Dynamics and diffusion in coherent vortices: closing the stability loop

Andrew Gilbert (University of Exeter)

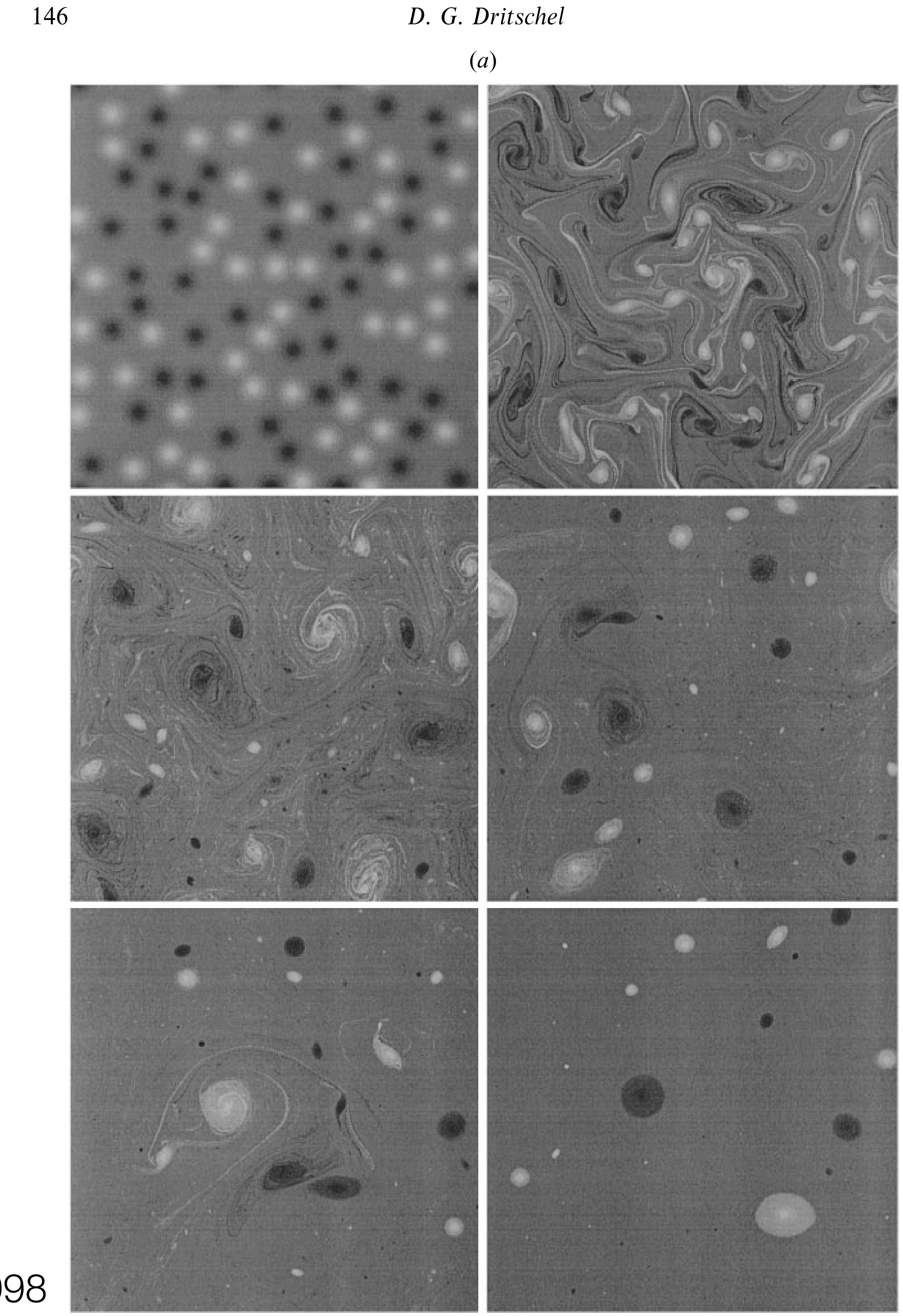
Matthew Turner (University of Brighton)

Andrew Bassom (University of Perth)

with EPSRC/Leverhulme Trust support

Vortices are prevalent in fluid dynamics

- two dimensional turbulence: vorticity arranges into coherent vortices which move and interact
- each vortex `feels' the effect of others in terms of motion and a time-dependent, irrotational straining field
- vortices occasionally interact closely and highly nonlinearly
- also vortices in three dimensions



Dritschel 1998

Geophysical applications

- contour dynamics simulations of stratospheric south polar vortex
- vortex forced by topography
- vortex has a coherent m=3 mode
- vortex has `surf zone' of enhanced mixing
- interactions between mode and mixing

1 MARCH 1993 WAUGH 727

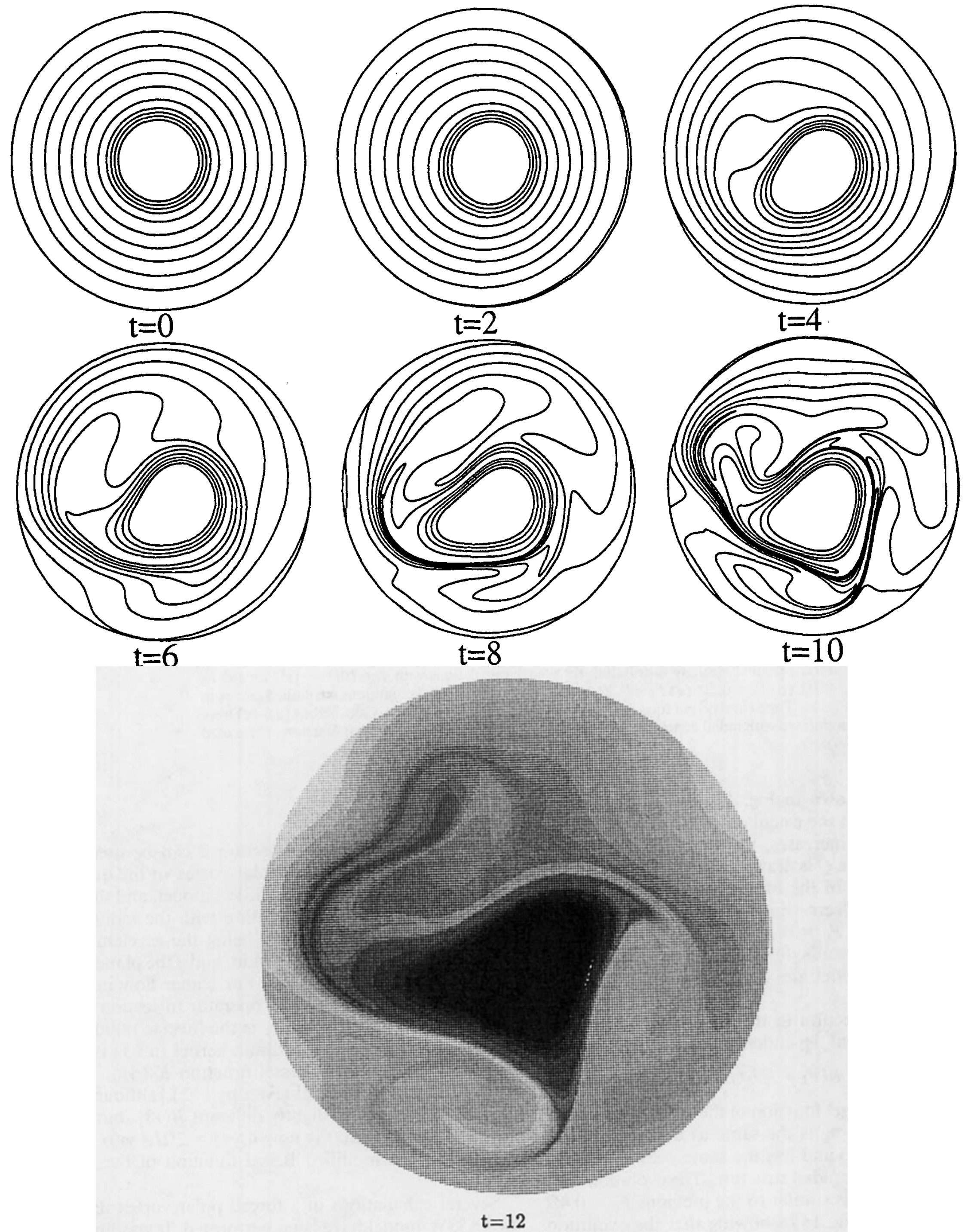


FIG. 13. Planar calculation with $f = 2\Omega(1 - \frac{1}{2}r^2)$, $N = 16$, $F_0 = 0.4\Omega$, and initial Q-profile (dashed curve in Fig. 11) corresponding to the initial profile in the spherical calculations [using transformation (9)]. The plots are the planar equivalent of polar stereographic projection, that is, $(r/\sqrt{2 - r^2}, \theta)$.

