Ferromagnetic microswimmers

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Swimming

- Motion of a body through a fluid by means of change of shape

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} \qquad \nabla \cdot \boldsymbol{u} = 0$$

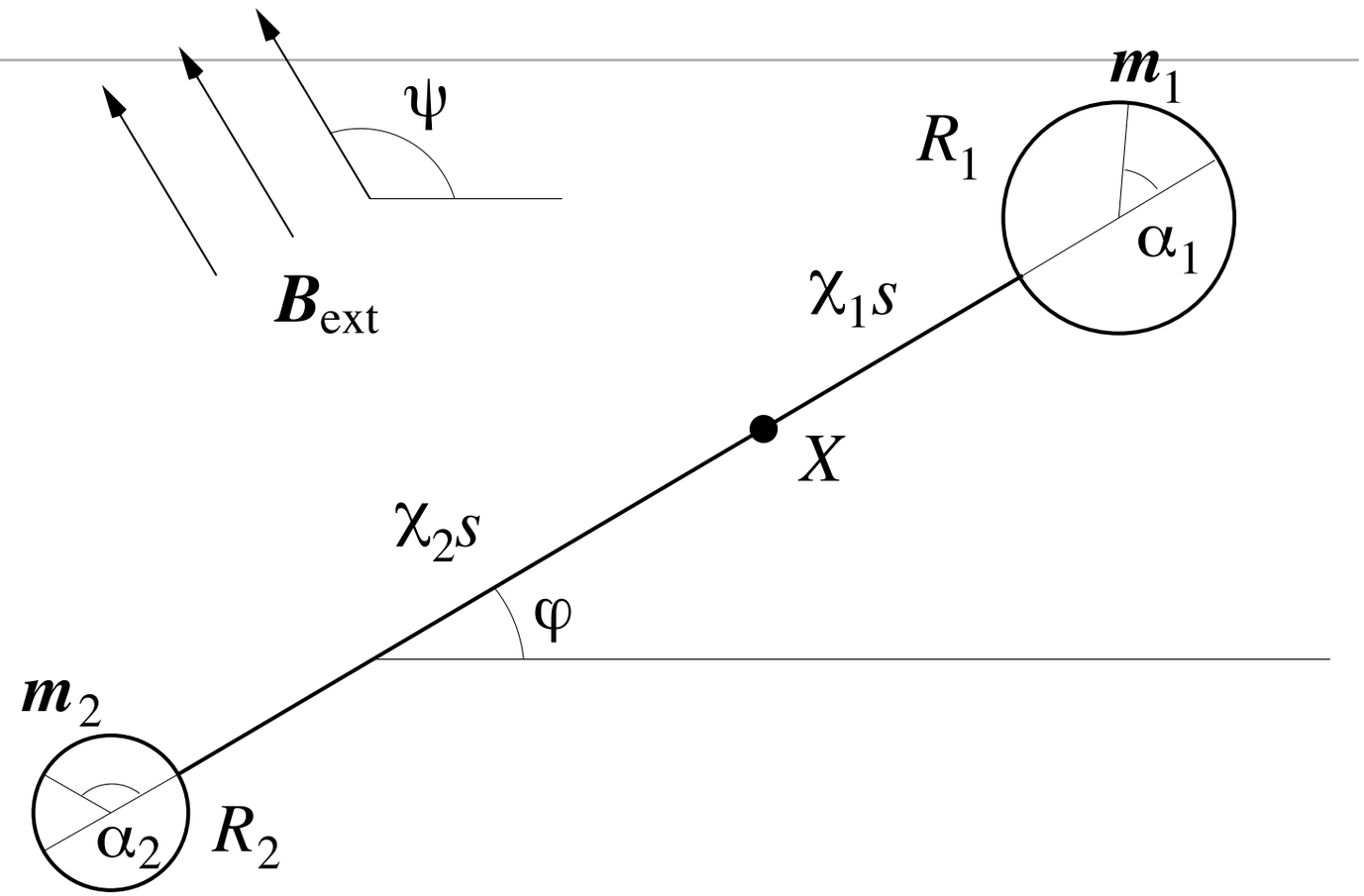
- Reynolds number Re = Length * Velocity / Viscosity
- Familiar with swimming at large-ish Re (also ships, planes, fish ...): inertia important
- On small scales (or slow motions, viscous fluids) Re is small and inertia irrelevant
- Flows are dominated by viscosity

Purcell's scallop theorem

- Simple, time-reversible motions do not generate persistent motion
- e.g. scallop opening and closing, tail flapping back and forth
- but a more complex motion, for example rotating a spiral tail, can do
- movies ....
- much work on how microorganisms swim, but how can we fabricate a swimmer? How can we provide an energy source?
- Why? Drug delivery, microscale mixers, pumps, valves, lab-on-a-chip devices

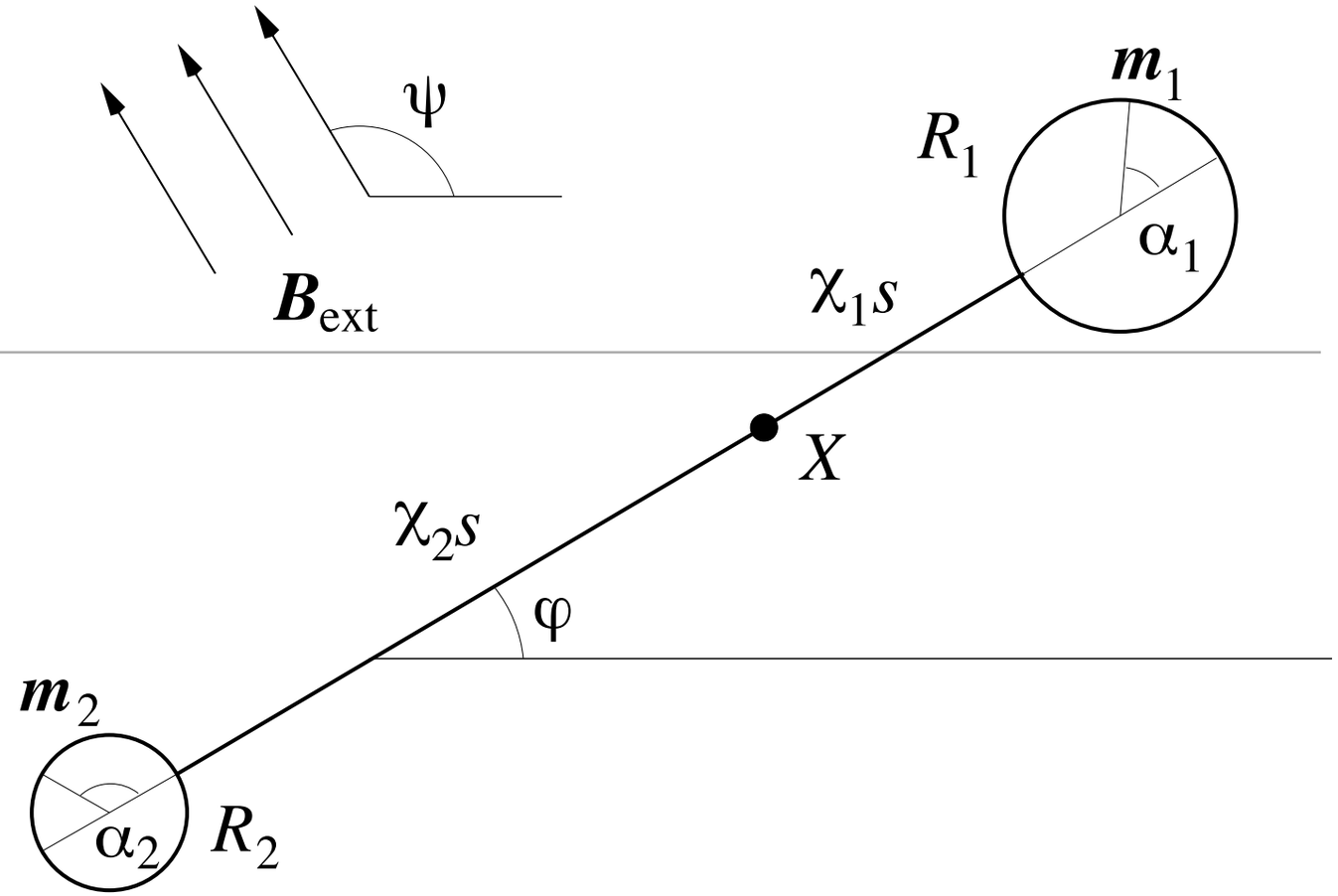
Magnetically driven swimmers

- two unequal beads
- elastically coupled
- in an external field



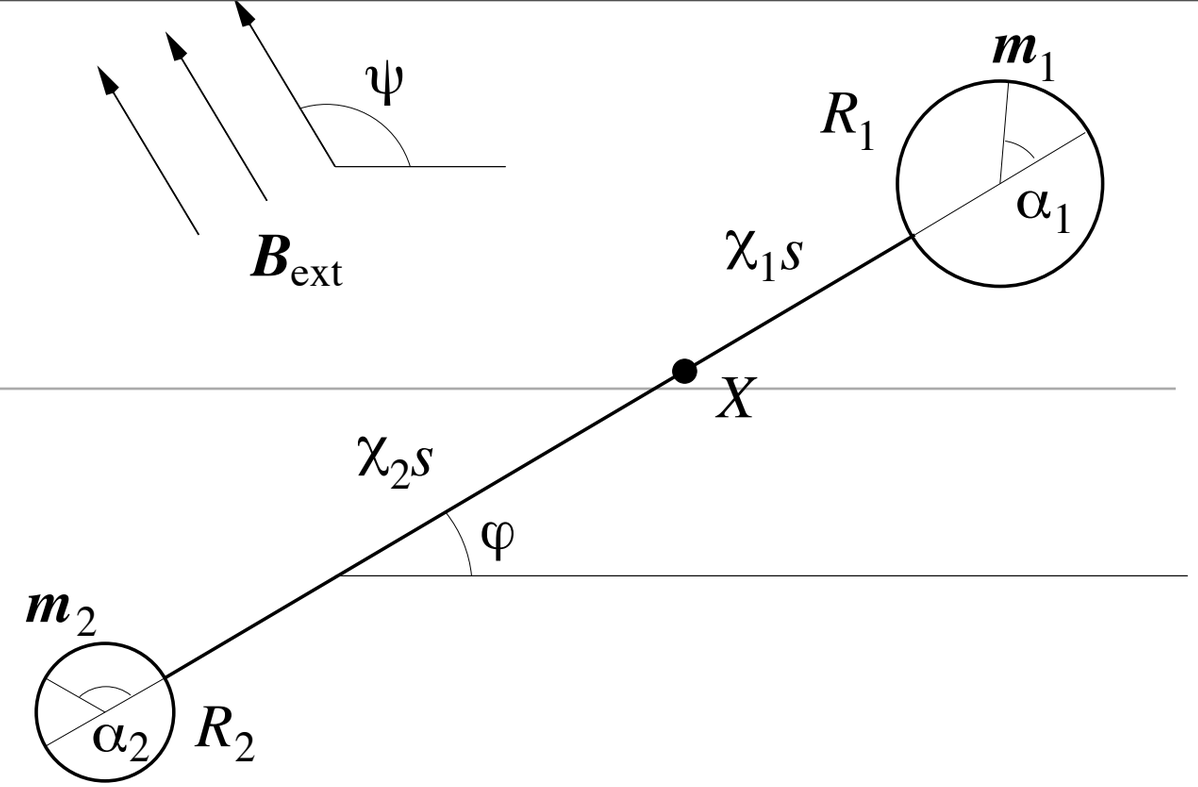
- both beads are magnetic dipoles and so there is a force between them
- one bead is `hard': magnetic direction fixed
- one bead is `soft' (ferromagnetic) and direction tends to follow external field

