' regime) to obtain

• internal dynamics of swimmer

\[ \omega \dot{s} + s - 1 = A_{\text{mag}} s^{-4} [\cos(\alpha_2 - \alpha_1) - 3 \cos \alpha_1 \cos \alpha_2], \]
\[ \omega s^2 \dot{\phi} = A_{\text{ext}} b [\sigma \sin(\psi - \phi - \alpha_1) + \sigma^{-1} \sin(\psi - \phi - \alpha_2)] \]

• motion of Centre of Reaction

\[ (\rho + \rho^{-1}) \mathbf{X} = -\frac{3\varepsilon}{4} (\chi_2 - \chi_1) \int_0^t \frac{1}{s} (2\dot{s}\hat{r} + s\dot{\phi}\hat{\phi}) \, dt \]
\[ (\rho + \rho^{-1}) \mathbf{X}_{\text{int}} = \frac{3\varepsilon}{4} (\chi_2 - \chi_1) \int_0^t 2 \log s \dot{\phi}\hat{\phi} \, dt. \]

• magnetic dipole angles

\[ \zeta \omega \dot{\alpha}_j = (1 - \kappa_j) b \sin(\psi - \phi - \alpha_j) - \kappa_j \sin 2\alpha_j, \]
Simulations: reference parameters

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<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
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<td>$l_0 = 4R_1 = 8R_2$</td>
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<td>$(\alpha, \beta)$</td>
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Simulation 1: full system

Swimming! NE direction
Simulation 2: soft-hard system

\( \kappa_2 = 1 \)

Swimming! NW direction
Simulation 3: intermediate \( \kappa_2 = 0.7 \).

Not swimming ... need to `trap' swimmer orientation \( \phi(t) \).
Simplify to model 0

- Full problem
  \[ \omega \dot{s} + s - 1 = A_{\text{mag}} s^{-4} \left[ \cos(\alpha_2 - \alpha_1) - 3 \cos \alpha_1 \cos \alpha_2 \right], \]
  \[ \omega s^2 \dot{\phi} = A_{\text{ext}} b \left[ \sigma \sin(\psi - \phi - \alpha_1) + \sigma^{-1} \sin(\psi - \phi - \alpha_2) \right] \]
  \[ (\rho + \rho^{-1}) \mathbf{X}_{\text{int}} = \frac{3\varepsilon}{4} \left( \chi_2 - \chi_1 \right) \int_0^t 2 \log s \, \dot{\phi} \dot{\phi} \, dt. \]

- take magnet 1 soft \( \alpha_j = \psi - \phi \) and magnet 2 hard \( \alpha_j = 0 \)

- simplify \( \omega \ll 1, \quad A_{\text{mag}} \ll 1 \quad s \simeq 1 \)

- and rescale to yield `model 0'
  \[ \dot{\phi} = Ab \sin(\psi - \phi), \quad \tilde{\mathbf{X}}_{\text{int}} = -\int_0^t 4 \cos(\psi - \phi) \dot{\phi} \dot{\phi} \, dt. \]

- just one parameter and a diagnostic: \( s = 1 - 2 A_{\text{mag}} \cos(\psi - \phi) \)
Does model 0 swim (unidirectional motion)?

- Parameter $A=1.46$ as in earlier runs:

- similar to intermediate case:
Does model 0 swim?

- Parameter A reduced by 75%
Does model 0 swim?

- Parameter A reduced further and further:
- moves more and more slowly in bigger and bigger path
- no `trapping' swimmer orientation
- straightforward multiple scales analysis

\[
\phi_0 = \bar{\phi}_0 = \frac{\alpha \beta \tau}{2} \equiv \Omega_0 t, \quad \Omega_0 \equiv \frac{1}{2} \alpha \beta A^2
\]

\[
\tilde{X}_{\text{int}}(t) = \frac{4 A \Lambda}{3 \Omega_0} (\sin^3 \Omega_0 t, 1 - \cos^3 \Omega_0 t).
\]

\[
\Lambda(\alpha, \beta) = \langle (\alpha^2 \cos^2 t - \beta^2 \sin^2 t)(\alpha^2 \cos^2 t + \beta^2 \sin^2 t)^{-1/2} \rangle
\]

\[
= \frac{2}{\pi \alpha} \frac{\alpha^2}{\alpha^2 - \beta^2} \left[ (\alpha^2 + \beta^2) E(1 - \alpha^2/\beta^2) - 2 \beta^2 K(1 - \beta^2/\alpha^2) \right]
\]
Model 0+?