Waugh 1993

Mixing and spreading of vortices

- basic mechanism:
- vortex is distended by external, irrotational strain flow
- vortex winds up fluctuations
- vorticity mixes and vortex spreads
- response to external strain depends on the vortex profile
- profile depends on the spreading

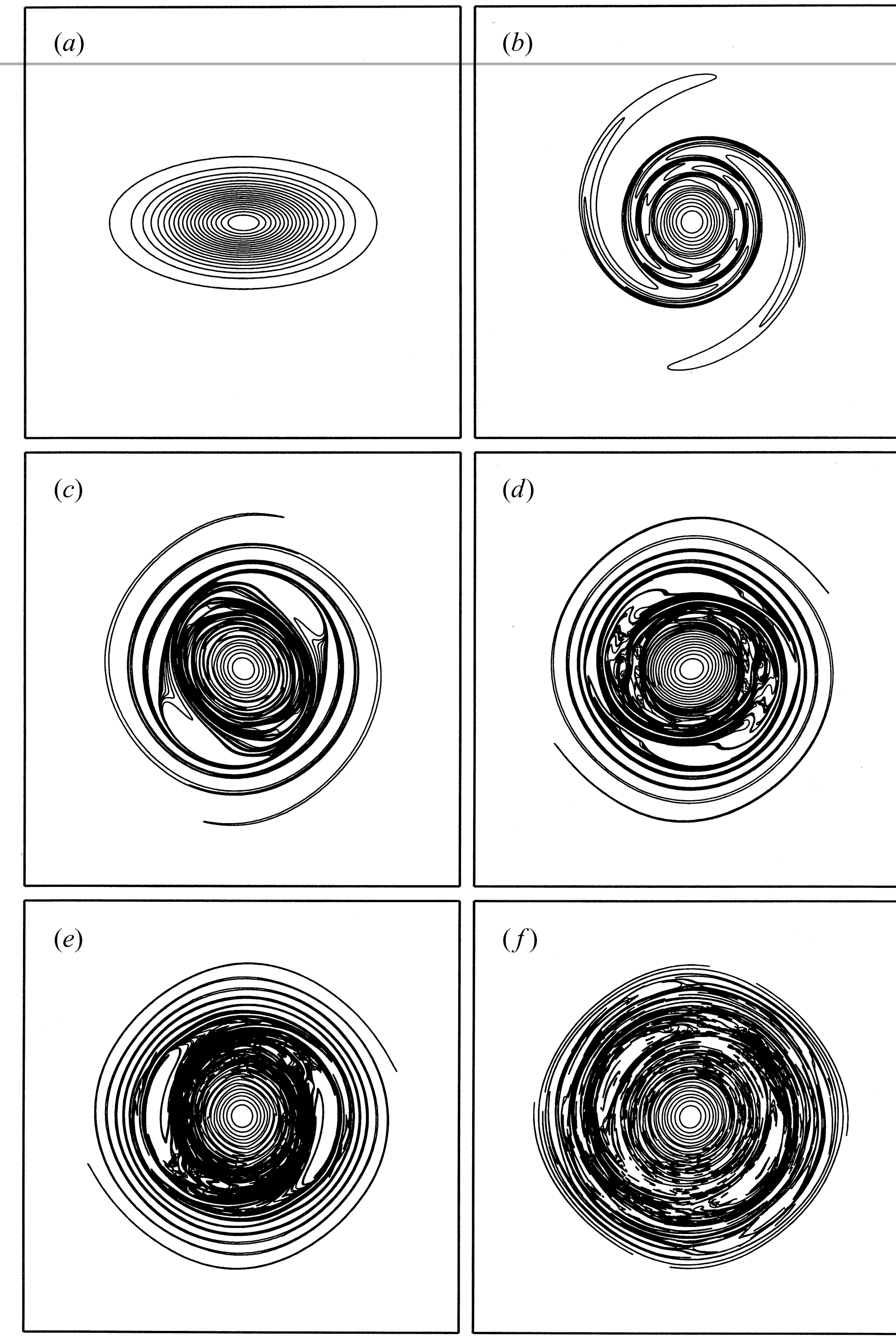


FIGURE 1. The evolution of a Gaussian vortex initially deformed into a 2 : 1 ellipse. The simulation is conducted at extremely high resolution using the numerical algorithm described in §3 in a doubly periodic box. A basic grid of 512^2 is used, but vorticity structures are retained down to a tenth of the grid scale. This resolution is used in all subsequent simulations except where noted. The times shown are $t = 0, 3, 11, 18, 27$ and 100 (a–f), in units of $T = 4\pi/\omega_{max}$. A time step of $\Delta t = 0.025$ was used in all simulations; the value of Δt is not constrained by numerical stability.

Dritschel 1998

Axisymmetrization or cat’s eye formation

- weak perturbations axisymmetrize
- stronger ones can leave persistent cat’s eyes, linked to an approximate normal mode on the vortex
- this is all for two-dimensional Euler/ Navier-Stokes equation

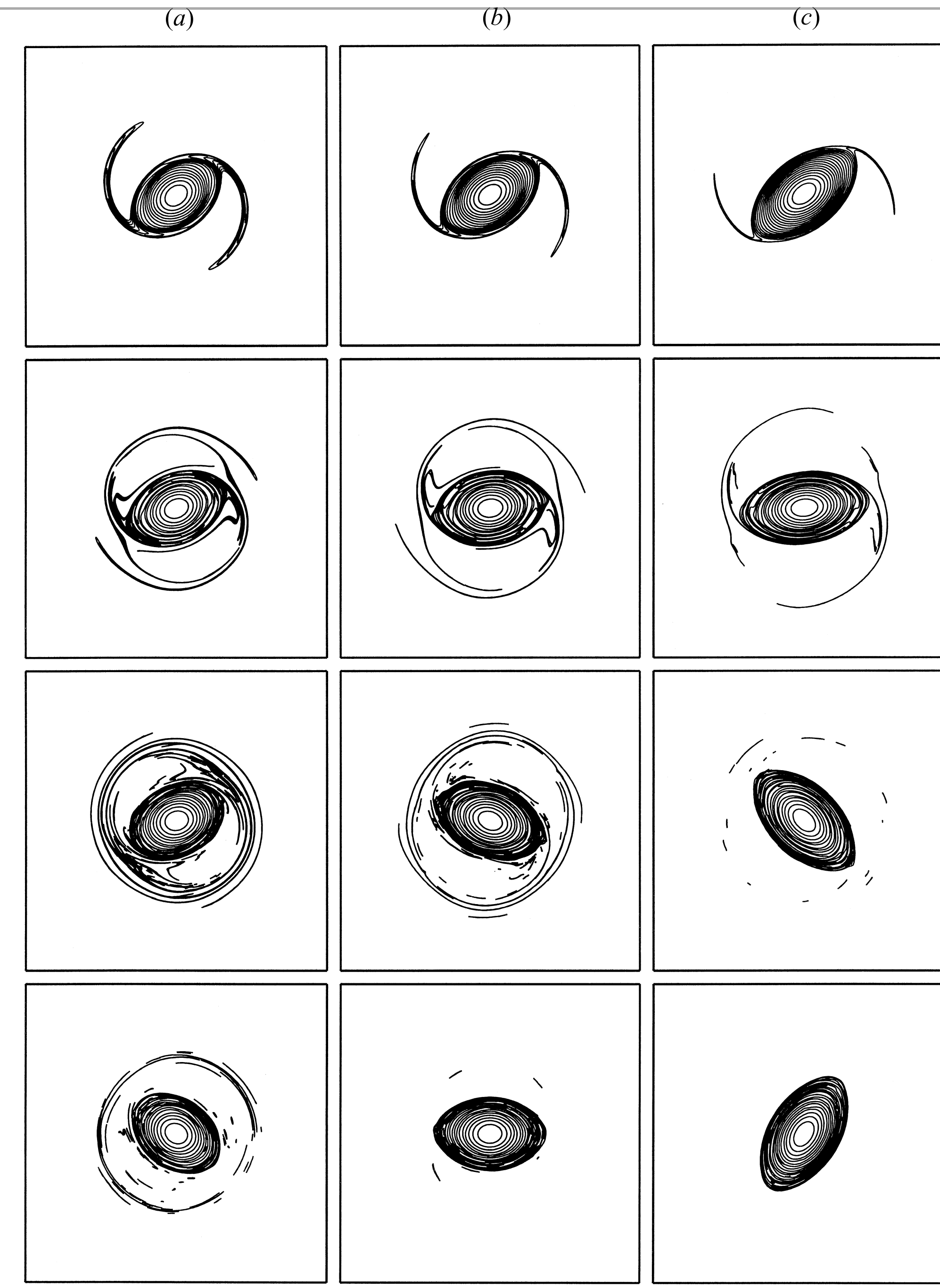


FIGURE 3. The evolution of three different initial vortices: (*a*) for $a = 0$, (*b*) for $a = 0.2$ and (*c*) for $a = 0.4$. Time proceeds downwards; the times shown are $t = 2$, 8, 20 and 100.

passive (note, the disturbance vorticity is not removed until it has reached a tenth of the grid scale, i.e. about a thousandth of the vortex radius, a scale which is far smaller than in any previous simulation of this process; higher resolution is considered immediately below).

Dritschel 1998

Our context and goals:

- role of dynamics in the mixing and spreading of vortices
- how does the vortex profile affect the mixing?
- how does the mixing affect the vortex profile?
- what does a Gaussian vortex evolve into?
- two-dimensional Euler/Navier-Stokes equation
- weak, random, external irrotational strain

Governing equations

Euler equation $\partial_t \omega = J(\psi, \omega), \quad \omega = -\nabla^2 \psi, \quad \mathbf{u} = (r^{-1}\partial_\theta \psi, -\partial_r \psi)$

Initial condition (Gaussian vortex) $\omega = \overline{\omega} = (4\pi)^{-1} e^{-r^2/4},$

Mean profiles

angular velocity

$$\overline{\omega} = -r^{-1}\partial_r(r\partial_r \overline{\psi}), \quad \alpha = -r^{-1}\partial_r \overline{\psi}, \quad \beta = r^{-1}\partial_r \overline{\omega}.$$

External irrotational forcing: strain, m=2

$$\psi(r, \theta, t) \sim (-2\pi)^{-1} \log r + \varepsilon q(t) r^m e^{im\theta} + \mathrm{c.\,c.} \quad (r \to \infty).$$

$$\langle q(t) q^*(t') \rangle = \delta(t - t'), \quad w(p) = 1;$$

Frequency p in the external forcing gives resonance
at a radius r where fluid elements co-rotate.....