Mechanisms



- torque from the external field on the beads
  - balanced by Stokes drag
- external field affects magnet dipole direction and so attraction/repulsion between beads
  - balanced by Stokes drag + elastic force

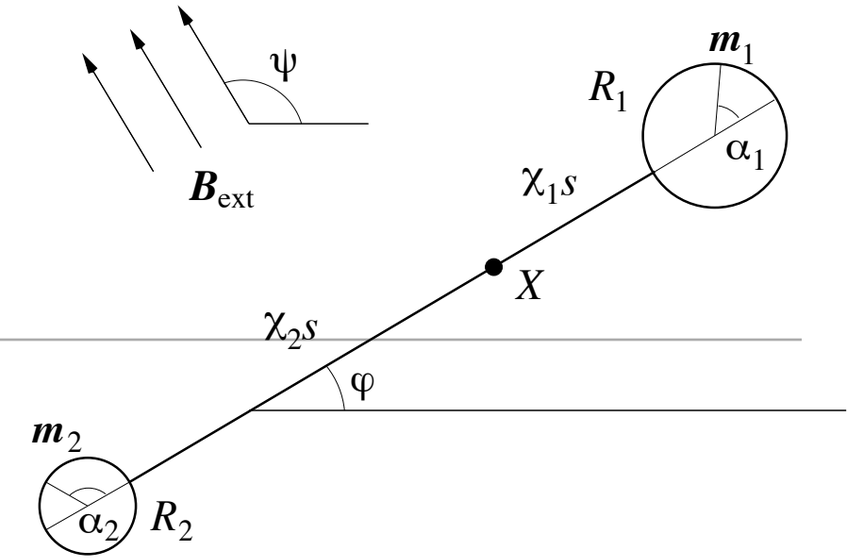
Lagrangian formalism



- generalised coordinates $\boldsymbol{q} = (X, Y, s, \phi)$
- positions $\boldsymbol{r}_1 = \boldsymbol{X} + \chi_1 s\hat{\boldsymbol{r}}, \quad \boldsymbol{r}_2 = \boldsymbol{X} - \chi_2 s\hat{\boldsymbol{r}}$
- velocities $\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{X}} + \chi_1(\dot{s}\hat{\boldsymbol{r}} + s\dot{\phi}\hat{\boldsymbol{\phi}}), \quad \dot{\boldsymbol{r}}_2 = \dot{\boldsymbol{X}} - \chi_2(\dot{s}\hat{\boldsymbol{r}} + s\dot{\phi}\hat{\boldsymbol{\phi}}).$
- choose $\boldsymbol{X} = (X, Y)$ as the Centre of Reaction (cf. Centre of Mass)

$$\chi_1 = R_2/(R_1 + R_2), \quad \chi_2 = R_1/(R_1 + R_2).$$

Lagrangian: conservative pieces



- Lagrangian $$L = T - V_{\text{spring}} - V_{\text{mag}} - V_{\text{ext}}$$

- K.E. $$T = \tfrac{1}{2}(\nu_1 + \nu_2)\dot{\boldsymbol{X}}^2 + \tfrac{1}{2}(\nu_1\chi_1 + \nu_2\chi_2)(\dot{s}^2 + s^2\dot{\phi}^2) + (\nu_1\chi_1 - \nu_2\chi_2)(\dot{s}\hat{\boldsymbol{r}} + s\dot{\phi}\hat{\boldsymbol{\phi}}) \cdot \dot{\boldsymbol{X}}$$

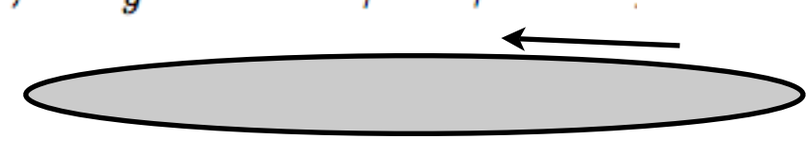
- spring energy $$V_{\text{spring}} = \tfrac{1}{2}k(s - l_0)^2$$

- dipole interaction energy
$$\begin{aligned} V_{\text{mag}} &= \frac{\mu_0}{4\pi s^3}\left[\boldsymbol{m}_1 \cdot \boldsymbol{m}_2 - 3(\boldsymbol{m}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{m}_2 \cdot \hat{\boldsymbol{r}})\right] \\ &= \frac{\mu_0 m_1 m_2}{4\pi s^3}\left[\cos(\alpha_2 - \alpha_1) - 3\cos\alpha_1\cos\alpha_2\right] \end{aligned}$$

- external field energy
$$\begin{aligned} V_{\text{ext}} &= -\boldsymbol{m}_1 \cdot \boldsymbol{B}_{\text{ext}} - \boldsymbol{m}_2 \cdot \boldsymbol{B}_{\text{ext}} \\ &= -B_{\text{ext}} b[m_1\cos(\phi + \alpha_1 - \psi) + m_2\cos(\phi + \alpha_2 - \psi)] \end{aligned}$$

- angles $\alpha_j(t)$ taken as prescribed (outside Lagrangian framework)

- external field $\boldsymbol{B}_{\text{ext}} = B_{\text{ext}}(b_x, b_y)$ $\quad b_x \equiv b\cos\psi = \alpha\cos t, \quad b_y \equiv b\sin\psi = \beta\sin t.$



Lagrangian: fluid drag and interaction

- Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$

- Euler-Lagrange equations $p_i = \dfrac{\partial L}{\partial \dot{q}_i}, \quad \dot{p}_i - \dfrac{\partial L}{\partial q_i} = Q_i, \quad Q_i = \sum_j \boldsymbol{F}_j \cdot \dfrac{\partial \boldsymbol{r}_j}{\partial q_i}$

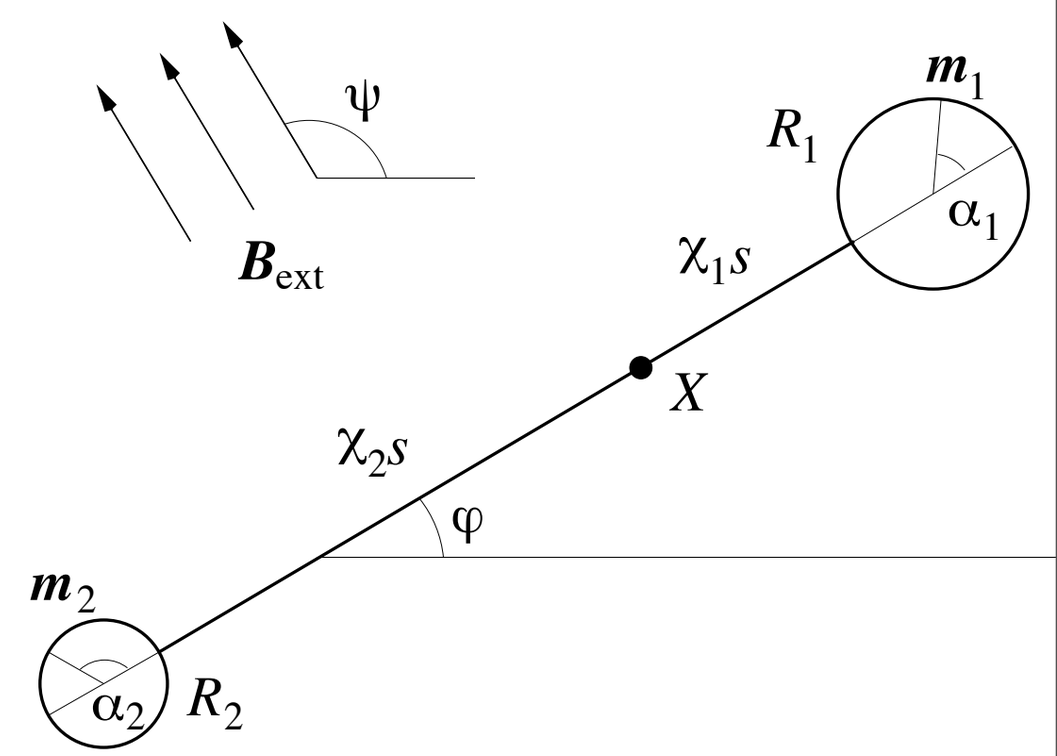
- Generalised forces $\boldsymbol{Q} = \boldsymbol{F}_1 + \boldsymbol{F}_2, \quad Q_s = (\chi_1 \boldsymbol{F}_1 - \chi_2 \boldsymbol{F}_2) \cdot \hat{\boldsymbol{r}}, \quad Q_\phi = (\chi_1 \boldsymbol{F}_1 - \chi_2 \boldsymbol{F}_2) \cdot s\hat{\boldsymbol{\phi}}$

