FAIR MEASURE AND FAIR ENTROPY FOR NON MARKOV INTERVAL MAPS

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Abstract. In a recent paper ([8]) the authors introduced the entropy of a special measure, the fair measure. The fair entropy is computed following backward trajectories in a way such that at each step every preimage can be chosen with equal probability. In this paper, we continue studying the fair measure and the fair entropy for non-invertible interval maps under the framework of thermodynamic formalism. We extend several results in [8] to the non-Markov setting, and we prove that for each symmetric tent map the fair entropy is equal to the topological entropy if and only if the slope is equal to 2. Moreover, we also show that the fair measure is usually an equilibrium state, which has its own interest in stochastic mechanics.

1. Introduction

Denote by $\mathcal{PMM}$ (piecewise monotone mixing) the class of continuous maps $f : [0,1] \to [0,1]$ which are piecewise monotone (with finitely many pieces) and topologically mixing.

It is well known (e.g.,[1])that given a function $f \in \mathcal{PMM}$ and $x \in [0,1]$ then we can compute the topological entropy by counting preimages of any point, this is

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(f^{-n}(x)). \tag{1}$$

Let X be a compact metric space and $f : X \to X$ a continuous map. We assume that f is a surjection, and that there exists a partition $\mathcal{X}$ of X into Borel sets X_i, $i = 1, 2, \dots s$, such that for every i the restriction of f to X_i is a homeomorphism onto its image (which is also a Borel set).

We define a partition $\mathcal{A}$ as the common refinement of the partitions $\{f(X_i), X \setminus f(X_i)\}$, $i = 1, 2, \dots s$. Clearly, $\mathcal{A}$ is a finite partition of X into Borel sets. For each $A \in \mathcal{A}$ denote by $p(A)$ the set of those numbers $i \in \{1, 2, \dots s\}$ for which $A \subset f(X_i)$. Thus, each $x \in A$ has a preimage in every X_i such that $i \in p(A)$ and no more preimages. In particular, the number $c(x)$ of preimages of x depends only on A (and therefore can be denoted $c(A)$). Since f is a surjection, this number is always positive.

Let $\mathfrak{M}$ be the space of all probability Borel measures on X. We define an operator Φ from $\mathfrak{M}$ to itself as in [8]. If $\mu \in \mathfrak{M}$, then we chop μ into pieces $\mu|_A$, $A \in \mathcal{A}$, divide each piece by $c(A)$ and push via $(f|_{X_i})^{-1}$ to each X_i with $i \in p(A)$. That is, we set

$$\Phi(\mu) = \sum_{A\in\mathcal{A}} \sum_{i\in p(A)} (f|_{X_i})^{-1}_{*} \left(\frac{\mu|_A}{c(A)} \right). \tag{2}$$

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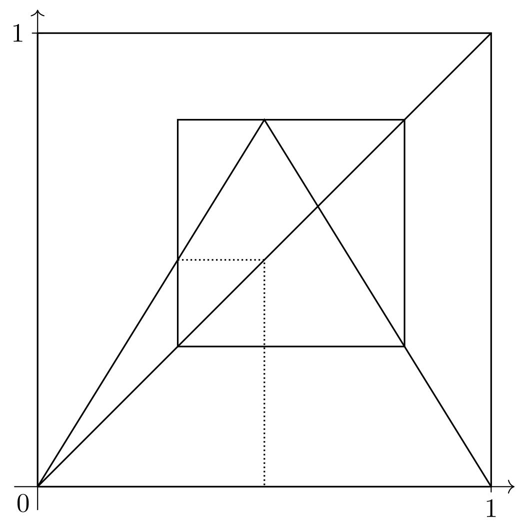

FIGURE 1. The symmetric tent map $\widetilde{T}_a$ with Fibonacci slope, i.e., $a = \frac{1+\sqrt{5}}{2}$. In particular $1/2$ is a 3-periodic point.

The operator Φ is a partial inverse of the operator f_* that carries the measure forward. Namely, it follows immediately from the definition of Φ that

$$f_* \circ \Phi = \mathrm{id}. \tag{3}$$

Definition 1.1. A measure $\mu \in \mathfrak{M}$ will be called a *fair measure* if $\Phi(\mu) = \mu$, and the *fair entropy* refers to maximum value of the measurable theoretic entropy of every fair measure.