Mathematical development: key steps

- weakly nonlinear approach

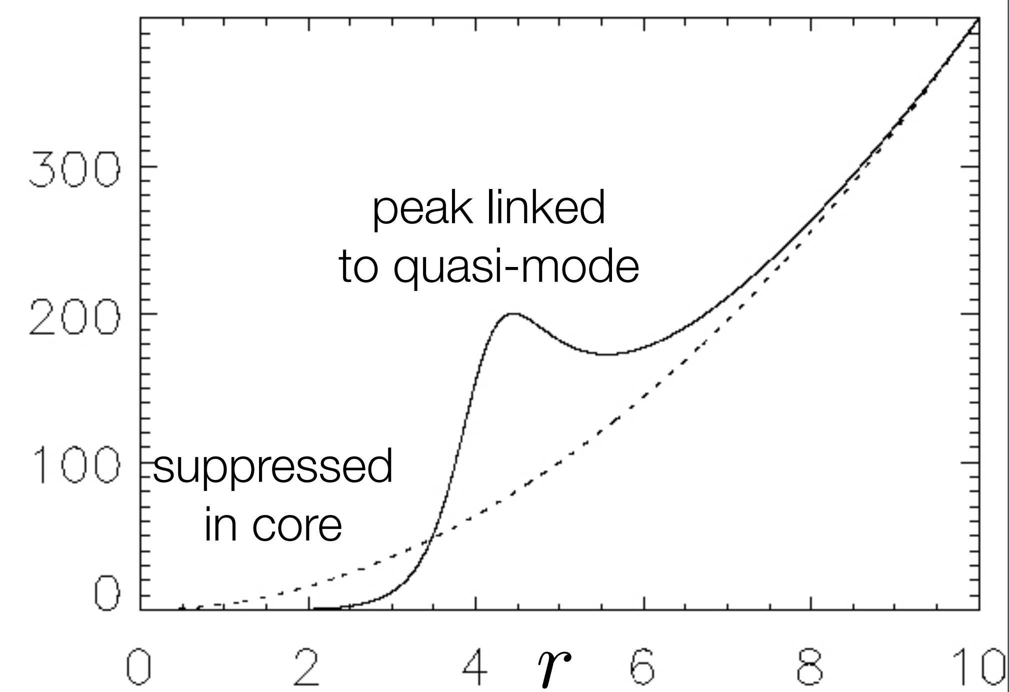
- for each p we have a stability problem: linear response of the vortex to mode m at frequency p. Large response at the resonant radius r with $p \equiv m\alpha(r)$

- we compute this numerically and then calculate the transport of vorticity

- result is a diffusion equation for the mean profile

$$\partial_\tau \overline{\omega} + r^{-1}\partial_r(rF) = 0, \quad F(r) = -\kappa(r)\partial_r\overline{\omega},$$

$$\kappa_*(r) = m^2 r^{-2}|C(p)|^2 w(p), \quad p \equiv m\alpha(r).$$



a)

FIGURE 1. Effective diffusivity (a) $\kappa(r)$ plo

For a passive scalar

$$\kappa_{\text{scalar}}(r) = m^2 r^{-2} r^{2m} w(p), \quad p \equiv m\alpha(r).$$

Inside the mathematics - formal problem

- expansion

$$\omega(y,t,\tau) = \overline{\omega}(y,\tau) + \varepsilon\hat{\omega}(y,t,\tau)e^{im\theta} + \mathrm{c.\,c.} + \cdots\ ,$$
$$\psi(y,t,\tau) = \overline{\psi}(y,\tau) + \varepsilon\hat{\psi}(y,t,\tau)e^{im\theta} + \mathrm{c.\,c.} + \cdots\ .$$

- linear problem (dropping hats)

$$\partial_t\omega + im\alpha\omega + im\beta\psi = 0, \quad -\omega = (\partial_r^2 + r^{-1}\partial_r - r^{-2}m^2)\psi$$

- feedback to mean profile, time and ensemble averaged

$$\tau = \varepsilon^2 t,$$

$$\partial_\tau\overline{\omega} + r^{-1}\partial_r(rF) = 0, \quad F = 2mr^{-1}\,\mathrm{Im}\langle\omega\psi^*\rangle.$$

Inside the mathematics - Laplace transform

- frequency p in LT space: we need to solve

$$\widetilde{\omega} \equiv -(\partial_r^2 + r^{-1}\partial_r - r^{-2}m^2)\widetilde{\psi} = \frac{\beta(r)}{p/m - \alpha(r)}\,\widetilde{\psi} \qquad \widetilde{\psi}(r,p) \sim r^m \widetilde{q}(p) \qquad (r \to \infty)$$

- and the solution is $\widetilde{\psi}(r,p) = M(r,p)\widetilde{q}(p), \quad \widetilde{\omega}(r,p) = N(r,p)\widetilde{q}(p).$

- where the local expansion near the resonant radius r for the given p is

$$N(r,p) = -C(p)\mu(s)(r-s)^{-1} + \cdots, \qquad \mu(s) = \beta(s)/\partial_s\alpha(s)$$

$$M(r,p) = C(p)[1 + \mu(s)(r-s)\log(s-r) + \cdots] + D(p)[(r-s) + \cdots]$$

for $r < s$, and

$$M(r,p) = C(p)[1 + \mu(s)(r-s)(\log(r-s) + i\chi) + \cdots] + D(p)[(r-s) + \cdots]$$

for $r > s$. χ is a phase shift

messy but can be done numerically for each p

Numerical problems ....

- results sensitive to time-stepping, grid spacing, and precision used! Also issues of neutral modes

- bad news: smooth and cap the diffusivity to see what is going on

$$\kappa = \kappa_{\text{scalar}} G_\delta F_\delta(\kappa_* / \kappa_{\text{scalar}}),$$

$$F_\delta(s) = \delta^{-1} \tanh(s\delta),$$

$$(G_\delta f)(s) = \pi^{-1/2} \delta^{-1} \int_{-\Lambda\delta}^{\Lambda\delta} e^{-(s-s')^2/\delta^2} f(s')\, ds.$$

- physical motivation: break down of two-scale system

- link smoothing/capping to value of epsilon used $\delta = O(\varepsilon^{1/2}).$

- not ideal!

Results - Gaussian

- profile steepens
- high diffusion around r=4
- steps appear for small smoothing
- steps merge
- high diffusion where flat
- low diffusion where steep
- runaway mechanism

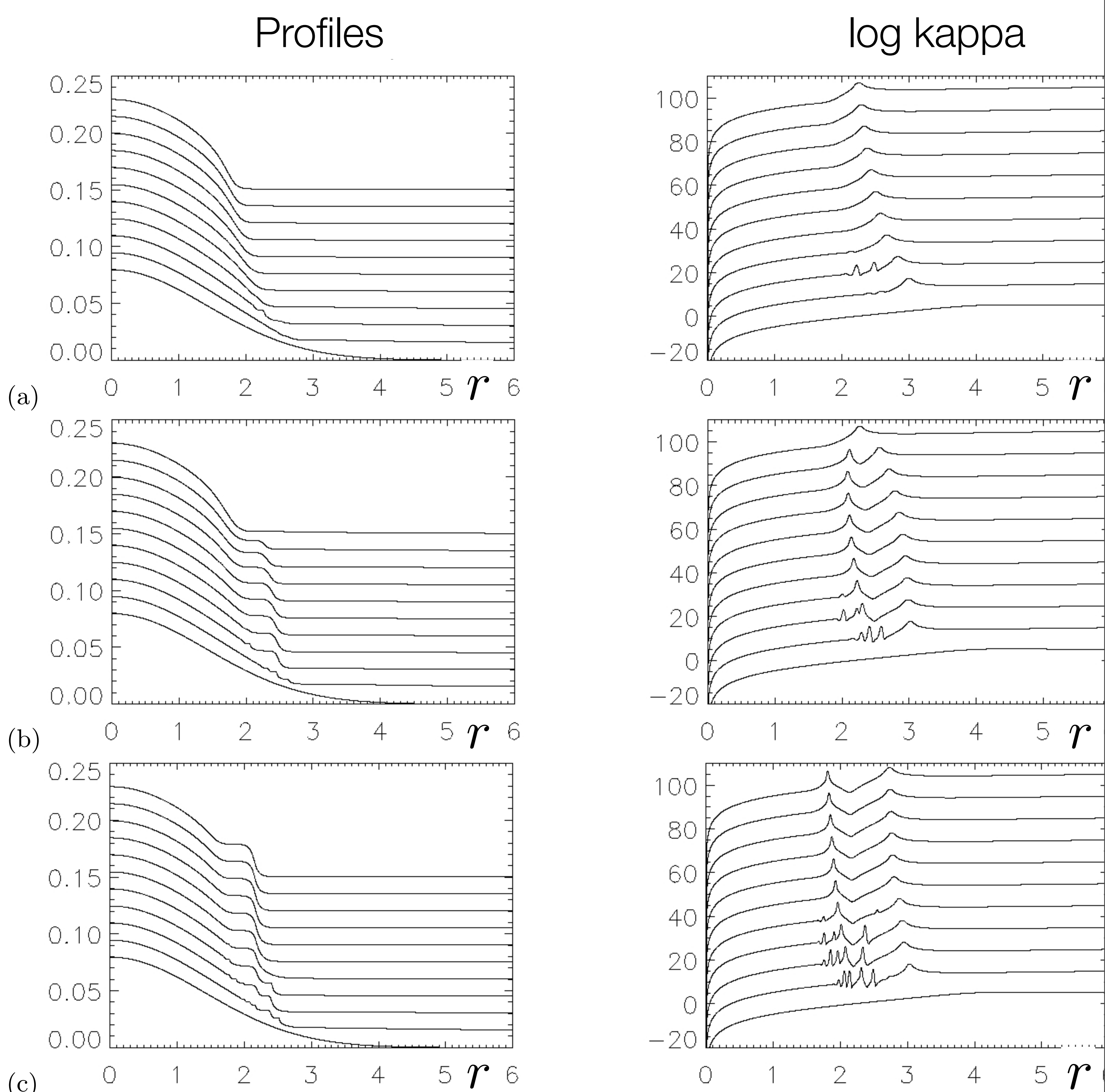


FIGURE 2. Evolution of a Gaussian vortex with smoothing (a) $\delta = 0.02$, (b) $\delta = 0.015$ and (c) $\delta = 0.01$. The curves show a sequence of vorticity profiles $\overline{\varpi}$ (left panels) and effective diffusivities $\log \kappa$ (right panels), plotted against r. In each panel the curves are separated by additive constants and given for $\tau = 0$, 0.05, 0.1, etc., reading up the curves.