- Fluid drag and interaction terms

$$\boldsymbol{F}_1 = -6\pi\eta R_1 \dot{\boldsymbol{r}}_1 + \frac{9\pi}{2}\frac{\eta R_1 R_2}{s}(\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + I) \cdot \dot{\boldsymbol{r}}_2 + \cdots$$

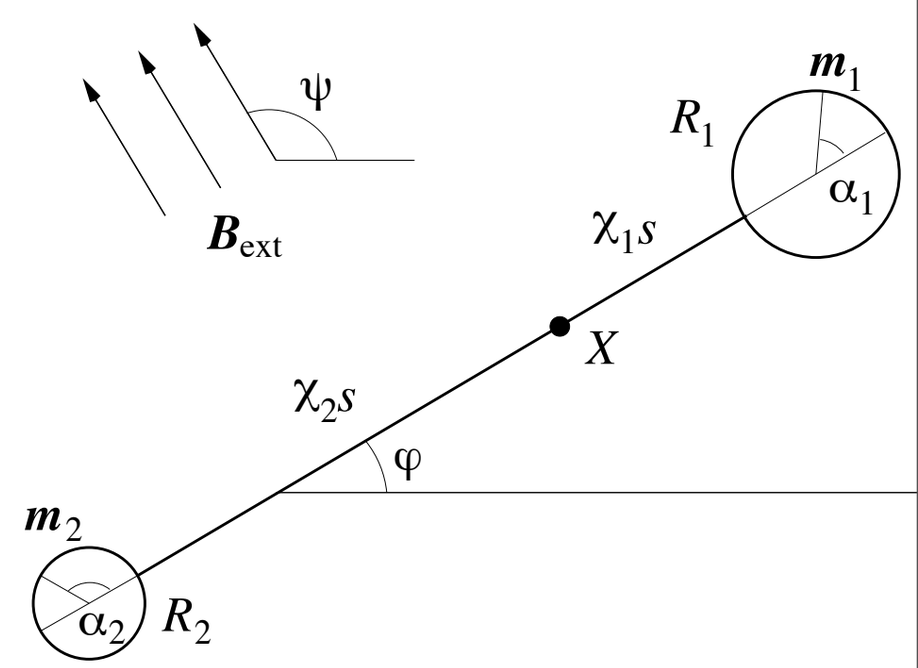
- expansion in a small parameter

$$\varepsilon = \sqrt{R_1 R_2}/l_0 \ll 1$$



Magnetic dipole evolution

- Soft magnets adjust to external field on a nanosecond timescale τ
- Minimise energy $E_j = -\boldsymbol{M}_j \cdot \boldsymbol{H}_{\text{ext}} + K_j \sin^2 \alpha_j$
- use gradient flow $\tau \dot{\alpha}_j = (1-\kappa_j) b \sin(\psi - \phi - \alpha_j) - \kappa_j \sin 2\alpha_j$
- $\alpha_j(t)$ angle with `easy axis'
- entirely soft magnet: $\kappa_j = 0$ $\alpha_j = \psi - \phi$
- entirely hard magnet: $\kappa_j = 1$ $\alpha_j = 0$
- possible hysteretic effects: jumps in direction



Simplify

- non-dimensionalise and drop inertia (Stokes' regime) to obtain

- internal dynamics of swimmer

$$\varpi \dot{s} + s - 1 = A_{\mathrm{mag}} s^{-4} \left[\cos(\alpha_2 - \alpha_1) - 3\cos\alpha_1 \cos\alpha_2\right],$$
$$\varpi s^2 \dot{\phi} = A_{\mathrm{ext}} b[\sigma \sin(\psi - \phi - \alpha_1) + \sigma^{-1} \sin(\psi - \phi - \alpha_2)]$$

- motion of Centre of Reaction

$$(\rho + \rho^{-1})\boldsymbol{X} = -\frac{3\varepsilon}{4}(\chi_2 - \chi_1) \int_0^t s^{-1}(2\dot{s}\hat{\boldsymbol{r}} + s\dot{\phi}\hat{\boldsymbol{\phi}})\, dt$$

$$(\rho + \rho^{-1})\boldsymbol{X}_{\mathrm{int}} = \frac{3\varepsilon}{4}(\chi_2 - \chi_1) \int_0^t 2\log s\, \dot{\phi}\hat{\boldsymbol{\phi}}\, dt.$$

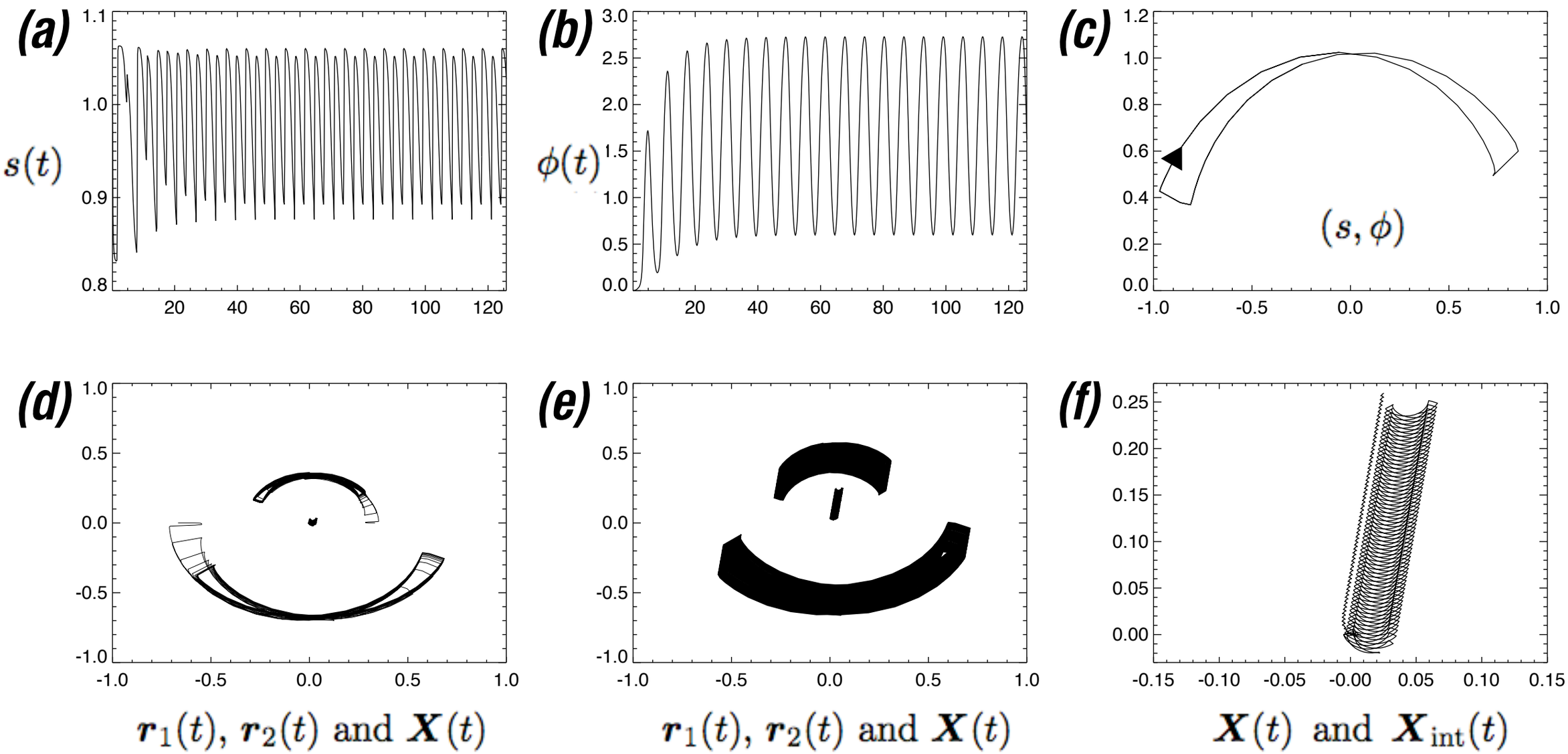
- magnetic dipole angles

$$\varsigma \varpi \dot{\alpha}_j = (1 - \kappa_j) b \sin(\psi - \phi - \alpha_j) - \kappa_j \sin 2\alpha_j,$$