- Focus on mechanisms for trapping: IF we can trap the orientation then we are finished. Focus only on equation for the angle

- Full equation:

  $$\omega s^2 \dot{\phi} = A_{\text{ext}} b [\sigma \sin(\psi - \phi - \alpha_1) + \sigma^{-1} \sin(\psi - \phi - \alpha_2)]$$

- Model 0 version (unrescaled):

  $$\omega \dot{\phi} = A_{\text{ext}} b \sigma^{-1} \sin(\psi - \phi)$$

  - reintroduce $s^2$ term - model 0+MoR

  - make magnet 2 not entirely hard - model 0+nH

  - make magnet 1 not entirely soft - model 0+nS

  - or any combination
Model 0+MoR
Model 0+MoR theory

- equations

\[ s = 1 - 2A_{\text{mag}} \cos(\psi - \phi), \quad \dot{\phi} = A \dot{s}^{-2} b \sin(\psi - \phi), \]
\[ \dot{X}_{\text{int}} = - \int_0^t 4 \cos(\psi - \phi) \dot{\phi} \dot{\phi} \, dt. \]

- multiple scale analysis

\[ \frac{d\phi_0}{dt} = \frac{1}{2} \alpha \beta A^2 - 2AA_{\text{mag}} A \sin 2\phi_0 \]

- trapping threshold

- moves N-NW or S-SE

\[ \dot{X}_{\text{int}} \sim (- \sin \phi_0, \cos \phi_0) \tilde{V} t, \quad \tilde{V} = 2A \Lambda \sin 2\phi_0 = \alpha \beta A^2 / 2A_{\text{mag}} \]

- cf angular momentum and drag term

\[ (\nu_1 \chi_1 + \nu_2 \chi_2) s^2 \dot{\phi} \quad 6\pi \eta (\chi_1^2 R_1 + \chi_2^2 R_2) s^2 \dot{\phi} \]

- cf soft-hard system
Model 0+nH
Model 0+nH theory

- equations

\[ s = 1 - 2A_{\text{mag}}[\cos(\psi - \phi) \cos \alpha_2 - \frac{1}{2} \sin(\psi - \phi) \sin \alpha_2], \]
\[ \dot{\phi} = Ab \sin(\psi - \phi - \alpha_2), \]
\[ \zeta \omega \dot{\alpha}_2 = (1 - \kappa_2)b \sin(\psi - \phi - \alpha_2) - \kappa_2 \sin 2\alpha_2, \]
\[ \tilde{X}_{\text{int}} = -\int_0^t 4[\cos(\psi - \phi) \cos \alpha_2 - \frac{1}{2} \sin(\psi - \phi) \sin \alpha_2] \dot{\phi} \dot{\Phi} \, dt \]

- multiple scale analysis

- trapping threshold

- moves E-NE or W-SW

- cf full system
Model 0+nS
Model 0+nS theory

- equations

\[
\begin{align*}
    s &= 1 - 2A_{\text{mag}} \cos \alpha_1, \\
    \dot{\phi} &= A b [\sigma^2 \sin(\psi - \phi - \alpha_1) + \sin(\psi - \phi)], \\
    \zeta \omega \dot{\alpha}_1 &= (1 - \kappa_1) b \sin(\psi - \phi - \alpha_1) - \kappa_1 \sin 2\alpha_1, \\
    \mathbf{\tilde{X}}_{\text{int}} &= -\int_0^t 4 \cos \alpha_1 \dot{\phi} \dot{\phi} \, dt,
\end{align*}
\]

- trapping threshold

\[
\frac{d\phi_0}{dt} = \frac{1}{2} \alpha_1 \beta A^2 - A \sigma^2 \kappa_1 \frac{\alpha - \beta}{\alpha + \beta} \sin 2\phi_0.
\]

- moves N-NW or S-SE

\[
\mathbf{\tilde{X}}_{\text{int}} \approx (- \sin \phi_0, \cos \phi_0) \tilde{V} \, t, \quad \tilde{V} = \frac{\alpha_1 \beta A^2 \Lambda}{\sigma^2 \kappa_1} \frac{\alpha + \beta}{\alpha - \beta}
\]
Discussion

- Set up simple model: isolated role of fluid interactions for swimming motion
- isolated trapping mechanisms and relevant parameters: rules of thumb
  - e.g. trapping from MoR, nS reinforce, from nH opposes
- ? comparison with experiment ?
- ? motion in 3-d
- ? collective motion, interaction with boundaries
- ? viscoelastic medium
- ? more complex magnetic/elastic/fluid systems