`Butterfly' diagram

- time tau horizontal
- radius r increases upwards
- left: vorticity gradient
- right: effective diffusivity
- waves, steps, merger, stripping
- small-scale steps for small smoothing parameters

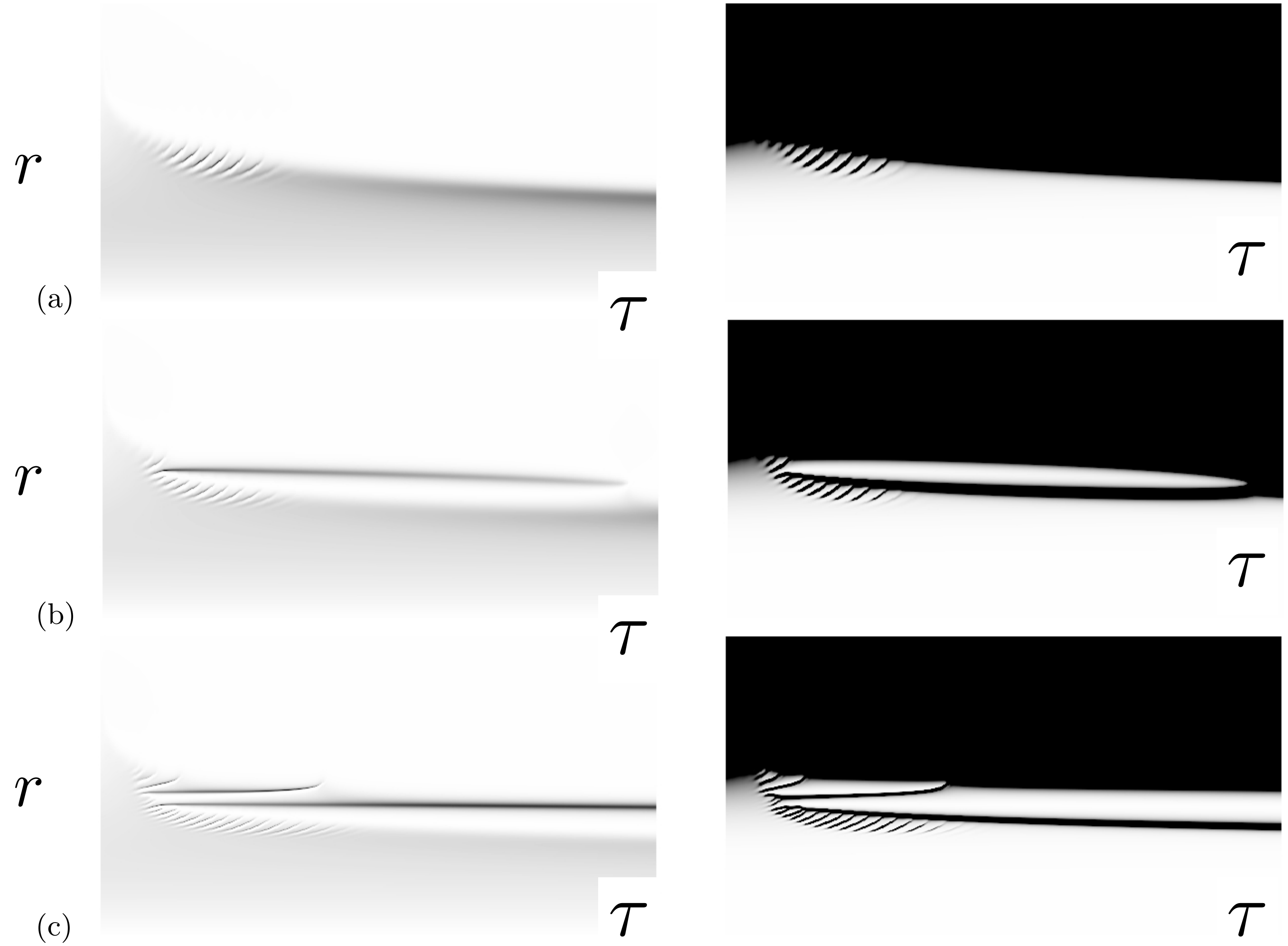


FIGURE 3. Space-time diagram of the evolution of the vorticity gradient, $-\partial_r\overline{\omega}(r,\tau)$, (left) panels and effective diffusivity $\kappa(r,\tau)$ (right panels), plotted in grey scales in the (τ,r)-plane for (a) $\delta=0.02$, (b) $\delta=0.015$ and (c) $\delta=0.01$, The grey scale coding is capped at levels $-\partial_r\overline{\omega}=0.25$, $\kappa=4$, corresponding to black; zero is white. The ranges are $0\leq r\leq 5$ (vertical) and $0\leq\tau\leq 0.5$ (horizontal).

Simplified model

- how can we understand the results above?
- what is the nature of the runaway step-making process?
- why does it occur only at a range of radii?
- Balmforth, Llewellyn Smith and Young (BLSY) model (2001):
- compact vortex, e.g. a Rankine, top-hat vortex, radius r=1
- supports a normal mode, elliptical m=2 distortion
- critical radius r = sqrt(2) where fluid co-rotates with the mode, lies beyond the edge of the vortex

BLSY model

- normal mode - wavelike disturbance - on the top-hat vortex
- interacts with weak vorticity in a thin layer around r = sqrt(2)
- model asymptotically exact (limit of weak vorticity)
- Gaussian vortex has very similar qualitative behaviour:
- `nearly' possesses a normal mode, which is damped
- this is a `quasi-mode' and has a resonant radius around r=4.

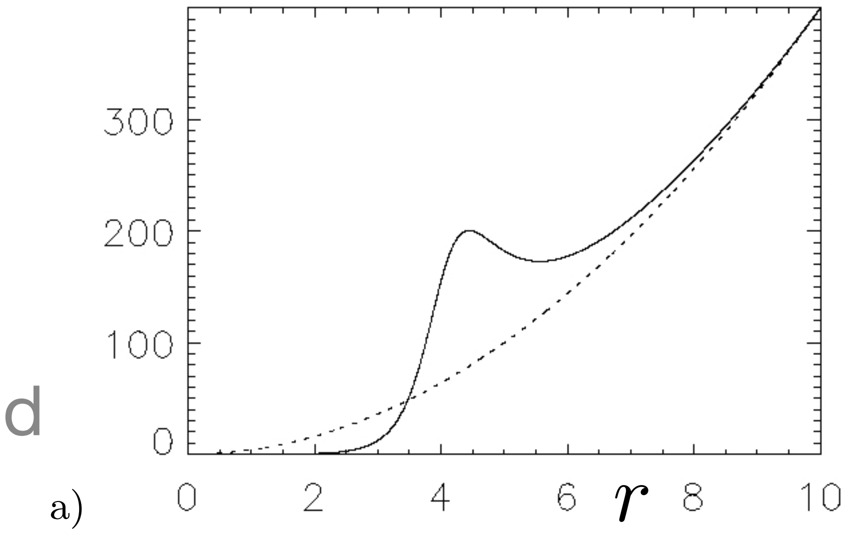


FIGURE 1. Effective diffusivity (a) $\kappa(r)$ plot

BLSY equations

$$\partial_t \zeta + y \partial_\theta \zeta + \partial_y (y + \zeta) \partial_\theta \varphi = 0,$$

$$i \partial_t \hat{\varphi} = \varepsilon q(t) + \mathcal{P} \int_{-\infty}^{\infty} dy \oint \frac{d\theta}{2\pi} \, \zeta e^{-im\theta},$$

$$\varphi(\theta, t) = \hat{\varphi}(t) e^{im\theta} + \mathrm{c.\,c.}$$

- zeta(y,theta,t) = vorticity in the layer
- phihat(t) = normal mode amplitude
- ODE for phihat(t)
- PDE for zeta(y,theta,t)
- thresholds for the formation of cat's eyes

382 *M. R. Turner and A. D. Gilbert*

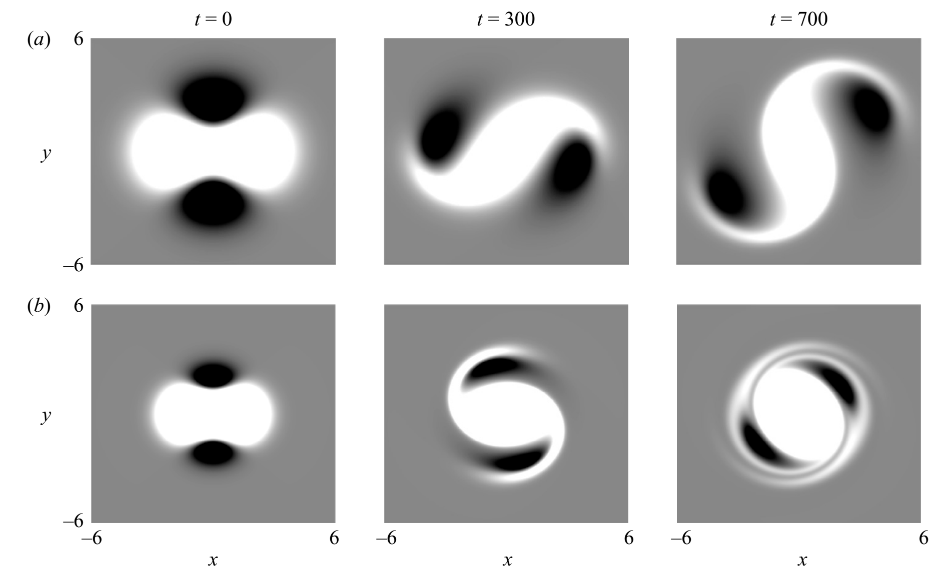


FIGURE 1. Plot of the Gaussian vortex (*a*) and the tanh profile vortex (*b*), with $\sigma = \sigma_2 = 0.2$, relaxing to a tripole structure with $R = 10^4$ and $\delta = 0.5$. Positive vorticity is shown as white, negative vorticity is black and zero vorticity is grey; the peak vorticity values are saturated at $|Z| = 0.005$.