It has been shown that each $f \in \mathcal{PMM}$ admits a unique fair measure (see [8, Theo5.11] or Proposition 3.1 in Sec 3).

On the other hand, if two functions belonging to the $\mathcal{PMM}$ class are topologically conjugated, the conjugacy will carry the fair measure of one to the fair measure of the other, and so the fair entropy is a topological invariant. It is also worth to mention that the fair entropy is in a sense equal to the backward trajectories randomly in the sense that at each step every preimage can be chosen with equal probability [8, Theo 3.4]. Based on this (comparing to (1)), the fair entropy provides a natural and computationally effective lower bound for the topological entropy.

For each $a \in (\sqrt{2}, 2]$, consider the tent map

$$\widetilde{T}_a(x) = \begin{cases} ax, & \text{if } 0 \le x < 1/2, \\ a(1-x), & \text{if } 1/2 \le x < 1. \end{cases}$$

(see Figure 1).

For simplicity, write $I_a := [\widetilde{T}_a(1/2), \widetilde{T}_a^2(1/2)]$, the *dynamical cone* of $\widetilde{T}_a$, and we concentrate on the restriction $T_a := \widetilde{T}_a|_{I_a}$. Of course, T_a and $\widetilde{T}_a$ are topologically conjugate, so it is enough to consider T_a in what follows.

In [MR] the authors proved the following theorem:

Theorem 1.2. *For Markov symmetric tent maps $T_a : I_a \to I_a$ with slope $a \in (\sqrt{2}, 2]$, the fair entropy is equal to the topological entropy if and only if $a = 2$.*

Also in [MR], it was conjectured that this result should also hold for all tent maps f_a with slope $a \in (\sqrt{2}, 2]$, not necessarily Markov.

Our main result is the next theorem:

Theorem 1.3. *For each symmetric tent map $T_a : I_a \to I_a$ with $a \in (\sqrt{2}, 2]$, there exists a unique non-atomic fair measure μ_a which is invariant and ergodic, supp$(\mu_a) = I_a$, and*

$$h_{\mu_a} = \int \log c(x) d\mu_a. \tag{4}$$

Moreover, the fair entropy is equal to the topological entropy if and only if $a = 2$.

Together with Milnor and Thurston's kneading theory [7], understanding the fair entropy for the symmetric tent maps will be the basis for understanding arbitrary unimodal interval maps (i.e., the map is monotonically increasing on the first lap and decreasing on the second one). We have the following corollary:

Corollary 1.4. *The fair entropy of a unimodal $\mathcal{PMM}$ is strictly less than* $\log 2$ *if and only if its fair entropy is strictly less than its topological entropy.*

The novelty of this paper is that we use thermodynamic formalism to study the fair measure as well as the fair entropy. Comparing to the previous methods in [8], there are at least two advantages:

- Thermodynamic formalism methods bypass the difficulties from the non-Markov structure;
- Thermodynamic formalism methods show the fair measure has many "good" statistic properties, such as exponential decay of correlation, central limit theorem and almost sure invariance principles.

The paper is organized as follows. In Section 2 we recall the basic technical results from thermodynamic formalism that we will use. In Section 3 we prove that every map $f : [0, 1] \to [0, 1]$ belonging to $\mathcal{PMM}$ has a unique fair measure. In Section 4 we prove the main result of this paper, namely, Theorem 1.3 (as well as Corollary 1.4).

2. Thermodynamic formalism

In this section, we briefly recall the basic setting of thermodynamic formalism. We recommended [2, 6] and the references therein for more detailed information. In the rest of this paper, denote by BV the set of all interval maps of bounded variation defined on [0,1]. For a given $\mathcal{PMM}$ $f : [0, 1] \to [0, 1]$, let $\mathrm{Inv}(f)$ be the set of all Borel probability invariant measure, and for each measure $\mu \in \mathrm{Inv}(f)$ and φ in BV, put

$$\mathcal{F}_\mu(f, \varphi) := h_\mu + \int \varphi d\mu, \tag{5}$$

and the pressure

$$P(f, \varphi) := \sup_{\mu \in \mathrm{Inv}(f)} \mathcal{F}_\mu(f, \varphi).$$

We say $\nu \in \mathrm{Inv}(f)$ is an *equilibrium state* for φ if $\mathcal{F}_\nu(f,\varphi) = P(f,\varphi)$.