Simulations: reference parameters

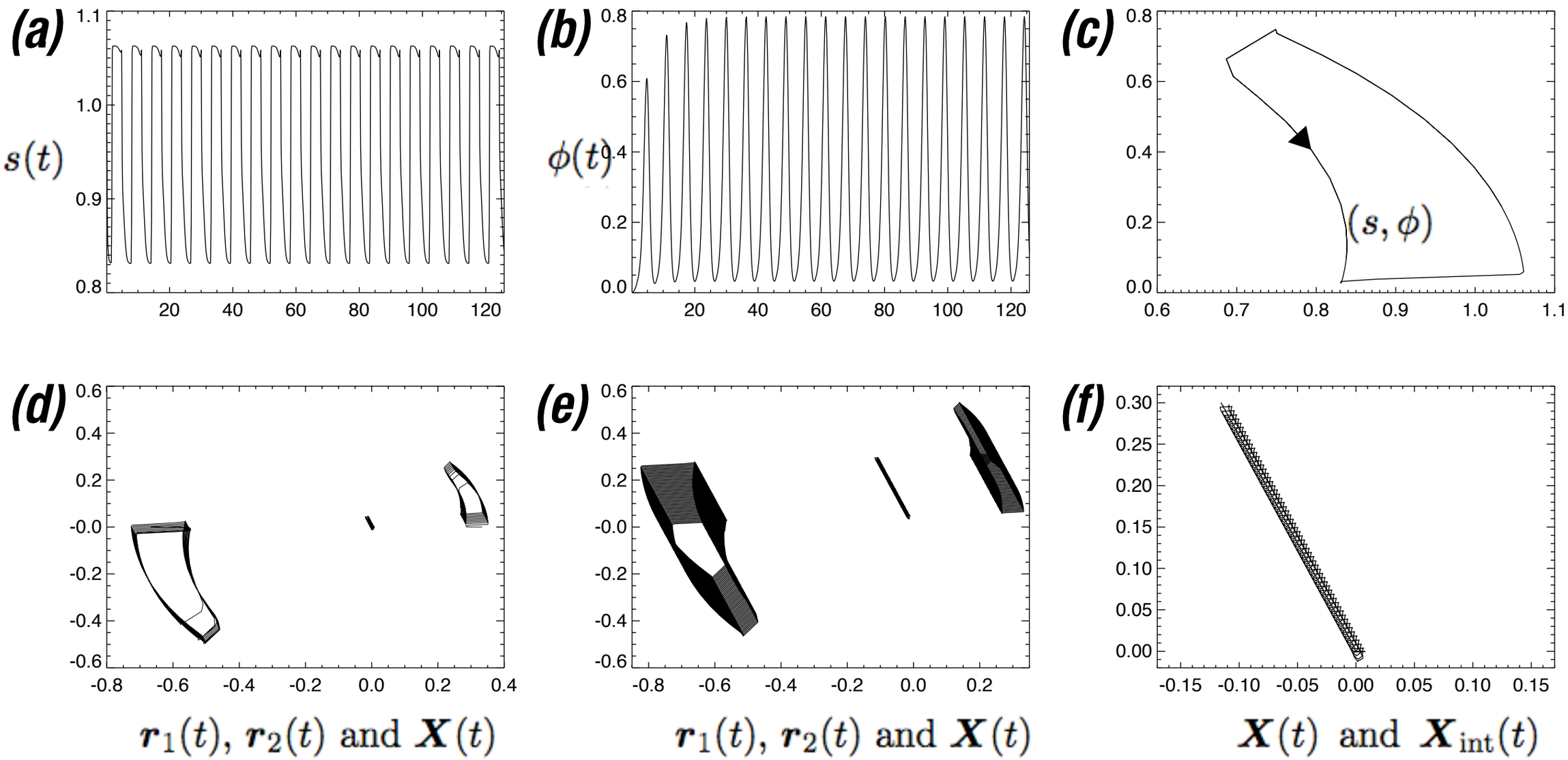
$l_0 = 4R_1 = 8R_2$	6.4×10^{-6}	m	ϖ	0.0503
k	0.05	N/m	υ	2.64×10^{-3}
η	0.1	Pa s	A_{mag}	0.0207
$\nu_1 = 8\nu_2$	1.51×10^{-13}	kg	A_{ext}	0.104
ω	2500	rad/s	ς	4.97×10^{-5}
B_{ext}	0.05	T	ρ	$\sqrt{2}$
$m_1 = 8m_2$	2.40×10^{-11}	A m^2	μ	$2\sqrt{2}$
τ	10^{-9}	s	σ	$2\sqrt{2}$
$\mathcal{L}$	6.4×10^{-6}	m	ε	$2^{-5/2}$
$\mathcal{T}$	4×10^{-4}	s	κ_1	0
$\mathcal{M}$	8×10^{-9}	kg	κ_2	0.5
			A	1.46
			(α, β)	$(1, 0.02)$

Simulation 1: full system



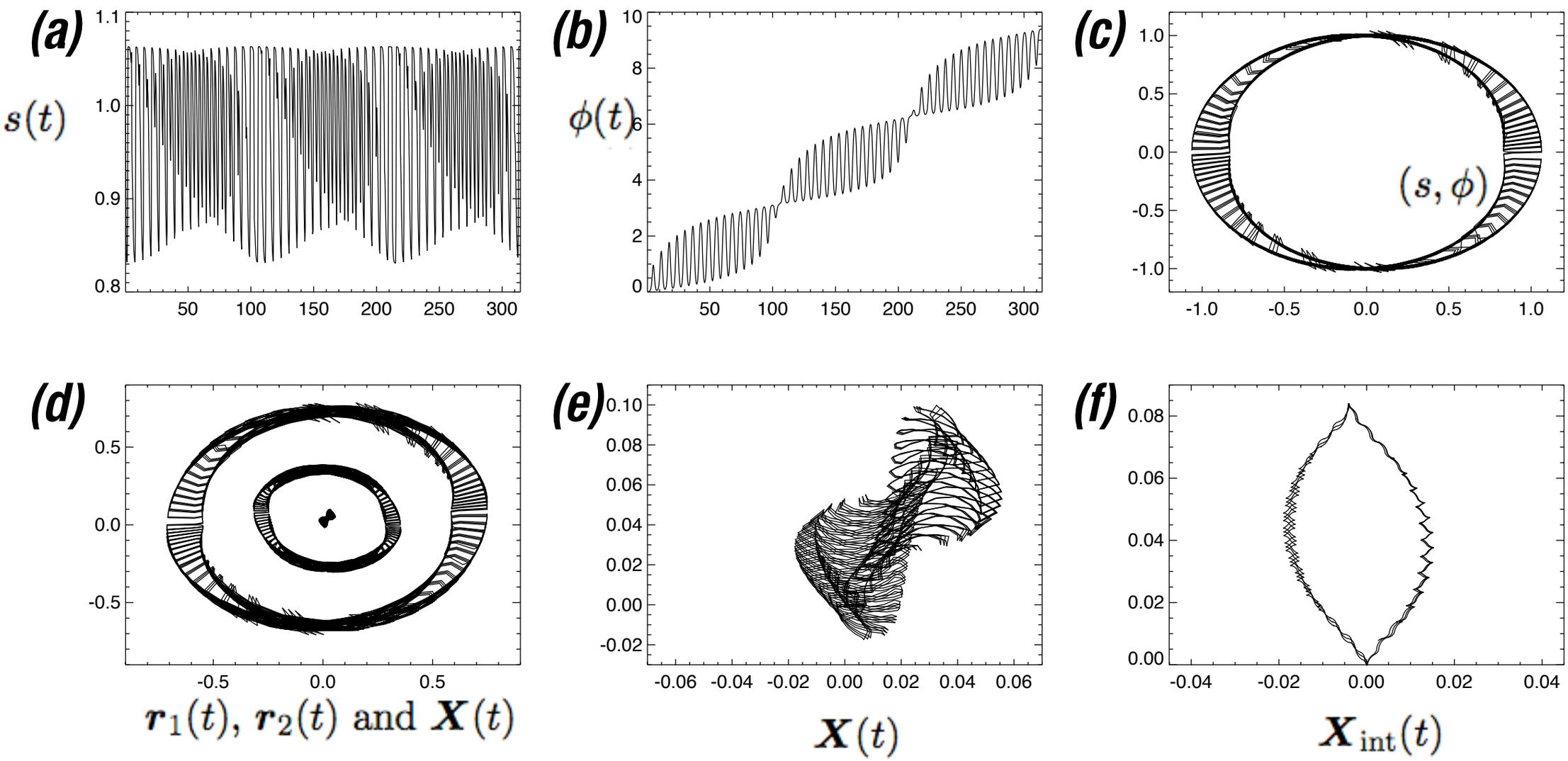
Swimming! NE direction

Simulation 2: soft-hard system $\kappa_2 = 1$



Swimming! NW direction

Simulation 3: intermediate $\kappa_2 = 0.7$



Not swimming ... need to `trap' swimmer orientation $\phi(t)$

Simplify to model 0

- Full problem

$$\varpi \dot{s} + s - 1 = A_{\rm mag} s^{-4} \left[\cos(\alpha_2 - \alpha_1) - 3\cos\alpha_1 \cos\alpha_2\right],$$
$$\varpi s^2 \dot{\phi} = A_{\rm ext} b[\sigma \sin(\psi - \phi - \alpha_1) + \sigma^{-1} \sin(\psi - \phi - \alpha_2)]$$
$$(\rho + \rho^{-1}) \boldsymbol{X}_{\rm int} = \frac{3\varepsilon}{4} (\chi_2 - \chi_1) \int_0^t 2 \log s \, \dot{\phi} \hat{\boldsymbol{\phi}} \, dt.$$

- take magnet 1 soft $\alpha_j = \psi - \phi$ and magnet 2 hard $\alpha_j = 0$

- simplify $\varpi \ll 1, \quad A_{\rm mag} \ll 1$ $\qquad s \simeq 1$

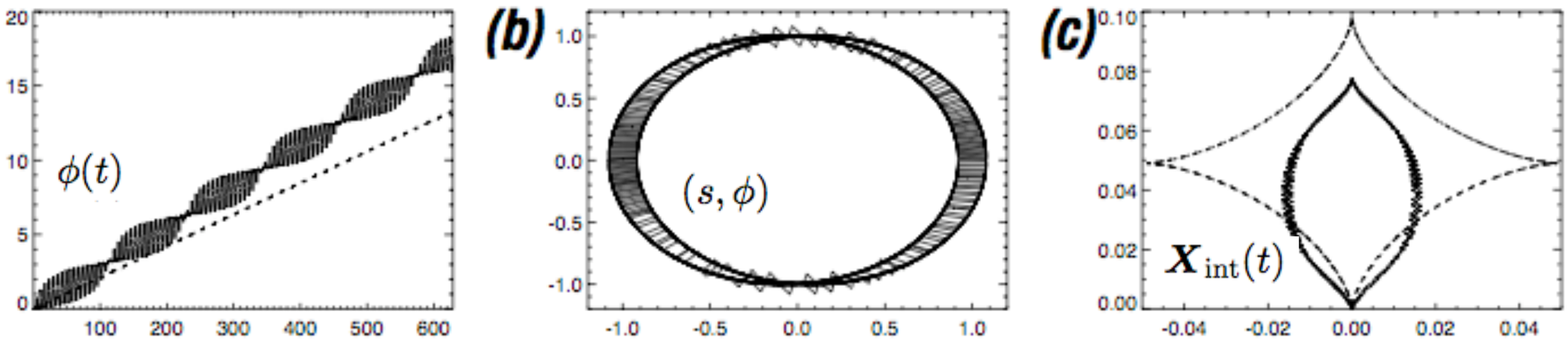
- and rescale to yield `model 0'

$$\dot{\phi} = Ab \sin(\psi - \phi), \quad \tilde{\boldsymbol{X}}_{\rm int} = -\int_0^t 4 \cos(\psi - \phi) \dot{\phi} \hat{\boldsymbol{\phi}} \, dt.$$