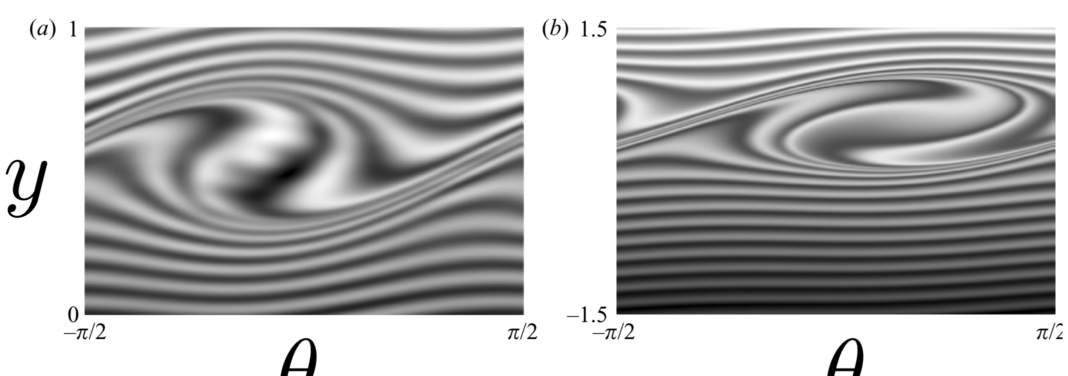


FIGURE 5. Grey-scale p … es of the total vorticity $\zeta_T = y + \zeta$ for (*a*) (,) = (0.4, 0.9) and (Γ, Δ) = (0.4, 1.1). These snapshots are taken at $t = 60$ in (*a*) and $t = 40$ in (*b*) where posi vorticity is white and negative vorticity is black. Note that the vertical scales are different, that the whole y domain is not displayed.


CAMBRIDGE
10 November 2008
Journal of
Fluid Mechanics
VOLUME 614

Effective diffusion in BLSY model, randomly forced

- total vorticity in critical layer $\overline{\omega} = y + \overline{\zeta}(y, \tau)$
- y gives uniform gradient
- obtain diffusion equation $\partial_\tau \overline{\zeta} + \partial_y F = 0, \quad F(y) = -\kappa(y)\partial_y(y + \overline{\zeta}),$
- effective diffusion $\kappa_*(y) = m^2 |M(p)|^2 w(p), \quad p \equiv my.$
- smoothing again: $\kappa = G_\delta F_\delta(\kappa_*),$
- effective diffusion determined by integral for each p

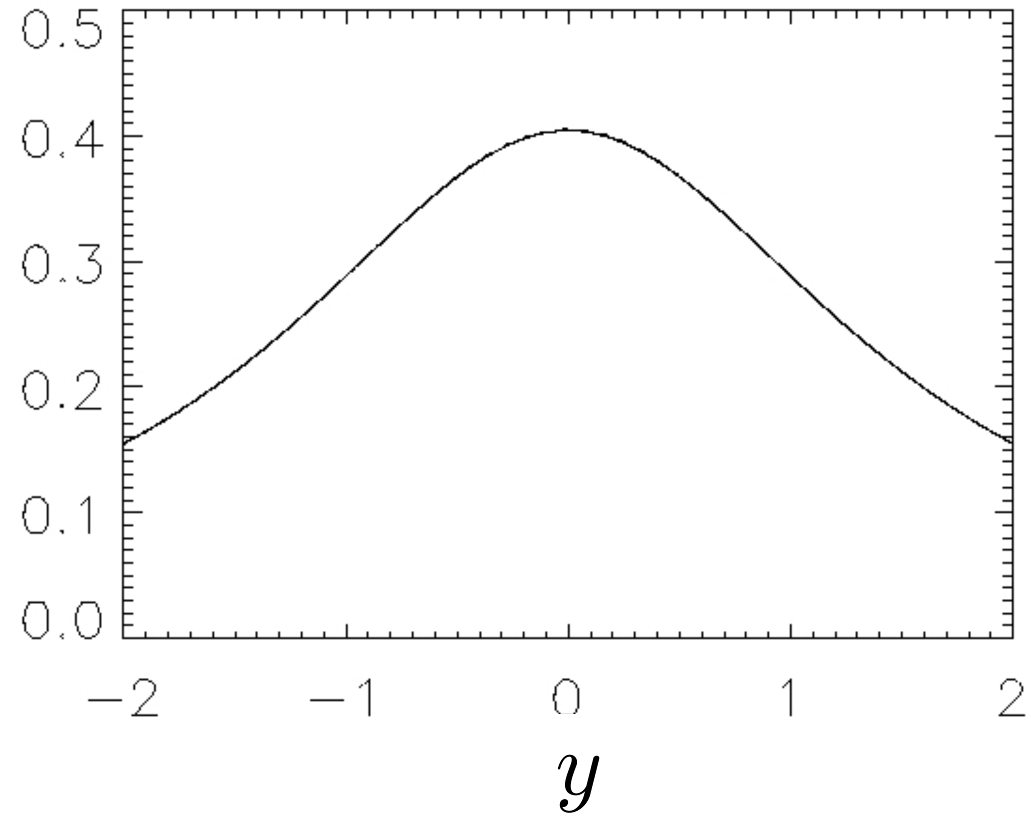
$$K(y,p) = \frac{1 + \partial_y \overline{\zeta}(y)}{p/m - y}, \quad L(p) = \mathcal{P}\int_{-\infty}^{\infty} dy\, K(y,p), \quad M(p) = \frac{1}{p - L(p)}$$

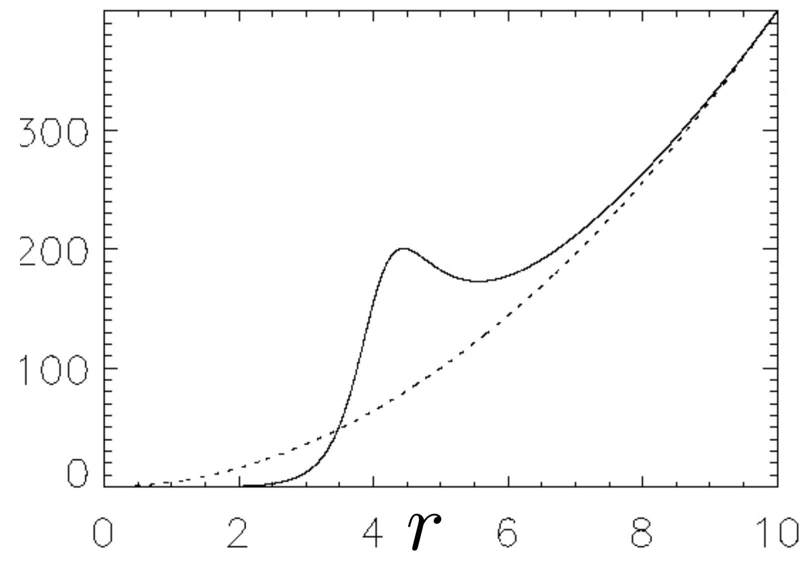
\`more' tractable analytically and numerically easy

Initially: uniform gradient in critical layer (m=2)

- vorticity in layer $\overline{\omega} = y + \overline{\zeta}(y,\tau) \qquad \overline{\zeta}(y) = 0, \quad M(p) = (p + i\pi)^{-1}$
- pole gives `quasi-mode damping rate' $p = -i\pi \qquad e^{-ipt} = e^{-\pi t}$
- effective diffusivity