The *transfer operator* $\mathcal{L}_\varphi : BV \to BV$ is defined by

$$\mathcal{L}_\varphi p(x) := \sum_{y=f^{-1}(x)} \exp(\varphi(y))p(y).$$

The following proposition is basically proved in [3, 4].

Proposition 2.1. *Let φ in BV be such that*

$$\sup\varphi - \inf\varphi < h_{top}(f). \tag{6}$$

Suppose further that there is a Borel probability measure μ on $[0,1]$ such that

$$\int \mathcal{L}_\varphi p d\mu = \int p d\mu, \ \forall p \in BV. \tag{7}$$

Then we have

- *μ is atom-free;*
- *There is a unique function $u : [0,1] \to (0,+\infty)$ in BV such that $\int u d\mu = 1$ and $\mathcal{L}_\varphi u = u$, and in particularly $ud\mu$ is T-invariant and ergodic and an equilibrium state for φ;*
- *μ is strongly mixing with exponential mixing rate. This implies central limit theorems and almost sure invariance principles for stochastic processes $(p \circ f^n)_{n\in\mathbb{N}}$ on $([0,1],\mu)$ for every $p \in BV$.*

Given a measure $\nu \in \mathfrak{M}$, we shall need a variation $v(p)$ for $p \in L^1(\nu)$, the set of all equivalence classes of ν-integrable function on $[0,1]$. We define

$$v(p) = \inf\{var(\widetilde{p}) : \widetilde{p} \text{ is a version of } p\}.$$

Now set $BV(\nu) = \{p \in L^1(\nu),\ v(p) < +\infty\}$, which is a linear subspace of $L^1(\nu)$. Define for $p \in BV(\nu)$ with the norm

$$\|p\|_{BV(\nu)} := \int p d\nu + v(p).$$

It is easy to see that $\big(BV(\nu), \|\cdot\|_{BV(\nu)}\big)$ is a Banach space.

With this convention, we have the following corollary.

Corollary 2.2. *For each $\mathcal{PMM}$ $f : [0,1] \to [0,1]$, assume φ, ψ are in BV, and satisfy*

$$\sup\varphi(resp.\psi) - \inf\varphi(resp.\psi) < h_{top}(f).$$

Suppose further that there are μ_1, μ_2 such that

$$\int \mathcal{L}_\varphi p(x) d\mu_1 = \int p(x) d\mu_1,\ \int \mathcal{L}_\psi p(x) d\mu_2 = \int p(x) d\mu_2,\ \forall p \in BV.$$

If φ, ψ admit the same equilibrium state (say μ), then μ_1, μ_2 have support on the same set, and there is a function $u \in BV(\mu)$ such that

$$\varphi - \psi = u \circ f - u + C.$$

Proof. Applying Proposition 2.1 to φ and ψ, then there exist two positive functions u_φ and u_ψ in BV such that $d\mu = u_\varphi d\mu_1 = u_\psi d\mu_2$, so $\text{supp}(\mu_1) = \text{supp}(\mu_2)$. Moreover, we write

$$\bar{\varphi} = \varphi - P(f,\varphi) + \log u_\varphi - \log u_\varphi \circ f, \text{ and } \bar{\psi} = \psi - P(f,\psi) + \log u_\psi - \log u_\psi \circ f,$$

then $\mathcal{L}_{\bar{\varphi}}(\mathbb{1}) = \mathcal{L}_{\bar{\psi}}(\mathbb{1}) = \mathbb{1}$. Applying [9, lem5.6.1] for $\bar{\varphi}$ and $\bar{\psi}$, we have

$$\bar{\varphi} = \bar{\psi} = J_\mu, \mu - a.e, \text{ where } J_\mu \text{ is the Jacobian of } \mu.$$

In other words,

$$\varphi - \psi = P(f,\varphi) - P(f,\psi) + u \circ f - u,$$

where $u = \log u_\psi - \log u_\varphi, \mu$-a.e. is in $BV(\mu)$, and $C = P(f,\varphi) - P(f,\psi)$ as we want. □

In the rest of the paper we apply Proposition 2.1 and Corollary 2.2, with the aim of making a better understanding on the fair measure, particularly for non-Markov $\mathcal{PMM}$.

