Rigorous numerical approximation of Ruelle–Perron–Frobenius operators and topological pressure of expanding maps

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Abstract

It is well known that for different classes of transformations, including the class of piecewise $C^2$ expanding maps $T : [0,1] \to [0,1]$, Ulam’s method is an efficient way to numerically approximate the absolutely continuous invariant measure of $T$. We develop a new extension of Ulam’s method and prove that this extension can be used for the numerical approximation of the Ruelle–Perron–Frobenius operator associated with $T$ and the potential $\phi_{\beta} = -\beta \log |T|$, where $\beta \in \mathbb{R}$.

In particular, we prove that our extended Ulam’s method is a powerful tool for computing the topological pressure $P(T, \phi_{\beta})$ and the density of the equilibrium state.

Mathematics Subject Classification: 37M25, 37D20, 37D35, 37E05

1. Introduction

Let $(X, \mathcal{B})$ be a measurable space and $T : X \to X$ a measurable transformation. Let $\mathcal{M}(X, T)$ denote the set of all $T$-invariant probability measures and $h_{\mu}(T)$ denote the metric entropy of $T$ with respect to $\mu$. An invariant probability measure $\mu_\phi \in \mathcal{M}(X, T)$ is said to be an equilibrium state for a continuous potential $\phi : X \to \mathbb{R}$ if it satisfies the variational principle, i.e.

$$P(T, \phi) := h_{\mu_\phi}(T) + \int_X \phi \, d\mu_\phi = \sup_{\mu \in \mathcal{M}(X, T)} (h_{\mu}(T) + \int_X \phi \, d\mu)$$

where $P(T, \phi)$ is the topological pressure associated with $\phi$ and $T$ (see for example [21]).

Within the mathematical framework of the thermodynamical formalism [21], a key ingredient in obtaining analytical expressions for the topological pressure $P(T, \phi)$ and related thermodynamic quantities is the Ruelle–Perron–Frobenius (RPF) operator $L_\phi : B(X) \to B(X)$, where $B(X)$ is the space of all measurable bounded functions on $X$, defined as $L_\phi f(x) = \sum_{y \in T^{-1}(x)} e^{\phi(y)} f(y)$. Ruelle [19] proved that the equilibrium state of a finite state
topologically mixing Markov shift is given by $\mu_\phi = hv_\phi$, where $v_\phi$ is a probability measure and $h$ is a density satisfying $L_\phi h = \lambda h$, $L_\phi^* v_\phi = \lambda v_\phi$ and $\log(\lambda) = P(T, \phi)$. Later on, with some extra conditions on the potential $\phi$, these results were extended to some other classes of transformations (see for instance [3, 10, 14, 20, 22, 26, 27]; see also [1] for other references). In all these settings the equilibrium measure $\mu_\phi = h \, dv_\phi$ is absolutely continuous w.r.t. the conformal measure (possibly non-Lebesgue) $v_\phi$ (see [4] for background on conformal measures).

Common choices of the potential are: $\phi_\beta = -\beta \log |T'|$, $\beta \in \mathbb{R}$, yielding the operator $L_\beta f(x) = \sum_{\phi \in \bigcap_{i=1}^{\infty} \mathcal{T}^{-1}(I_a)} \frac{f(y)}{|T'(x)|}$, which is used to study the existence of phase transitions in certain classes of transformations (e.g. [17, 23]) and $\phi = -\log |T'|$ which yields the well-known Perron–Frobenius (PF) operator $L f(x) = \sum_{\phi \in \bigcap_{i=1}^{\infty} \mathcal{T}^{-1}(I_a)} \frac{f(y)}{|T'(x)|}$. The densities of absolutely continuous (w.r.t. Lebesgue) invariant measures are fixed points of $L$.