$$\kappa_*(y) = m^2(m^2y^2 + \pi^2)^{-1}w(p),$$





Gaussian case from earlier

vorticity profiles

log kappa

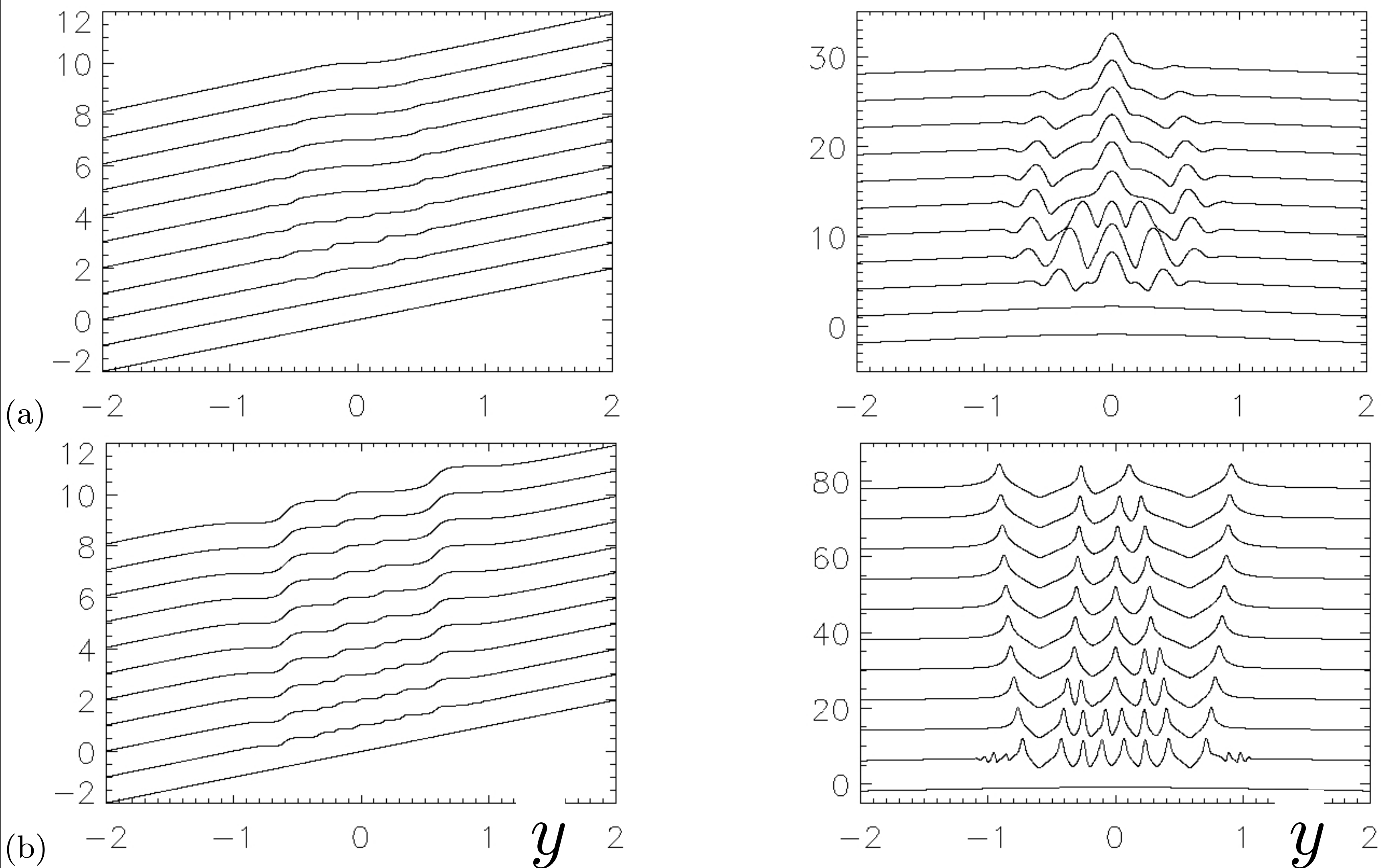


FIGURE 4. Evolution in the BLSY model with smoothing (a) $\delta = 0.05$ and (b) $\delta = 0.01$. The curves show a sequence of vorticity profiles $\overline{\omega} = y + \overline{\zeta}$ (left panels) and effective diffusivities $\log \kappa$ (right panels), plotted against y. In each panel the curves are separated by additive constants and given for $\tau = 0$, 0.1, 0.2, etc., reading up the curves.

Gaussian case from earlier

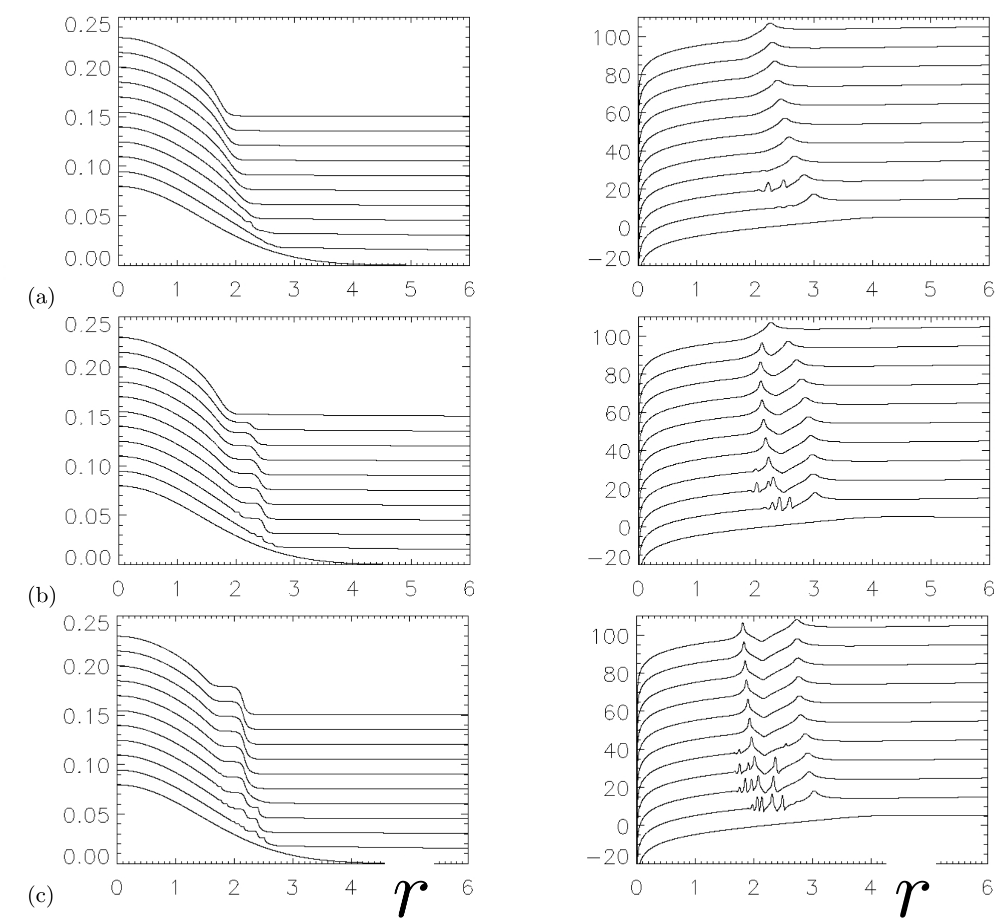


FIGURE 2. Evolution of a Gaussian vortex with smoothing (a) $\delta = 0.02$, (b) $\delta = 0.015$ and (c) $\delta = 0.01$. The curves show a sequence of vorticity profiles $\overline{\omega}$ (left panels) and effective diffusivities $\log \kappa$ (right panels), plotted against r. In each panel the curves are separated by additive constants and given for $\tau = 0$, 0.05, 0.1, etc., reading up the curves.

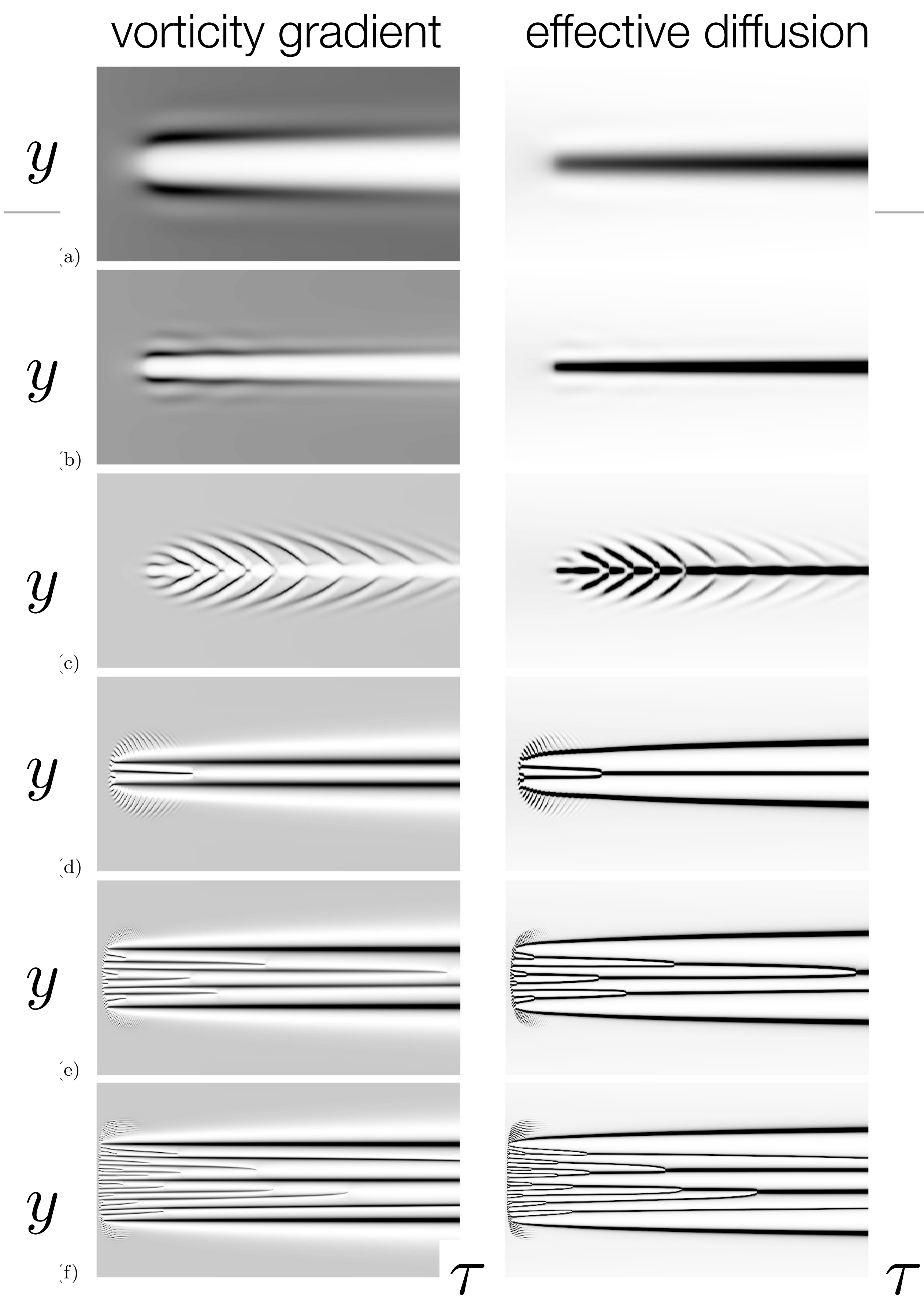


FIGURE 5. Space-time diagram of the evoluti he perturbation vorticity gradient $\partial_y\overline{\zeta}(y,\tau)$ left panels) and the effective diffusivity $\kappa(y,\tau)$ (right panels), plotted in grey scales in the $\tau, y)$-plane for (a) $\delta = 0.2$, (b) $\delta = 0.1$, (c) $\delta = 0.05$, (d) $\delta = 0.02$, (e) $\delta = 0.01$ and (f) $\delta = 0.005$. he grey scale coding is capped at the level of 4, corresponding to black; zero is white. The anges are $-2 \leq y \leq 2$ (vertical) and $0 \leq \tau \leq 1$ (horizontal), except for (a) $0 \leq \tau \leq 4$ and (b)