3. The existence of fair measure

This section is devoted to reprove the following proposition, which is stated as [8, Theo5.11].

Proposition 3.1. *Every $\mathcal{PMM}$ $f : [0,1] \to [0,1]$ admits a unique and non-atomic fair measure, which is fully supported on $[0,1]$.*

The strategy is mainly based on the framework of transfer operator, which is originally from Hoflbauer and Keller in [3]. Since this proposition is the basis of the substantial discussions, we provide a detailed proof for the convenience of the reader.

Proof. Denote by $0 = c_0 < c_1 < \cdots < c_N = 1$ the boundary points of the finite partition on which f is monotone. We have that

$$f^j(c_k+) = \lim_{t \downarrow c_k} f^j(t), \text{ and } f^j(c_k-) = \lim_{t \uparrow c_k} f^j(t)$$

exist for all $k = 0, 1, \cdots N$ and $j \in \mathbb{N}$. On the other hand, $\Psi(x) := c(f(x))^{-1}$ is a piecewise constant map with $\Psi(x+), \Psi(x-)$ exists for all $x \in [0,1]$, and is such that $\Psi(x+) \neq \Psi(x-)$ occurs only at $f(c_k), k = 0, 1, \cdots, N$.

Set

$$W := \left(\bigcup_{i=0}^{\infty} f^{-i} \big(\bigcup_{j=1}^{\infty} \{ f^j(c_k+), f^j(c_k-) : 1 < k \leq N \} \big) \right) \setminus \{0,1\}.$$

W is f-invariant and countable. If $x \in W$, substitute x by $x+$ and $x-$ in $[0,1]$, and we obtain a new space $\widehat{[0,1]}$ with order relation by $y < x- < x+ < z$ if $y < x < z$. This order topology makes $\widehat{[0,1]}$ compact. In particular, $[0,1] \backslash W$ is dense in $\widehat{[0,1]}$ and $\Psi(x+), \Psi(x-), f(x+), f(x-)$ exist, so Ψ, f can be extended continuously from $[0,1] \backslash W$ to $\widehat{[0,1]}$, and they are both continuous functions on $\widehat{[0,1]}$, because all of their discontinuities in $[0,1]$ are in W. This implies that the *transfer operator*

$$\mathcal{L}p(x) := \frac{1}{c(x)} \sum_{y = f^{-1}(x)} p(y)$$

maps $C(\widehat{[0,1]})$ into $C(\widehat{[0,1]})$, and $\mathcal{L}$ is continuous on $(C(\widehat{[0,1]}), \|\|_\infty)$. Therefore, the dual operator $\mathcal{L}^*$ is continuous on the dual space $C(\widehat{[0,1]})^*$, i.e., the Banach space of probability measures on $\widehat{[0,1]}$ with respect to weak*-topology. Note that for each $\nu \in C(\widehat{[0,1]})^*$, we have

$$\mathcal{L}^*(\nu)(\mathbb{1}) = \mathcal{L}(\mathbb{1}) = \mathbb{1}.$$

Applying the Schauder-Tychonoff-theorem to the continuous map $\nu \to \mathcal{L}^*\nu$ on the compact convex set of all positive Borel probability measure on $\widehat{[0,1]}$, there is a fixed point μ, namely $\mathcal{L}^*\mu = \mu$. Furthermore, for each $p \in C(\widehat{[0,1]})$, we have

$$\int pdf^{-1}\mu = \int p \circ f d\mu = \int p \circ f d\mathcal{L}^*\mu = \int \mathcal{L}(p \circ f) d\mu = \int pd\mu.$$

This implies that μ is also an invariant measure with respect to f.

Meanwhile, note also that $\mathfrak{M} \subseteq C(\widehat{[0,1]})^*$, and operator Φ in (2) equals to the restriction $\mathcal{L}^*$ on $\mathfrak{M}$. So it follows from [8, Lem5.3,5,4] that μ is non-atomic, and fully supported. Thus, we can put $\bar{\mu} := \mu|_{\widehat{[0,1]}\setminus W} \in \mathfrak{M}$, and $\Phi(\bar{\mu}) = \bar{\mu}$ (that is $\bar{\mu}$ is a non-atomic fair measure), and it is supported on $[0,1]$. Finally, the uniqueness of the fair measure $\bar{\mu}$ follows from [8, Prop 5.10], and the proof of this proposition is completed.