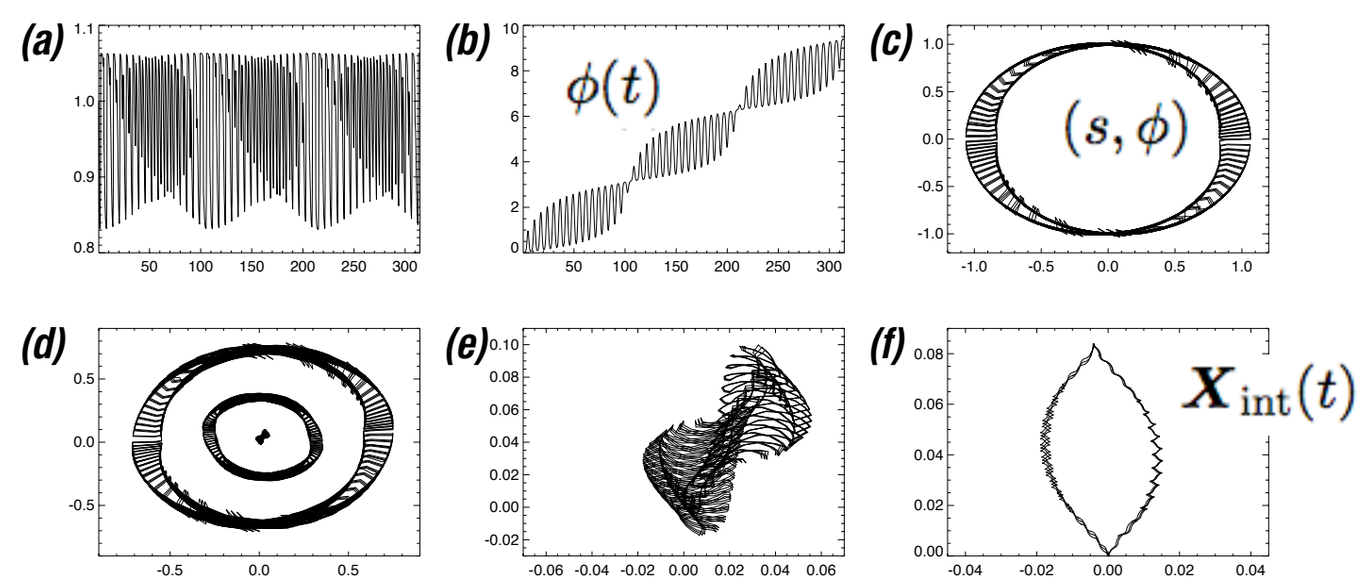
- just one parameter and a diagnostic: $s = 1 - 2A_{\rm mag} \cos(\psi - \phi)$

Does model 0 swim (unidirectional motion)?

- Parameter A=1.46 as in earlier runs:

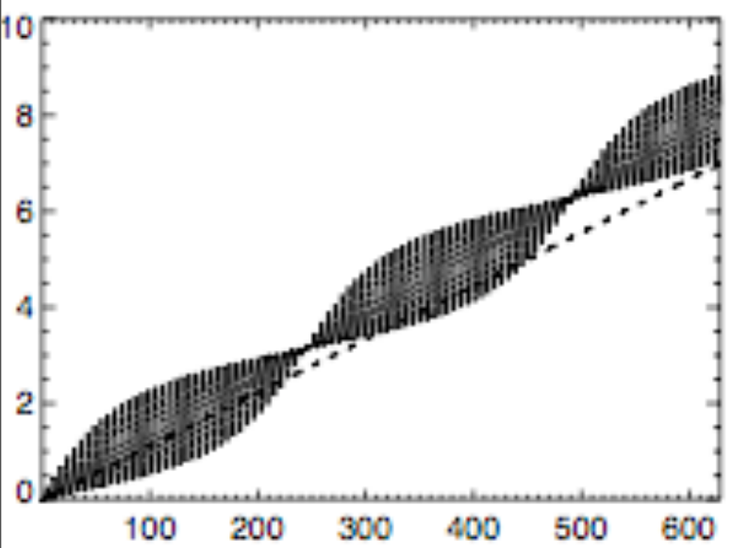


- similar to intermediate case:

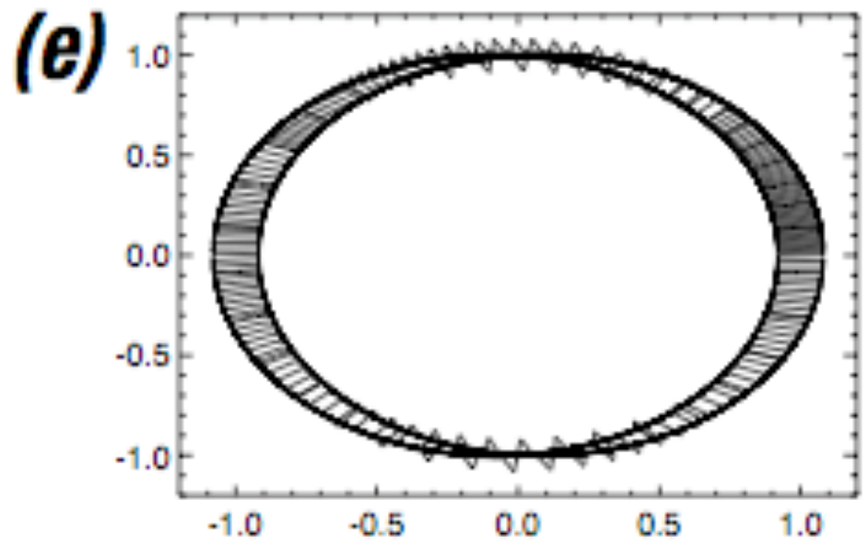


Does model 0 swim?

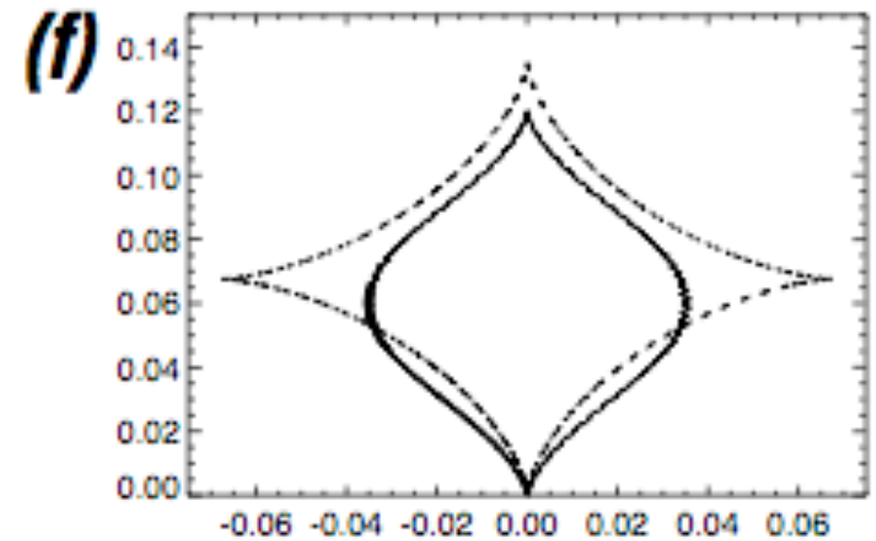
- Parameter A reduced by 75%



$\phi(t)$



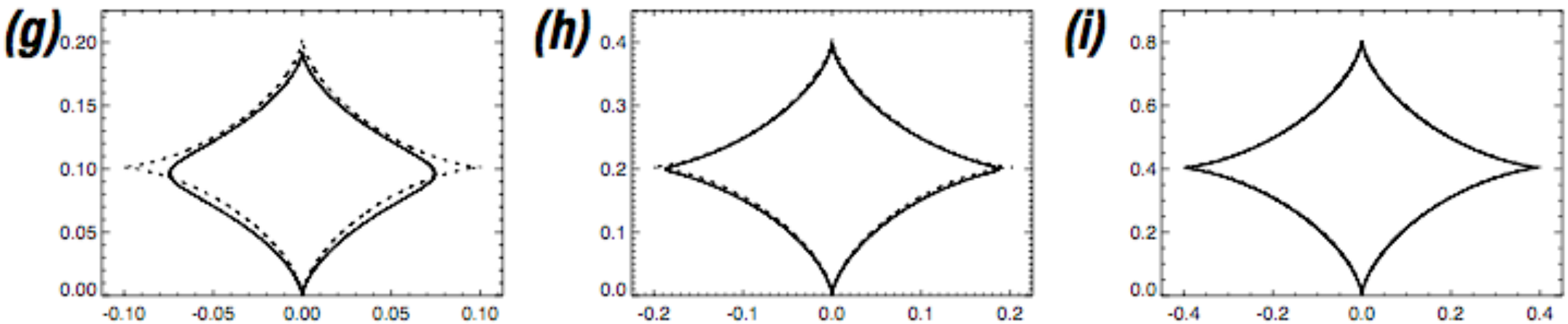
(s, ϕ)



$\boldsymbol{X}_{\mathrm{int}}(t)$

Does model 0 swim?

- Parameter A reduced further and further:



$\boldsymbol{X}_{\mathrm{int}}(t)$

- moves more and more slowly in bigger and bigger path

- no `trapping' swimmer orientation

$$\phi_0 = \overline{\phi}_0 = \alpha\beta\tau/2 \equiv \Omega_0 t, \quad \Omega_0 \equiv \tfrac{1}{2}\alpha\beta A^2$$

$$\tilde{\boldsymbol{X}}_{\mathrm{int}} = \frac{4A\Lambda}{3\Omega_0}\left(-\sin^3\Omega_0 t, 1-\cos^3\Omega_0 t\right).$$

- straightforward multiple scales analysis

$$\Lambda(\alpha,\beta) = \left\langle (\alpha^2\cos^2 t - \beta^2\sin^2 t)(\alpha^2\cos^2 t + \beta^2\sin^2 t)^{-1/2}\right\rangle$$
$$= \frac{2}{\pi\alpha}\frac{\alpha^2}{\alpha^2-\beta^2}\left[(\alpha^2+\beta^2)\,\mathrm{E}(1-\alpha^2/\beta^2) - 2\beta^2\,\mathrm{K}(1-\beta^2/\alpha^2)\right]$$

Model 0+?