It is well known that for different classes of transformations, including the class of expanding maps of the unit interval, Ulam’s method (see section 5 for details) gives good estimates of the PF operator and thus of the absolutely continuous $T$ invariant measure [2, 5, 7, 12]. In this work we show that Ulam’s method can be used to approximate the leading eigenvalue and corresponding eigenfunction of the RPF operator $L_\beta$ for expanding, piecewise monotonic maps $T : [0, 1] \to [0, 1]$ with a finite number of monotonicity intervals. More importantly, we show that the approximated eigenfunction is exactly the density of the equilibrium state and that its associated eigenvalue gives the value of $P(T, \phi_\beta)$, where $\phi_\beta = -\beta \log(|T'|)$. Our approach has also been successfully used to study non-uniformly expanding maps that exhibit phase transitions [8].

The outline of the paper is as follows. In the first part, we develop a suitable Lasota–Yorke (LY) inequality that allows us to prove that a normalized version of $L_\beta$ preserves a cone of non-negative functions in $L^1$. Related inequalities have been produced in [14] in terms of a limiting measure $v$ that is not explicitly known. To our knowledge the explicit $BV-L^1$ form of the LY inequality developed in section 3 below has not been previously published. Next, we prove that $L_\beta$ has a positive eigenfunction $h$, establish that the positive eigenvalue associated with $h$ satisfies $\lambda_\beta = e^{P(T, \phi_\beta)}$ and that $h$ is the density of the equilibrium state for $(T, \phi_\beta)$ with respect to the corresponding conformal measure. Finally, we recall Ulam’s method and state our main result on the numerical approximation of the density $h$ and of the topological pressure $P(T, \phi_\beta)$.

2. Class of transformations considered

Let $I$ be the unit interval $[0, 1]$ and let $T : I \to I$ be a piecewise $C^2$ transformation. Let $\mathcal{P} = \{I_a\}$ be a finite partition of $I$ such that $I_a$ are closed intervals, $I = \bigcup_a I_a$ and $\text{int}(I_a) \cap \text{int}(I_{a'}) = \emptyset$, $\forall a \neq a'$. The restriction of $T$ to $I_a$, $T_a = T \mid I_a : I_a \to T(I_a)$ is assumed to be strictly monotone. $T^{-1}_a : T(I_a) \to I_a$ represents the inverse branches of $T$. The $n$th iterate of $T$ is defined by $T^n_{a^{(n)}} = T \mid I_{a^{(n)}} : I_{a^{(n)}} \to I$ where $I_{a^{(n)}} \in \bigcap_{i=0}^{n} T^{-i} \mathcal{P}$ and its inverse is defined by $T^{-n}_{a^{(n)}} : T^n(I_{a^{(n)}}) \to I_{a^{(n)}}$. Where necessary, we define $T^n(x)$ at the endpoints of $I_a$ by taking an appropriate one-sided derivative. We assume that there exists $\alpha > 1$ such that

$$|T^n(x)| \geq \alpha, \quad \forall x \in I. \tag{1}$$

Note that under the above assumptions, $T^n(x)$ is finite and bounded away from zero for all $x \in I$. Thus, there exists $s \geq 0$ such that

$$\frac{|T^n(x)|}{|T^n(x)|} \leq s, \quad \forall x \in I. \tag{2}$$
From (1) and (2) we have that there exists $D \geq 0$ such that
\[ \frac{|(T_{\varphi}^{n\beta})(x)|}{|(T_{\varphi}^{n\beta})(y)|} \leq D \leq e^{c_\beta}, \quad \forall n \geq 1, \forall x, y \in I. \tag{3} \]

We further assume that $T$ is covering (see [14, 13]), i.e. for each $n \in \mathbb{N}$ there exists $N(n) > 1$ such that $T^{N(n)}(I^{\varphi}) = [0, 1], \forall I^{\varphi} \in \bigcup_{i=0}^{m} T^{-i} \varphi$. Under the above assumptions, we choose $c' > 0$ and $c_N(0) > 0$ such that
\[ m(T(I_a)) \geq c', \quad \forall I_a \in \varphi, \tag{4} \]
\[ m(I_{a,N(0)}) \geq c_N(0), \quad \forall I_{a,N(0)} \in \bigcup_{i=0}^{N(0)} T^{-i} \varphi, \tag{5} \]
where $m$ is Lebesgue measure.