□

4. Proof of Theorem 1.3

In this section we prove the main result of this paper, namely, Theorem 1.3. The proof is based on a couple of lemmas, and will be given in the end of the section.

Given a surjective continuous interval map $f : I \to I$, we say f is *topologically exact* if for every non-empty open interval $U \in I$, there exists $n \geq 0$ such that $f^n(U) = I$. This implies that for every $x \in I$, the set $\bigcup_{i=0}^{\infty} f^{-i}(x)$ is dense in I.

The following lemma plays a central role in this section.

Lemma 4.1. *Let $f : I \to I$ be a surjective continuous interval map, which is topologically exact on I. Denote by $a_{\max} := \max_{x\in I}\{f(x)\}$, $a_{\min} := \min_{x\in I}\{f(x)\}$, and $\Lambda_1 := \big\{x \in I, c(x) = 1\big\}$, then*

$$f(\Lambda_1) \cap \Lambda_1 \subset \{a_{\max}, a_{\min}, f(a_{\max}), f(a_{\min})\}. \tag{8}$$

Proof. Without loss of generality, we assume $I = [0,1]$. With this convention, $a_{\min} = 0$ and $a_{\max} = 1$. We prove this lemma by contradiction. Suppose there is an extra point $x \in (f(\Lambda_1)\cap\Lambda_1)\setminus\{0, 1, f(0), f(1)\}$, then there is a unique preimage $y \in \Lambda_1\setminus\{0,1\}$ such that $f(y) = x$. Meanwhile, there is a unique preimage z such that $f(z) = y$. We first show that x, y, z are pointwise different. Otherwise, suppose $x = y$, then x is a fixed point and $\bigcup_{i=0}^{\infty} f^{-i}(x) = \{x\}$. This is a contraction to the topological exactness of f, and thus $x \neq y$. Following the same arguments, we also obtain the remaining cases $y \neq z$ and $z \neq x$.

Next we will show that

1. x must be an extreme value for both $f|_{[0,y]}$ and $f|_{[y,1]}$;
2. x is a maximum (resp. minimum) of $f|_{[0,y]}$ if and only if x is a minimum (resp. maximum) of $f|_{[y,1]}$.

For Claim (1), we only prove x is an extreme value of $f|_{[0,y]}$, the other assertion is similarly supplied. To this end, suppose on the contrary that x is not an extreme value. Using the intermediate value theorem on $f|_{[0,y]}$, then there is a point $\omega \in [0, y]$, with $\omega \neq y$, but $f(\omega) = f(y) = x$. This is a contraction to the fact that $x \in \Lambda_1$. On the other hand, we note that $x \notin \{0, 1\}$, together with Claim (1), it directly yields Claim (2).

Analogously, we also have

(1) y must be an extreme value for both $f|_{[0,z]}$ and $f|_{[z,1]}$;
(2) y is a maximum (resp. minimum) of $f|_{[0,z]}$ if and only if y is a minimum (resp. maximum) of $f|_{[z,1]}$.

In the rest of the proof, we will construct a proper sub-interval with x, y, z as its ending points. This will lead to a contradiction to the topological exactness of f. To this end, we need to distinguish the order of x, y, z on the real line. By symmetry, it is sufficient to prove the case "$z < y$". The remaining case "$y < z$" is simply supplied.

Based on the hypothesis that f is subjective and $z < y$, it follows that

$$I = [0, z] \cup [z, y] \cup [y, 1].$$

We now distinguish two cases.

Case 1: Suppose x is a minimum of $f|_{[0,y]}$. By Claim (2), x must be a maximum of $f|_{[y,1]}$. We claim that y must be a minimum of $f|_{[0,z]}$ and a maximum of $f|_{[z,1]}$. Otherwise, y is a maximum of $f|_{[0,z]}$ and a minimum of $f|_{[z,1]}$. However, by our hypothesis $z < y$, we have $[0, z] \subset [0, y]$ and $[y, 1] \subset [z, 1]$. The former statement yields that $x < y$, while the latter statement yields that $x > y$, which is impossible. Therefore, we obtain the claim and thus $x < y$.

We further distinguish two sub-cases.