- Focus on mechanisms for trapping: IF we can trap the orientation then we are finished. Focus only on equation for the angle

- Full equation: $$\varpi s^2 \dot{\phi} = A_{\rm ext} b[\sigma \sin(\psi - \phi - \alpha_1) + \sigma^{-1} \sin(\psi - \phi - \alpha_2)]$$

- Model 0 version (unrescaled): $$\varpi \dot{\phi} = A_{\rm ext} b \sigma^{-1} \sin(\psi - \phi)$$

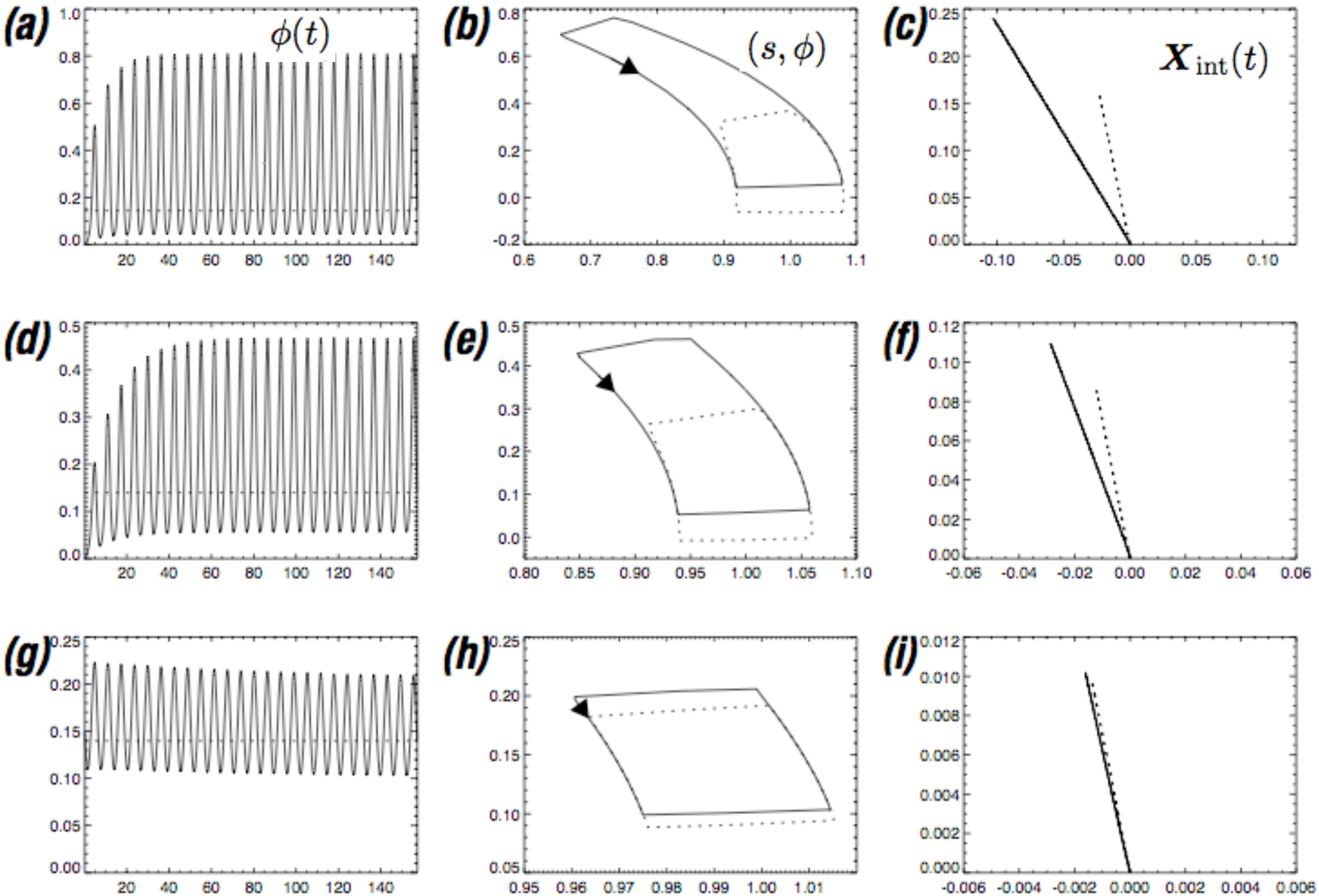
  - reintroduce s^2 term - model 0+MoR

  - make magnet 2 not entirely hard - model 0+nH

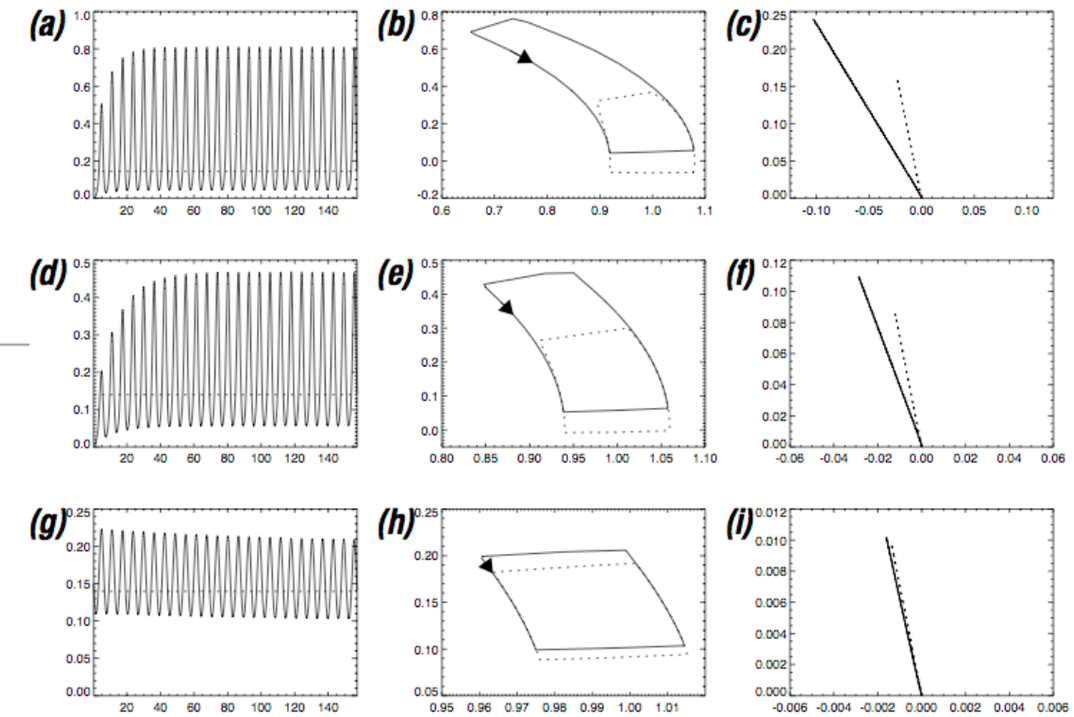
  - make magnet 1 not entirely soft - model 0+nS

  - or any combination

Model 0+MoR



Model 0+MoR theory



- equations

$$s = 1 - 2A_{\rm mag}\cos(\psi - \phi), \quad \dot{\phi} = As^{-2}b\sin(\psi - \phi),$$

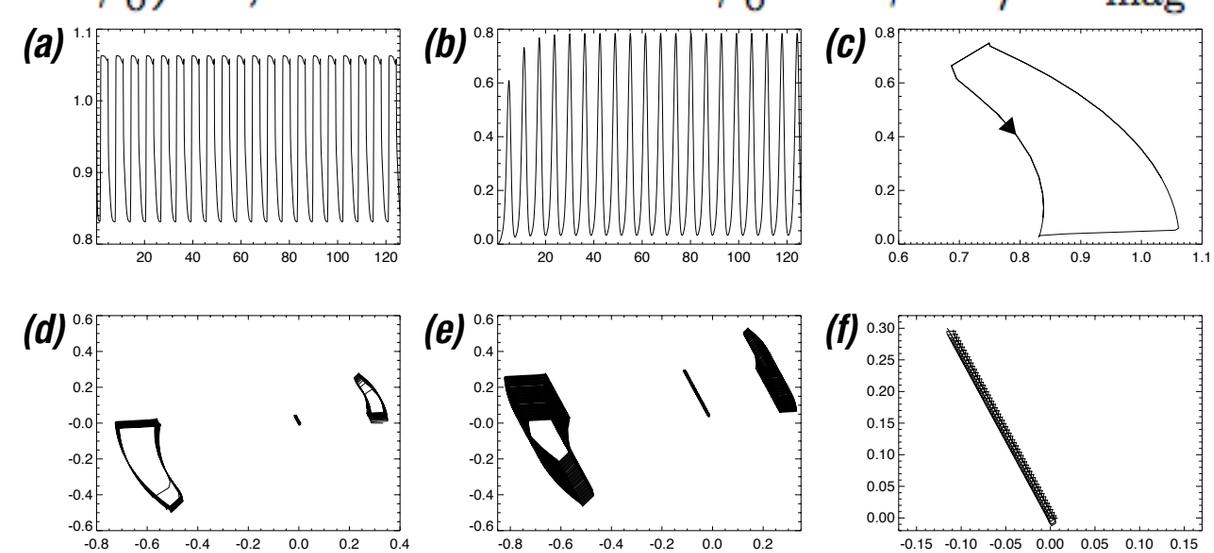
$$\tilde{\boldsymbol{X}}_{\rm int} = -\int_0^t 4\cos(\psi - \phi)\,\dot{\phi}\hat{\boldsymbol{\phi}}\,dt.$$

- multiple scale analysis

$$\frac{d\overline{\phi}_0}{dt} = \tfrac{1}{2}\alpha\beta A^2 - 2AA_{\rm mag}\Lambda\sin 2\overline{\phi}_0$$

- trapping threshold

- moves N-NW or S-SE $\quad \tilde{\boldsymbol{X}}_{\rm int} \simeq (-\sin\overline{\phi}_0, \cos\overline{\phi}_0)\tilde{V}t, \quad \tilde{V} = 2A\Lambda\sin 2\overline{\phi}_0 = \alpha\beta A^2/2A_{\rm mag}$