For $\beta \in \mathbb{R}$ we consider the potential $\phi_\beta : I \to \mathbb{R}$ defined as $\phi_\beta(x) = -\beta \log(|T'(x)|)$ and the corresponding weight $g_\beta : I \to (0, 1), g_\beta(x) = \exp(\phi_\beta(x))$. In this setting, conditions (1) and (2) are enough to guarantee that $\phi_\beta : I \to \mathbb{R}$ (and consequently $g_\beta : I \to (0, 1)$) is a function of finite variation, i.e. $V_1(\phi_\beta) < \infty$ where $V_1(\phi_\beta) = \sup\{k \sum_{i=0}^{k} |\phi_\beta(x_i) - \phi_\beta(x_{i-1})| : k \geq 1, x_0 < \cdots < x_k, x_i \in I\}$.

**Notation.** Throughout the paper $\| \cdot \|_1$ will stand for the $L^1$ norm and $f \in L^1$ will refer to functions $f$ that are Lebesgue integrable. $BV(I)$ is the space of functions of bounded variation acting on $I$, i.e. $BV(I) = \{ f : I \to \mathbb{R} : V_1(f) < \infty \}$ and is endowed with the norm $\|f\|_{BV} = V_1(f) + \|f\|_{\infty}$.

### 3. Lasota–Yorke inequalities and cones for $L_\beta$

Cone techniques have been used to establish the existence of the invariant density of $T$ as a fixed point of the PF operator [13] and to obtain the density of the equilibrium measure (possibly not absolutely continuous w.r.t. Lebesgue) as an eigenfunction of the more general RPF operator [14]. The rough idea behind this technique is to choose a cone$^1$ of functions, typically defined via a LY-type inequality on which the operator is a contraction. In section 4 we develop a convex set of BV functions that is compact in $L^1$ and apply standard fixed point theorems to establish the existence of the required $L^1$ eigenfunction of the RPF operator. This approach may be viewed as an extension of [15], which showed that the standard PF operator associated with transformations similar to the ones introduced in section 2 preserves a suitable cone of $L^1$ functions and used this to prove convergence of Ulam’s approximation.

Our aim for the rest of this section is to build a LY inequality for $L_\beta$ associated with the transformations introduced in section 2 in terms of BV functions in $L^1$. Because $L_\beta$ is not a Markov operator (see lemma 4), we need to treat the $\beta < 1$ and $\beta \geq 1$ cases separately. Also, for technical reasons that will become obvious in the proofs we need to treat the $\beta < 0$ situation as a third separate case.

#### 3.1. Properties of $L_\beta$

We collect some properties of $L_\beta$ that will be used later to obtain the cone contraction. Under the assumptions of the previous section we write
\[ L_\beta f = \sum_a (g_\beta \circ T_a^{-1})(f \circ T_a^{-1}) \chi_{T(I_a)} = \sum_a \frac{f \circ T_a^{-1}}{|T_a \circ T_a^{-1}|} \chi_{T(I_a)}. \tag{6} \]

$^1$ A convex subset $\mathcal{P}$ of a real vector space $X$ is a cone if for any $t > 0$ and for all $f \in \mathcal{P}, tf \in \mathcal{P}$. 

Lemma 1.
(i) $L_\beta$ is a positive operator; that is, $L_\beta f \geq 0$ for all $f \in L^1$, $f \geq 0$.
(ii) $L_\beta : L^1(I) \rightarrow L^1(I)$ is a bounded operator.

Proof. See proofs section. □

Define the cone $B_k$, $0 \leq k < \infty$ by
$$B_k = \{ f \in L^1 : f \geq 0, V_I(f) \leq k \| f \|_1 \},$$
and note that $B_k$ is a subset of $BV(I)$.