Gaussian case from earlier

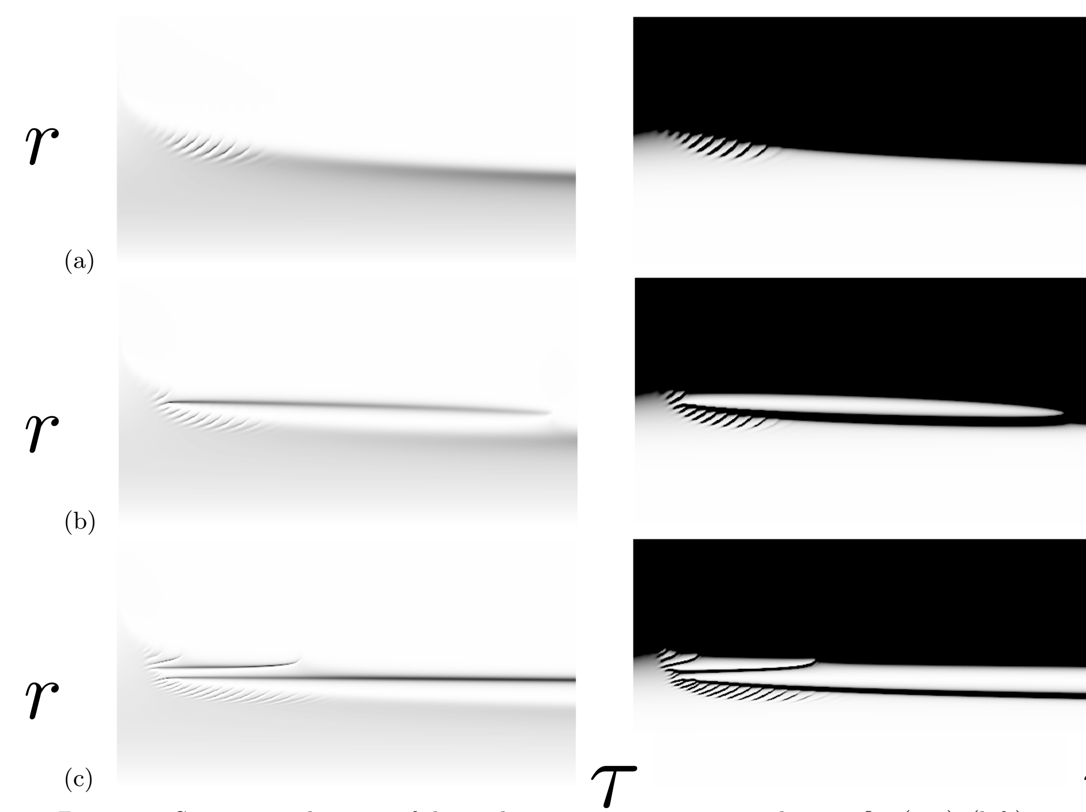


FIGURE 3. Space-time diagram of the evolution orticity gradient, $-\partial_r\overline{\omega}(r,\tau)$, (left) pan and effective diffusivity $\kappa(r,\tau)$ (right panels), plotted in grey scales in the (τ, r)-plane for (a) $\delta = 0.02$, (b) $\delta = 0.015$ and (c) $\delta = 0.01$, The grey scale coding is capped at levels $-\partial_r\overline{\omega} = 0.25$, $\kappa = 4$, corresponding to black; zero is white. The ranges are $0 \leq r \leq 5$ (vertical) and $0 \leq \tau \leq 0.5$ (horizontal).

Nature of fine-scale instability in BLSY model

- can add a weak, fine-scale perturbation to a uniform profile
- linearise and look at growth rate in BLSY model $\overline{\omega} = y$

$$\overline{\zeta}(y) = \mu \sin ky \qquad \mu \ll 1 \text{ and } k \gg 1.$$

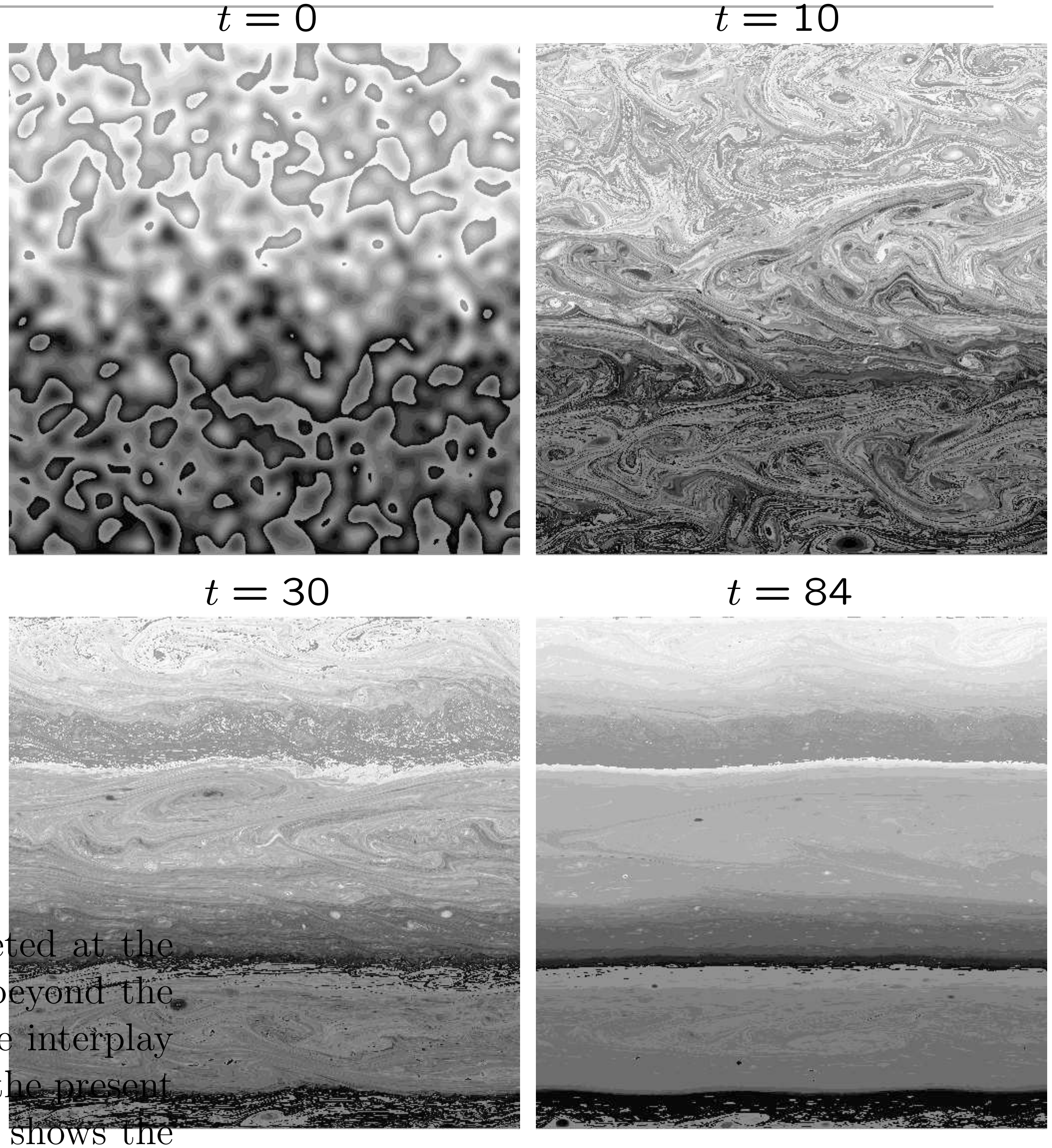
- local growth rate

$$\gamma(y) \simeq k^2 m^2 (m^2 y^2 + \pi^2)^{-2} w(my)(\pi^2 - m^2 y^2),$$

- very unstable growth for $y \leq \pi/m.$
- arises from term $(\partial_y \kappa)\partial_y(y + \overline{\zeta})$ in $\partial_y(\kappa \partial_y(y + \overline{\zeta}))$
- kappa always positive although k^2 growth rate
- observed in BLSY and Gaussian simulations

Links to other work: PV steps and banded flows

- McIntyre 1982, Dritschel/McIntyre 2007



To our knowledge, however, a systematic study has yet to be completed at the level of today's state of the art. One reason is the difficulty of getting beyond the simple positive-feedback heuristic and of quantifying the wave–turbulence interplay in this situation, including the role of Rossby-wave radiation stresses. In the present review, therefore, we limit ourselves to just one illustrative example that shows the PV Phillips effect particularly clearly. The example is taken from an ongoing series of numerical experiments to be reported elswhere. These simulate freely decaying quasigeostrophic shallow water turbulence in a beta-plane channel, starting with random vortices on an approximately uniform background PV gradient β. The experiments use a very accurate "contour-advective semi-Lagrangian" (CASL) algorithm

Giant planets (?)