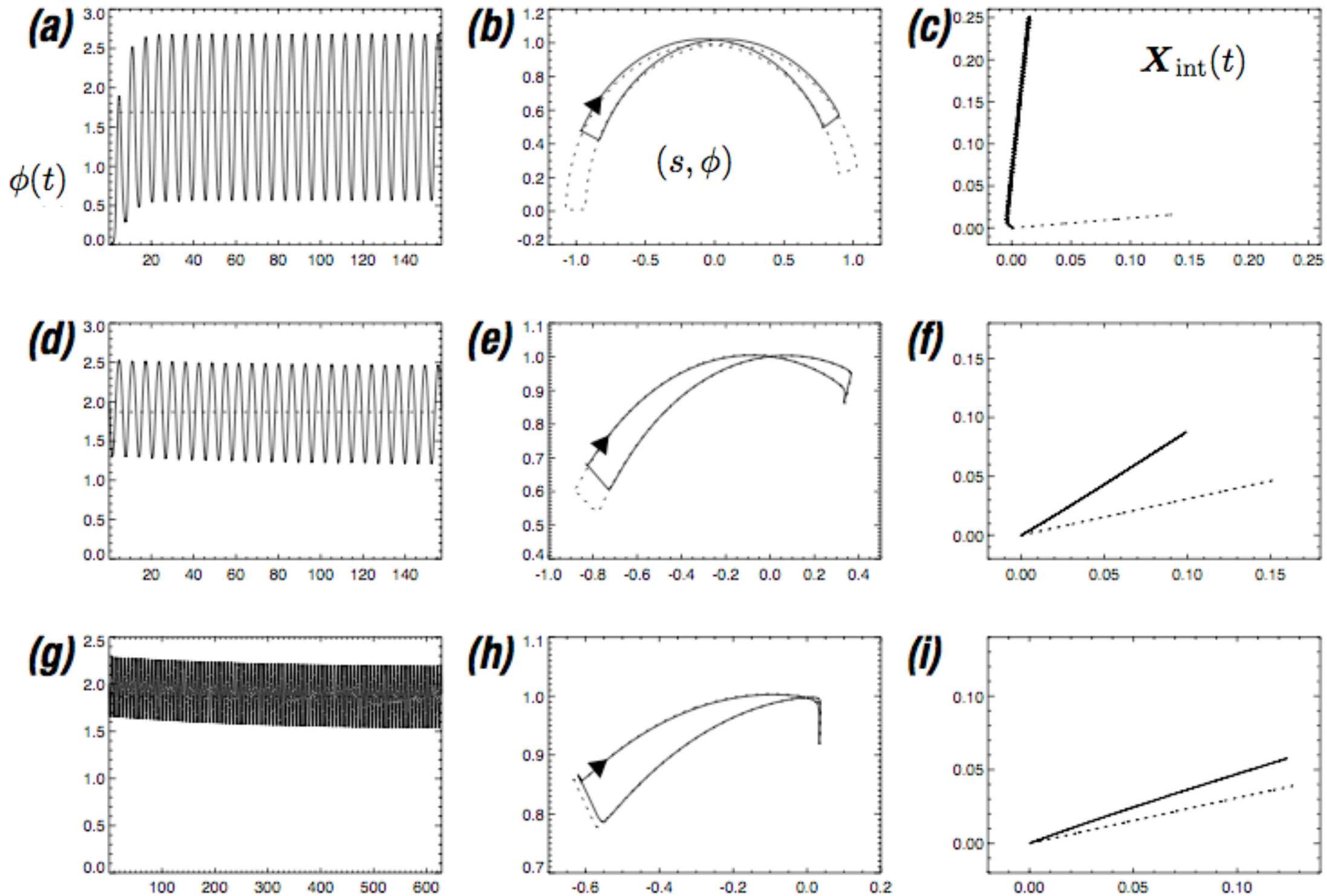


- cf angular momentum and drag term

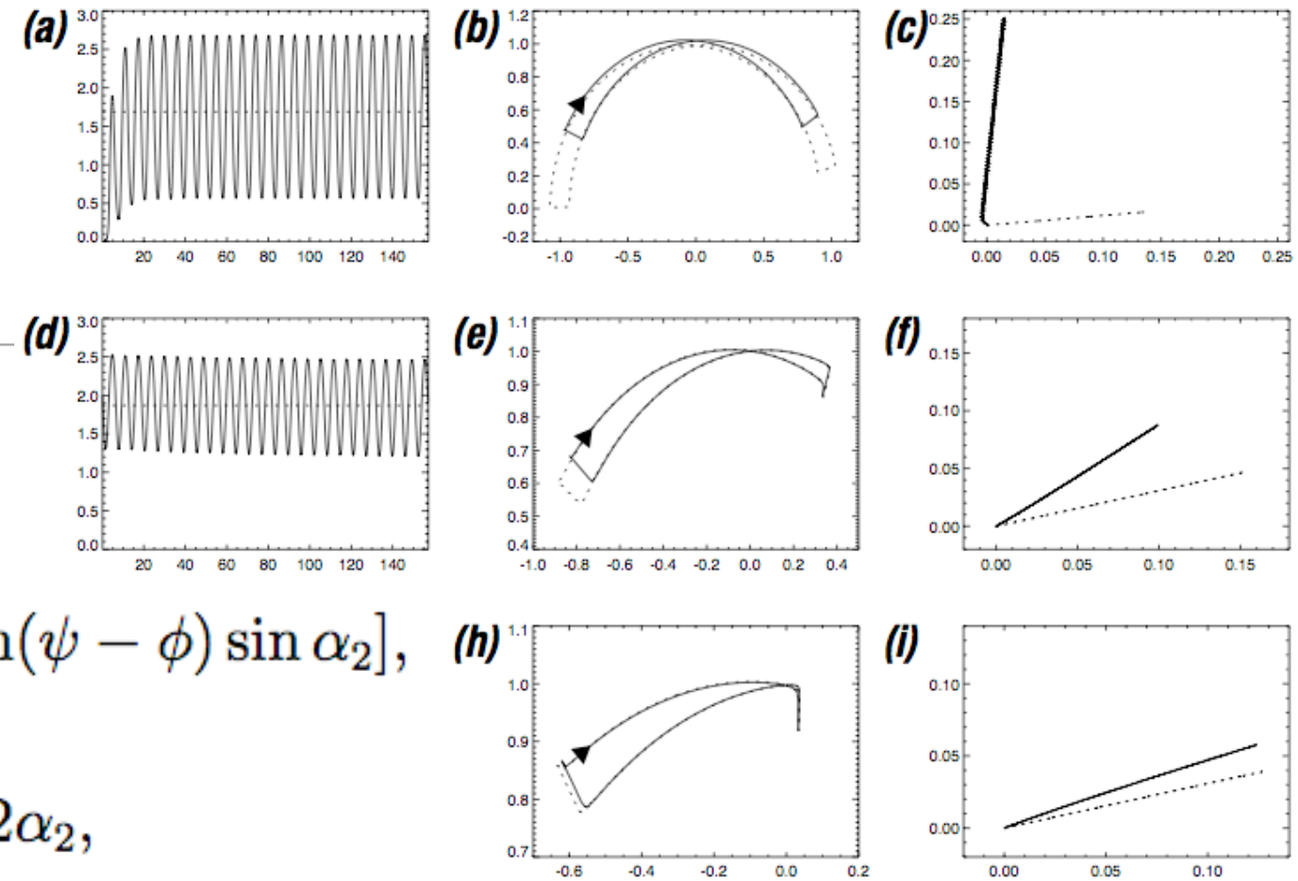
$$(\nu_1\chi_1 + \nu_2\chi_2)s^2\dot{\phi} \qquad 6\pi\eta(\chi_1^2 R_1 + \chi_2^2 R_2)s^2\dot{\phi}$$

- cf soft-hard system

Model 0+nH



Model 0+nH theory



- equations

$$s = 1 - 2A_{\rm mag}[\cos(\psi-\phi)\cos\alpha_2 - \tfrac{1}{2}\sin(\psi-\phi)\sin\alpha_2],$$
$$\dot\phi = Ab\sin(\psi-\phi-\alpha_2),$$
$$\varsigma\varpi\dot\alpha_2 = (1-\kappa_2)b\sin(\psi-\phi-\alpha_2) - \kappa_2\sin 2\alpha_2,$$
$$\tilde{\boldsymbol{X}}_{\rm int} = -\int_0^t 4[\cos(\psi-\phi)\cos\alpha_2 - \tfrac{1}{2}\sin(\psi-\phi)\sin\alpha_2]\dot\phi\hat{\boldsymbol{\phi}}\,dt$$

- multiple scale analysis

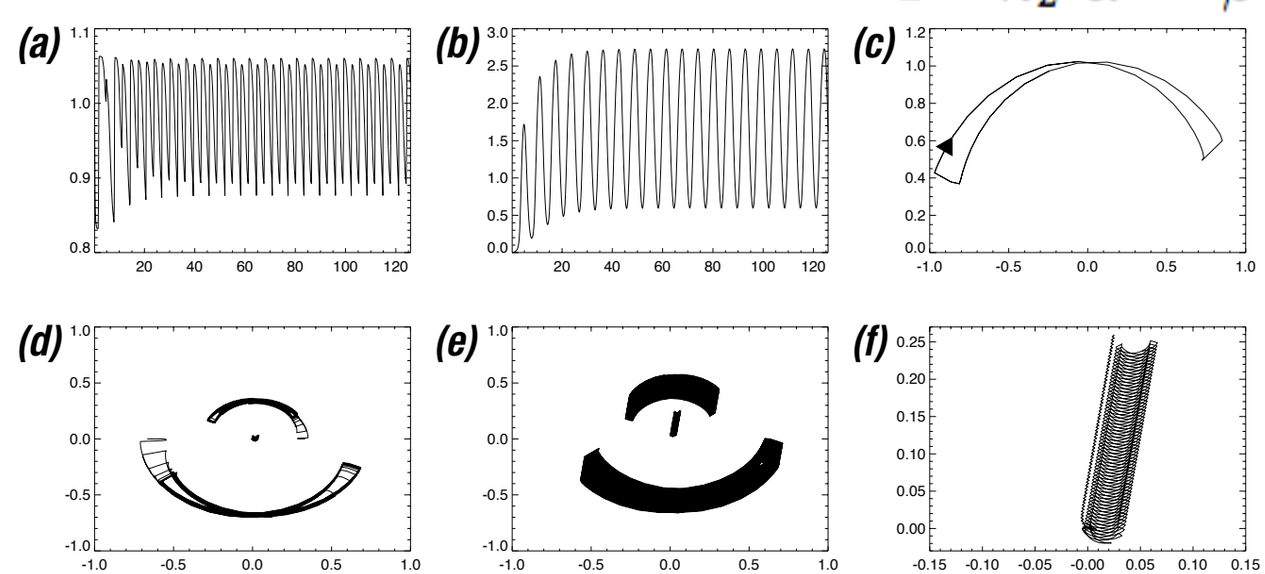
$$\frac{d\overline{\phi}_0}{dt} = \tfrac{1}{2}\alpha\beta A^2 + \tfrac{1}{8}A(1-\kappa_2)(\alpha^2-\beta^2)\sin 2\overline{\phi}_0.$$