3.2. Lasota–Yorke inequality

Lemma 2. Let $\alpha$, $s$, $D$, $c'$ and $c_{N(0)}$ be given as in (1), (2), (3), (4) and (5), respectively.
(i) When $\beta \geq 1$, for all $f \in B_k$
$$V_I(L_\beta f) \leq \frac{2}{\alpha^\beta} V_I(f) + M_1 \| f \|_1 \leq \left( \frac{2}{\alpha^\beta} k + M_1 \right) \| f \|_1,$$
where $M_1 = \frac{2}{\alpha^\beta}(s\beta + \frac{1}{\beta})$.
(ii) When $0 \leq \beta < 1$ for all $f \in B_k$
$$V_I(L_\beta f) \leq \frac{2}{\alpha^\beta} V_I(f) + M_2 \| f \|_1 \leq \left( \frac{2}{\alpha^\beta} k + M_2 \right) \| f \|_1,$$
where $M_2 = 2\left( \frac{D}{c_{N(0)}} \right)^{1-\beta}(s\beta + \frac{1}{\beta})$.
(iii) When $\beta < 0$ for all $f \in B_k$
$$V_I(L_\beta f) \leq 2\left( \frac{c_{N(0)}}{D} \right)^\beta V_I(f) + M_3 \| f \|_1 \leq \left( 2 \left( \frac{c_{N(0)}}{D} \right)^\beta k + M_3 \right) \| f \|_1,$$
where $M_3 = 2\left( \frac{D}{c_{N(0)}} \right)^{1-\beta}(s|\beta| + \frac{1}{\beta})$.

Proof. See proofs section. □

By choosing $k$ large enough, we can ensure that $L_\beta B_k \subseteq B_k$. However, in order to obtain a fixed point, we need to consider a normalized operator.

4. A normalized operator and a fixed point theorem

In this section, we obtain an eigenfunction of $L_\beta$ by demonstrating the existence of a fixed point of a normalized operator in a suitable convex set. Below we briefly summarize the method of proof. The normalized operator we consider is $L'_\beta : H \ni f$, where $H = \{ f \in L^1 : f \geq 0, \| f \|_1 = 1 \}$, defined as
$$L'_\beta f = \frac{L_\beta f}{\| L_\beta f \|_1}. \quad (7)$$
We prove that for some suitable $k$, the operator $L'_\beta$ becomes a contraction for the convex set
$$B'_k = B_k \cap H = \{ f \in L^1, f \geq 0 : V_I(f) \leq k, \| f \|_1 = 1 \}. \quad (8)$$
In this sense we first establish the following lemma.

Lemma 3. For each $0 < k < \infty$, $B'_k$ is compact in $L^1$. 

Proof. Let \( f_n \) be a sequence in \( B_k' \). Then \( V_\beta(f_n) \leq k \) and \( \| f_n \|_\infty \leq k + 1 \). By Helly’s selection principle, there exists \( n_k \) s.t. \( f_n \to f^* \) everywhere. Thus \( \| f_{n_k} - f^* \|_L^1 \leq \| f_{n_k} - f^* \|_\infty \to 0 \) as \( n_k \to \infty \). It is easily checked that \( f^* \in B_k' \). □

We obtain the fixed point of \( L_\beta' \) via a standard fixed point theorem. We start by collecting some properties of \( L_\beta' \). To do so we use the following lemma that describes basic properties on the relative sizes of \( \| f \|_1 \) and \( \| L_\beta f \|_1 \).

Lemma 4. For all \( f \in L^1, f \neq 0 \) the following hold:

(i) When \( \beta \geq 1 \),
\[
\frac{\| f \|_1}{\| L_\beta f \|_1} \leq \left( \frac{D}{c_{N(0)}} \right)^{\beta-1}.
\]

(ii) When \( \beta < 1 \),
\[
\frac{\| f \|_1}{\| L_\beta f \|_1} \leq \frac{1}{\alpha^{1-\beta}}.
\]

Proof. See proofs section. □

Lemma 5. For all \( \beta \in \mathbb{R} \), \( L_\beta' : H \circlearrowleft \) is

(i) well defined and
(ii) continuous.

Proof. Follows immediately from lemma 1, the definition of \( L_\beta' \) and lemma 4. □

We can now obtain explicit bounds for the variation of \( L_\beta' f \).

Lemma 6. Let \( B_k' \) be as defined in (8). For all \( f \in B_k' \) we have

(i) When \( \beta \geq 1 \),
\[
V_\beta(L_\beta' f) \leq \left( \frac{k}{\alpha^\beta} + M_1 \right) \left( \frac{D}{c_{N(0)}} \right)^{\beta-1},
\]
where \( M_1 = \frac{2}{\alpha^\beta} (s\beta + \frac{1}{\alpha}) \).