Subcase 1: Suppose $x \in [z, y]$, then x is a minimum of $f|_{[z,y]}$, and y is a maximum of $f|_{[z,y]}$. Hence

$$f([z, y]) \subset [x, y] \subset [z, y] \subsetneq I.$$

Subcase 2: Suppose $x \in [0, z]$, then x is a maximum of $f|_{[y,1]}$, and y is a minimum of $f|_{[0,z]}$. Hence

$$f^2([0, z]) \subset f([y, 1]) \subset [0, x] \subset [0, z] \subsetneq I.$$

In both sub-cases, f admits a proper forward invariant subinterval in I.

Case 2: Suppose x is a maximum of $f|_{[0,y]}$. By Claim (2), x must be a minimum of $f|[y, 1]$. Followed by the same arguments in Case 1, y must be a maximum of $f|_{[0,z]}$ and a minimum of $f|_{[z,1]}$. Therefore, $y < x$, and

$$f([y, 1]) \subset [x, 1] \subset [y, 1].$$

As a conclusion, we show that f always admits a proper forward invariant subinterval in I. This is a contradiction of the topological exactness of f, and we finish the proof. □

The following lemma is probably well known but we provide its simple proof for completeness (it is actually an exercise in [5, Ex3.4.16]).

Lemma 4.2. *The symmetric tent map $T_a : I_a \to I_a$ is topologically exact for every $a \in (\sqrt{2}, 2]$.*

Proof. We start by proving that for every open interval U there exists $K \in \mathbb{N}$, such that

$$\frac{1}{2} \in T_a^K(U) \cap T_a^{K+1}(U). \tag{9}$$

We prove (9) by contradiction. Suppose $\frac{1}{2} \notin T_a^k(U) \cap T_a^{k+1}(U)$ for every $k \in \mathbb{N}$, then

$$|T_a^{2k}(U)| \geq (a^2/2)^k |U|,$$

where $|\cdot|$ denotes the length of the interval U. By our choice of $a \in (\sqrt{2}, 2]$, $T_a^k(U) = I_a$, providing that k is sufficiently large. This is a contraction. So we have $T_a^{K+2}(U) = I_a$, in other words T_a is topologically exact. □

Lemma 4.3. *For each $a \in (\sqrt{2}, 2]$ with $T_a : I_a \to I_a$ non-Markov, there exists $N := N(a) \in \mathbb{N}$ such that*

$$\sup_{I_a} \left(\frac{1}{N} S_N(\log \Psi) \right) - \inf_{I_a} \left(\frac{1}{N} S_N(\log \Psi) \right) < h_{top}(T_a),$$

where $\Psi(x) = c(T_a(x))^{-1}$, and $S_N(f) = \sum_{i=0}^{N-1} f \circ T_a^i$.

Proof. Fix $a \in (\sqrt{2}, 2]$ with T_a non-Markov. By Lemma 4.2, T_a is topologically exact. Applying Lemma 4.1 for T_a, then we have $\Lambda_1 = [T_a^2(1/2), T_a^3(1/2)) \cup \{T_a(1/2)\}$, and $T_a(\Lambda_1) \cap \Lambda_1 \subset \{T_a(1/2), T_a^2(1/2), T_a^3(1/2)\}$. However, it is obvious to see that $T^3(1/2) \notin \Lambda_1$, so

$$T_a(\Lambda_1) \cap \Lambda_1 \subset \left\{ T_a(1/2), T_a^2(1/2) \right\}. \tag{10}$$

In other words, we have the following dichotomy

Case 1: If $x \neq T_a(1/2)$ and $x \in \Lambda_1$, then $T_a(x) \notin \Lambda_1$;
Case 2: If $x = T_a(1/2)$, then $T_a^2(1/2) \in \Lambda_1$, but $T_a^3(1/2) \notin \Lambda_1$.