- Scott and Polvani 2007
- forced dissipative shallow water equations on a sphere

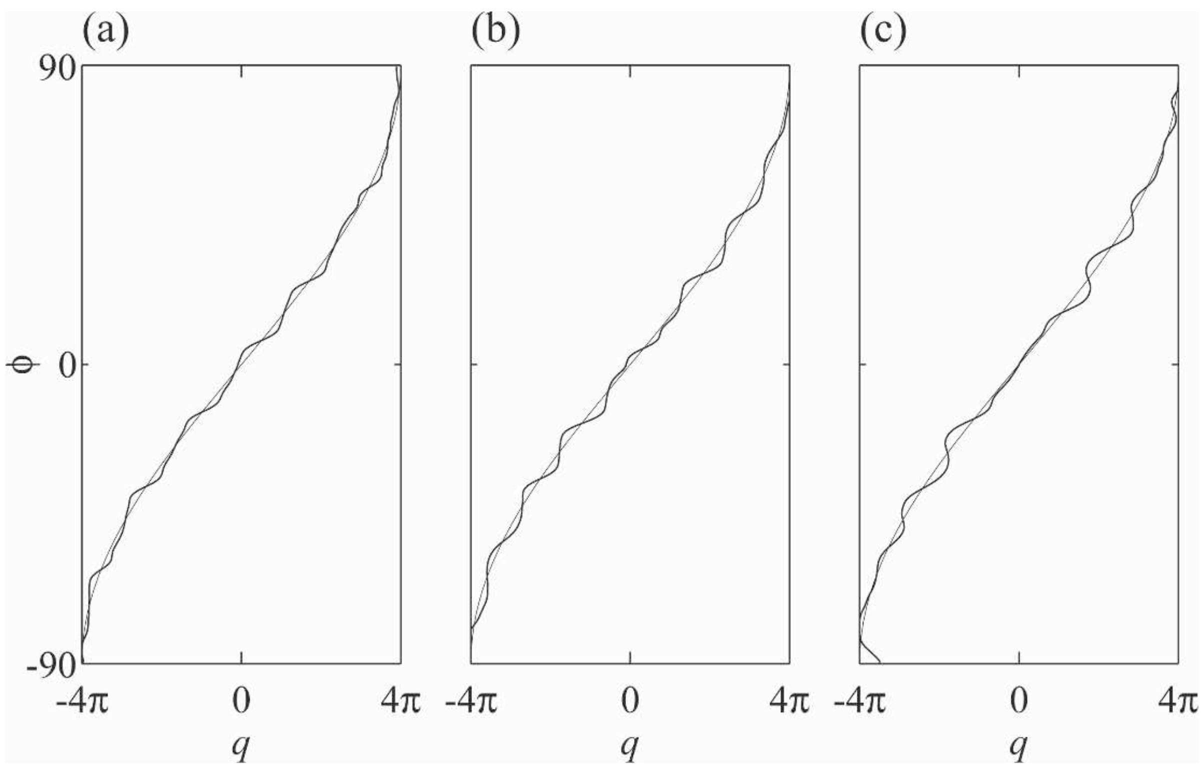


FIG. 10. Zonal mean potential vorticity vs latitude for the case $\varepsilon_0 = 1.0 \times 10^{-6}$ and (a)–(c) L_D = 0.1, 0.03, and 0.01.

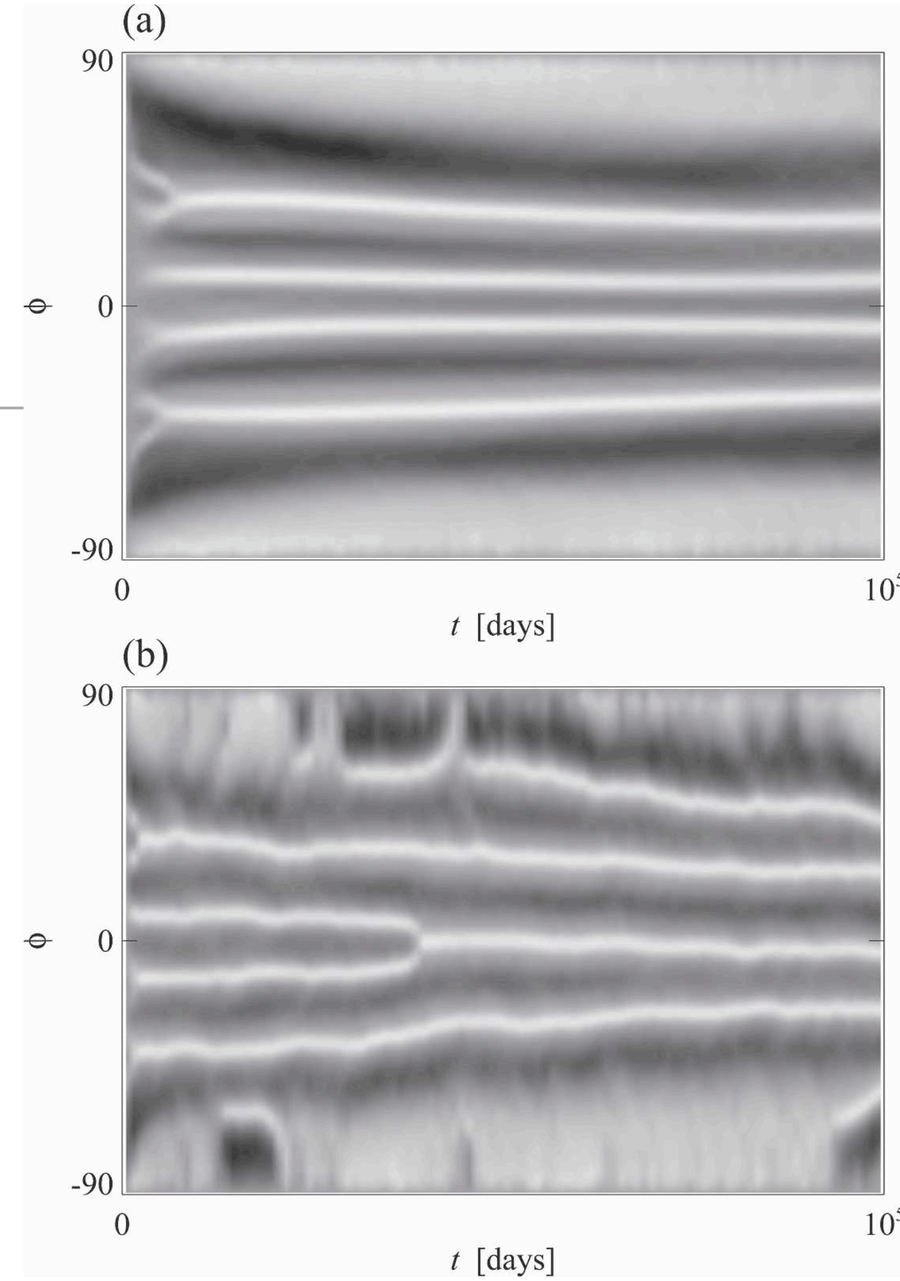


FIG. 2. Zonal velocity vs latitude and time for large-scale dissipation modeled by Rayleigh friction, with energy input and fric-

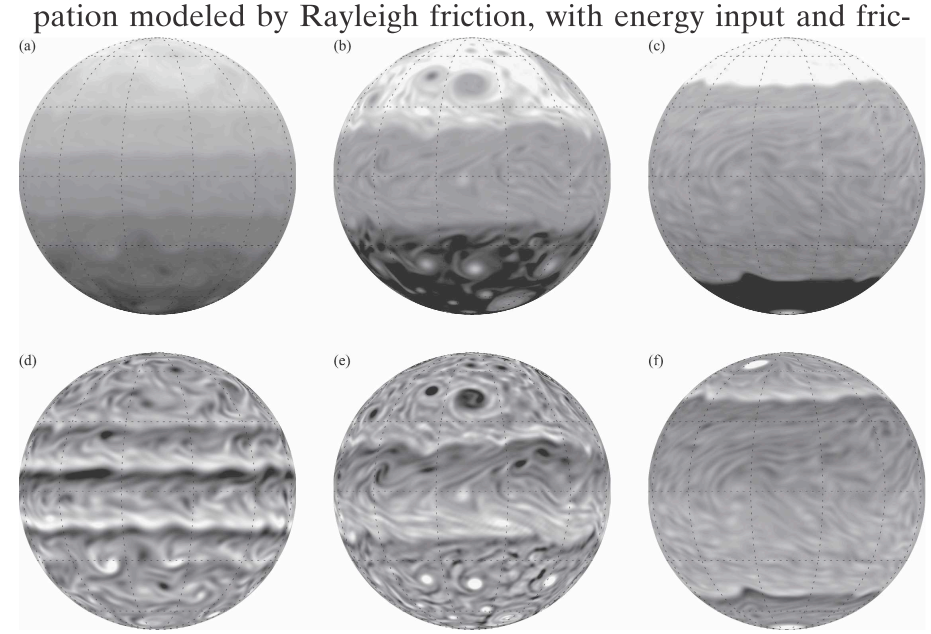


FIG. 14. (a)–(c) Potential vorticity and (d)–(f) vorticity for the three simulations with planetary parameters: (left) Jupiter, (middle) Saturn, and (right) Uranus/Neptune. Shading is same as in Fig. 9.

Our study

- Euler equation with random forcing
- systematic mathematical derivation: mixing linked to stability - novel
- obtained creation and merger of vorticity steps
- no assumptions about feedback or damping
- but had to specify random forcing (doesn't matter that much)
- and had to smooth profiles to make problem well posed and avoid fine-scale instability

Further directions

- obtain the effect in brute-force numerical simulations: (?PhD project)
- compare with theory and obtain insights into appropriate smoothing
- other flows (e.g. on a sphere, deep or shallow)
- ingredients in our work:
- mean profile, forced externally, stability problem for each frequency, feedback on the profile
- possible geophysical/astrophysical applications - any interest?
- more theoretical underpinning of work of Dritschel, McIntyre, Scott, Polvani,...

The end....