- trapping threshold

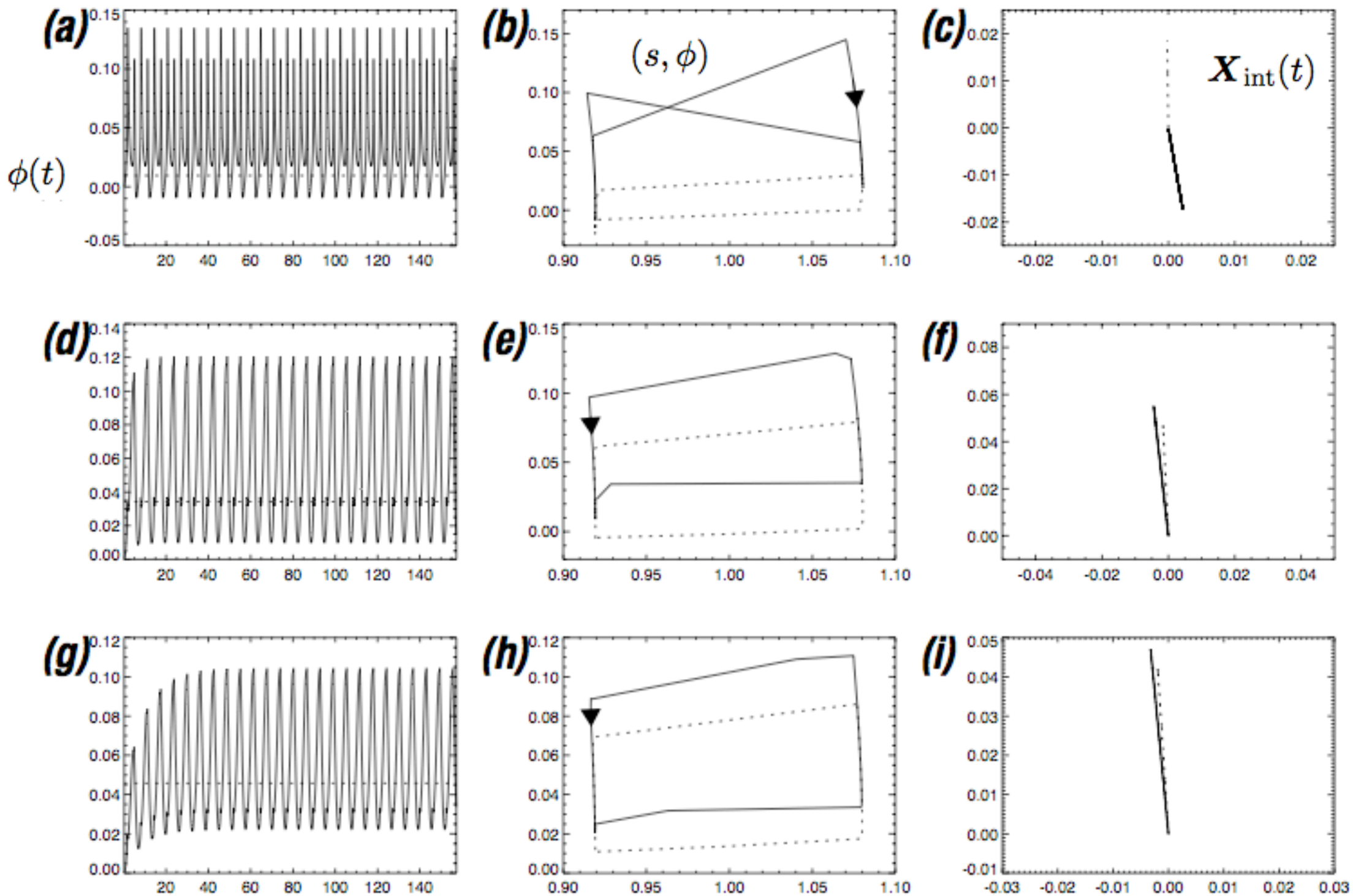
$$\tilde{\boldsymbol{X}}_{\rm int} \simeq -(-\sin\overline{\phi}_0, \cos\overline{\phi}_0)\tilde{V}t, \quad \tilde{V} = \frac{8A^2\Lambda}{1-\kappa_2}\frac{\alpha\beta}{\alpha^2-\beta^2}$$

- moves E-NE or W-SW

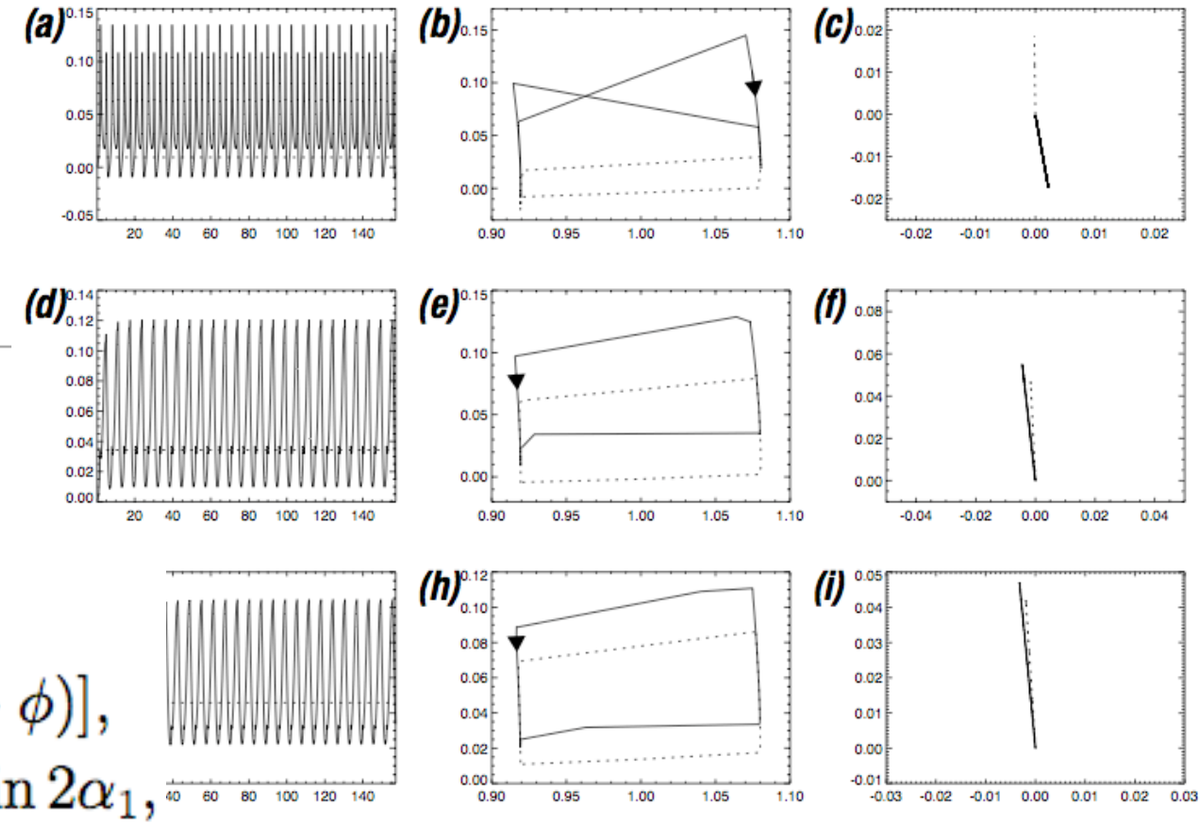
- cf full system



Model 0+nS



Model 0+nS theory



- equations

$$
\begin{aligned}
s &= 1 - 2A_{\mathrm{mag}} \cos\alpha_1, \\
\dot{\phi} &= Ab[\sigma^2 \sin(\psi - \phi - \alpha_1) + \sin(\psi - \phi)], \\
\varsigma\varpi\, \dot{\alpha}_1 &= (1-\kappa_1) b \sin(\psi - \phi - \alpha_1) - \kappa_1 \sin 2\alpha_1, \\
\tilde{\boldsymbol{X}}_{\mathrm{int}} &= -\int_0^t 4\cos\alpha_1\, \dot{\phi}\, \hat{\boldsymbol{\phi}}\, dt,
\end{aligned}
$$

- trapping threshold

$$
\frac{d\overline{\phi}_0}{dt} = \tfrac{1}{2}\alpha\beta A^2 - A\sigma^2\kappa_1 \frac{\alpha-\beta}{\alpha+\beta} \sin 2\overline{\phi}_0.
$$

- moves N-NW or S-SE

$$
\tilde{\boldsymbol{X}}_{\mathrm{int}} \simeq (-\sin\overline{\phi}_0, \cos\overline{\phi}_0)\tilde{V}t, \quad \tilde{V} = \frac{\alpha\beta A^2 \Lambda}{\sigma^2\kappa_1} \frac{\alpha+\beta}{\alpha-\beta}
$$

Discussion

- Set up simple model: isolated role of fluid interactions for swimming motion
- isolated trapping mechanisms and relevant parameters: rules of thumb
- e.g. trapping from MoR, nS reinforce, from nH opposes
- ? comparison with experiment ?
- ? motion in 3-d
- ? collective motion, interaction with boundaries
- ? viscoelastic medium
- ? more complex magnetic/elastic/fluid systems