Note also that T_a is non-Markov, so the forward trajectory of $1/2$ is not pre-periodic, i.e., $T_a^i(1/2) \neq T_a^j(1/2)$, providing that $i \neq j$. This implies that Case 2 can occur at most once in the forward trajectory of any point $x \in I_a$. By the pigeonhole principle, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \left(\sup_{I_a}\left(\frac{1}{n} S_n(\log \Psi)\right) - \inf_{I_a}\left(\frac{1}{n} S_n(\log \Psi)\right) \right) \\
&= \lim_{n\to\infty} \left(-\inf_{x \in I_a}\left(\frac{1}{n} S_n(\log c(T_a(x)))\right) + \sup_{x \in I_a}\left(\frac{1}{n} S_n(\log c(T_a(x)))\right) \right) \\
&\leq \lim_{n\to\infty} \left(\frac{1}{n}\Big(-\log 2 - \frac{(n-3)}{2} \log 2 \Big) + \log 2 \right) \\
&= \log \sqrt{2} < \log a = h_{top}(T_a).
\end{aligned}$$

Therefore, there exists an integer $N := N(a)$ such that

$$\sup_{I_a} \left(\frac{1}{N} S_N(\log \Psi) \right) - \inf_{I_a} \left(\frac{1}{N} S_N(\log \Psi) \right) < h_{top}(T_a),$$

as we want. □

Based on this, we have

Proposition 4.4. *For the symmetric tent map $T_a : I_a \to I_a$ with $a \in (\sqrt{2}, 2]$ and T_a non-Markov, T_a admits a unique non-atomic fair measure $\mu_a \in Inv(T_a)$. Moreover,*

(1) μ_a is the unique equilibrium state for the potential $\log \Psi$;

(2) The Jacobian $J_{\mu_a} = c(T_a)$, $\mu_a - a.e.$, and the measurable theoretic entropy

$$h_{\mu_a} = \int \log c(x) d\mu_a;$$

(3) Central limit theorems and almost sure invariance principles hold for stochastic processes $(p \circ T_a^n)_{n\in\mathbb{N}}$ on (I_a, μ_a) for every $p \in BV$.

Proof. By Proposition 3.1, T_a admits a unique fair measure $\mu_a \in \mathrm{Inv}(T_a)$. On the other hand, by Lemma 4.3, we can put a bounded variation function $g := \frac{1}{N} S_N(\log \Psi)$ such that

$$\sup_{I_a} g - \inf_{I_a} g < h_{top}(T_a).$$

It is easy to see that

$$\mathcal{L}_g \mathbb{1} = \sqrt[N]{\mathcal{L}^N_{\log \Psi} \mathbb{1}} = \mathbb{1}. \tag{11}$$

So we have

$$\int \mathcal{L}_g p(x) d\mu_a = \int p(x) d\mu_a, \ \forall p \in BV.$$

Applying Proposition 2.1 together with (11), we have that μ_a is the unique non-atomic equilibrium state for the potential g. Note also from the definition that g and $\log \Psi$ shares the same equilibrium state with same stochastic properties, so we obtain assertions (1) and (3).

Next we turn to prove assertion (2). Denote by J_{μ_a} the Jacobian of μ_a, then we have

$$\begin{aligned}
1 = \int \mathbb{1} d\mu_a &\geq \int \mathcal{L}_{\log \Psi}(J_{\mu_a} \Psi) d\mu_a \\
&= \int (J_{\mu_a}\Psi) d\mu_a \geq \int \left(1 + \log(J_{\mu_a}\Psi)\right) d\mu_a \ (\text{since } 1 + \log x \leq x) \\
&= 1 + \int \log \Psi d\mu_a + \int \log J_{\mu_a} d\mu_a \\
&= 1 + \underbrace{\int \log \Psi d\mu_a + h_{\mu_a}}_{(1)} \ (\text{Rohlin entropy formula}). \qquad (12)
\end{aligned}$$

We will estimate term (1) from below.

Recall that $\mathcal{X}$ is the monotonic partition of T_a, and put $\eta := \min\{|X_i| : X_i \in T_a^{-n}(\mathcal{X})\}$. Based on T_a is uniformly expanding (i.e., $\inf_{I_a} |T_a'| = a > 1$), so for every point $x \in I_a$, the set $T_a^{-n}(x)$ is (n, η) separated. Applying [9, Theo 3.3.2], and also note that $\mu_a \in \mathrm{Inv}(T_a)$ is an equilibrium state for $\log \Psi$, this yields that

$$\begin{aligned}
(1) =& P(T_a, \log \Psi) \\
\geq& \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y \in T_a^{-n}(x)} \exp S_n(\log \Psi(y)) \geq 0, \ \forall x \in I_a.
\end{aligned}$$