(ii) When \( 0 \leq \beta < 1 \),
\[
V_\beta(L_\beta' f) \leq \left( \frac{k}{\alpha} + M_2 \frac{1}{\alpha^{1-\beta}} \right),
\]
where \( M_2 = 2 \left( \frac{D}{c_{\alpha^0}} \right)^{1-\beta} (s\beta + \frac{1}{\alpha}) \).

(iii) When \( \beta < 0 \),
\[
V_\beta(L_\beta' f) \leq \left( \frac{1}{\alpha} \right)^{1-\beta} \left( 2 \left( \frac{c_{N(0)}}{D} \right)^{\beta} k + M_3 \right),
\]
where \( M_3 = 2 \left( \frac{D}{c_{\alpha^0}} \right)^{1-\beta} (s|\beta| + \frac{1}{\alpha}) \).

Proof. The result follows immediately from lemma 2(i) and lemma 4(i) when \( \beta \geq 1 \), lemma 2(ii) and lemma 4(ii) when \( 0 \leq \beta < 1 \) and lemma 2(iii) together with lemma 4(ii) when \( \beta < 0 \). □
Lemma 7. Let $B'_k$ be as introduced in (8). Then

(a) For each $\beta \geq 1$, if $\frac{\alpha}{2} > \frac{D c N(0)}{c N(0)} \beta^{-1}$, then

$$L'_\beta B'_k \subseteq B'_k, \quad \forall k \geq k(\beta),$$

where

$$k(\beta) = M_1 \frac{D c N(0) \beta^{-1}}{1 - \frac{2}{\alpha} \beta^{1 - \beta}}.$$  \hfill (12)

(b) For each $0 \leq \beta < 1$,

$$L'_\beta B'_k \subseteq B'_k, \quad \forall k \geq k(\beta),$$

where

$$k(\beta) = M_2 \frac{1 - \frac{2}{\alpha}}{1 - \frac{2}{\alpha^\beta}}.$$  \hfill (13)

(c) For each $\beta < 0$, if $\alpha^{1-\beta} > 2(c N(0)/D)\beta$,

$$L'_\beta B'_k \subseteq B'_k, \quad \forall k \geq k(\beta),$$

where

$$k(\beta) = M_3 \frac{\alpha^{1-\beta}}{\alpha^{1-\beta} - 2(c N(0)/D)\beta}.$$  \hfill (14)

Proof. Follows directly from (9)–(11). \hfill □

As $T$ is covering we can choose $N(0) > 1$ such that $T^{N(0)}(I_a) = [0, 1], \forall I_a \in \varphi$ and prove the existence of lower bounds for $L^{N(0)}_{\beta} f, f \in B'_k$.

Lemma 8. For all $f \in B'_{\lambda}$ there exist $M(k) > 0$ such that $L^{N(0)}_{\beta} f > M(k)$.

Proof. See proofs section. \hfill □

This allows us to demonstrate positivity of the eigenfunction of $L_{\beta}$ in the main result of this section, which we state below:

Theorem 9. For $k > k(\beta)$, with $k(\beta)$ defined as in (12)–(14), $L_{\beta}$ has a positive eigenfunction $h$ in $B_k$ with a positive eigenvalue $\lambda_{\beta}$.

Proof. For $k > k(\beta)$, we apply the Schauder theorem to the continuous operator $L'_\beta$ and the compact, convex set $B'_k$ to conclude that there exists $h \in B'_k$ with $L'_\beta h = h$. This fixed point equation yields: there exists $h \in B'_k$ such that $L_{\beta} h = ||L_{\beta} h||_1 h$. By lemma 5(i) we know that $\lambda_{\beta} = ||L_{\beta} h||_1 > 0$. The fact that $h$ is positive follows immediately from lemma 8 and positivity of $\lambda_{\beta}$. \hfill □

5. Topological pressure, equilibrium measure for $(T, \phi_{\beta})$

So far we have demonstrated that for our class of interval maps, under the conditions of theorem 9, the operator $L_{\beta}$ has a positive eigenvalue and a corresponding positive $L^1$ eigenfunction. In this section, we verify that the logarithm of this eigenvalue is equal to the topological pressure $P(T, \phi_{\beta})$, and obtain the equilibrium measure for $(T, \phi_{\beta})$. Moreover, we show that the eigenfunction $h$ corresponding to $\lambda_{\beta}$ is the only eigenfunction of $L_{\beta}$ in $B'_k$ and is a multiple of the density of the unique equilibrium state.