Substituting the above estimate into (12), we obtain $h_{\mu_a} + \int \log \Psi d\mu_a = 0$. In other words,

$$h_{\mu_a} = \int \log c(T_a(x)) d\mu_a = \int \log c(x) d\mu_a,$$

and $J_{\mu_a} = \Psi^{-1} = c(T_a)$, μ_a-a.e. as we want. □

Proposition 4.5. *For each symmetric tent map $T_a : I_a \to I_a$ with $a \in (\sqrt{2}, 2)$, let μ_a be the unique fair measure. Then for every $N \in \mathbb{N}$, there are no $u \in L^\infty(\mu_a)$ and a constant C such that*

$$\frac{1}{N} S_N(\log \Psi) = u \circ T_a - u + C.$$

Remark 4.6. As a comparison, it is easy to see that $\log \Psi = -\log 2$ for T_2, and that the Lebesgue measure on $[0,1]$ is the unique fair measure with maximum entropy measure, and the fair entropy and topological entropy coincide and are equal to $\log 2$.

Proof. Without loss of generality, we can assume that $N = 1$. We will prove this result by contradiction. Suppose there exist $u \in L^\infty(\mu_a)$ and a constant C, such that

$$\log \Psi - C = u \circ T_a - u.$$

This implies that

$$\|S_n(\log \Psi - C)\|_\infty = \|u \circ T_a^n - u\|_\infty \leq 2 \parallel u \parallel_\infty < +\infty, \ \forall n \in \mathbb{N}. \tag{13}$$

On the other hand, for each $n \in \mathbb{N}$ fixed, we distinguish the following two cases

Case 1: "$C \neq -\log 2$". Note that $\{T_a^i\}_{i=1}^n$ has a common fixed point $\alpha := \frac{a}{1+a}$, so there exists a sufficiently small open neighborhood U containing α such that $\mu_a(U) > 0$, and that for every point $x \in U$, $T_a^i(x)$ has two pre-images for every $i = 1, \cdots, n$. Therefore,

$$\begin{aligned} \parallel S_n(\log \Psi - C) \parallel_\infty &\geq |S_n(\log \Psi - C)(x)| \\ &= n|\log 2 - C| = n|\log 2 + C| \to \infty, \ \text{as } n \to \infty. \end{aligned}$$

Case 2: "$C = -\log 2$". Note that every point $x \in [T_a^2(1/2), T_a^3(1/2))$ has a unique pre-image, and $\mu_a\{[T_a^2(1/2), T_a^3(1/2))\} > 0$. Put

$$W = \{x \in [T_a^2(1/2), T_a^3(1/2)), \ x \text{ is infinitely many recurrent}\}.$$

By Poincaré recurrent theorem, $\mu_a(W) = \mu_a\{[T_a^2(1/2), T_a^3(1/2))\} > 0$. Therefore, we have

$$\begin{aligned} \parallel S_n(\log \Psi - C) \parallel_\infty &= \parallel S_n(\log \Psi + \log 2) \parallel_\infty \\ &\geq |S_n(\log \Psi + \log 2)(x)|, \ \forall x \in W \\ &= n' \log 2 \to +\infty, \ \text{as } n \to \infty, \ \text{where } n' \text{ is the recurrent time.} \end{aligned}$$

In both cases, $S_n(\log \Psi - C)$ is not uniformly bounded in $\parallel \cdot \parallel_\infty$-norm, which is a contradiction to (13), and the proof of the Lemma is completed. □

We are now ready to prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. The existence, atom freeness of the fair measure μ_a and assertion (4) come from Proposition 4.4. On the other hand, Proposition 4.5, Corollary 2.2 and Remark 4.6 directly imply that the fact that fair entropy equals to the topological entropy for T_a if and only if $a = 2$. □

Proof of Corollary 1.4. Since f is a unimodal $\mathcal{PMM}$, it is easy to see that f is not renormalizable, has no attracting periodic point, and has no wandering intervals. By [5, Theo 3.4.27], f is topologically conjugate to a symmetric tent map T_a with slope $a \in (\sqrt{2}, 2]$. By applying Theorem 1.3, the fair entropy is strictly less than its topological entropy if and only if the fair entropy of f is equal to $\log 2$. So the proof of this corollary is completed. □

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