(i) The map $T$ is covering. This has been dealt with in section 2.

(ii) The potential $\phi_{\beta} = \log(1/|T'|^\beta)$ is contracting (see Def. 3.4 in [14]).

Lemma 10. Under the conditions of theorem 9, the eigenvalue $\lambda_{\beta}$ of $L_{\beta}$ can be identified with the exponential of the pressure, i.e. $\lambda_{\beta} = e^{P(T, \phi_{\beta})}$. Moreover, $h$ is the only eigenfunction of $L_{\beta}$ in $B_k$ and is a multiple of the density of the unique equilibrium state.
Proof. Let the functional $\nu$ be defined as in [14] and let $h_\ast$ denote the density of the (unique) equilibrium state $\mu = h_\ast \nu$, the existence of $h_\ast$ is guaranteed by lemma 4.8 in [14]. Then, a direct application of theorem 3.2 in [14] (in particular, of footnote 5) implies that $|| \exp(n(\log \lambda_\beta - P(T, \phi_\beta))) h - \nu(h) h_\ast ||_\infty \to 0$ as $n \to \infty$. Thus $\log \lambda_\beta - P(T, \phi_\beta) = 0$ and $h = \nu(h) h_\ast$. Therefore, $h$ is the unique (up to scalar multiples) eigenfunction for $L_\beta$ in $B_k$ and $h \nu$ is the unique equilibrium state for suitably scaled $h$. □

6. Approximating $L_\beta$ by Ulam’s method

We begin by briefly recalling Ulam’s method in its original setting, the approximation of the Perron–Frobenius operator $L := L_1$, obtained by setting $\beta = 1$. A problem in ergodic theory that is still relevant today is the numerical approximation of absolutely continuous invariant measures (acims). If $f$ is a fixed point of $L$, then $f$ is the density of an acim. The approach suggested by Ulam [24] was to build a finite-dimensional approximation of $L$ and solve a linear system to obtain an approximation for $f$. Convergence of the approximate acim to the true acim, including error bounds in some cases, has been proved in a variety of settings [2, 5, 7, 12, 16].

We extend the Ulam construction to RPF operators $L_\beta$ and prove convergence of (i) the leading numerical Ulam eigenvalues to $e^{P(T, \phi_\beta)}$ and (ii) the corresponding numerical Ulam eigenfunctions to the density of the equilibrium state. In contrast to the standard Ulam approach, the leading eigenvalue of $L_\beta$ is unknown; moreover, the nature of the action of $L_\beta$ varies with $\beta$. Our method of proof proceeds as follows: we implicitly approximate the normalized operator $L_\beta'$ introduced in section 4 and demonstrate the existence of approximate fixed points of $L_\beta'$. We then extract a limit of these approximate fixed points and using the results of section 5 show that this limit is unique. Finally, this limit is identified with an eigenfunction of $L_\beta$ and the eigenvalue convergence is demonstrated. In practical terms, all that is required is the relatively straightforward construction of a matrix approximation of $L_\beta$.

Let $\xi^n = \{A_1, A_2, \ldots, A_n\}$ be a finite partition of $I = [0, 1]$ into intervals and define $\Delta_n = \left\{ f \in L^1 : f = \sum_{i=1}^n a_i \chi_{A_i}, a_i \in \mathbb{R} \right\}$. We will shortly consider a sequence of partitions $\{\xi^n\}_{n=0}^\infty$, and will assume that as $n \to \infty$, the maximal length of any interval in $\xi^n$ approaches zero.

Define $\Pi_n f = \sum_{i=1}^n \frac{1}{m(A_i)} \int_{A_i} f dm \chi_{A_i}$ as the canonical projection of $L^1$ onto $\Delta_n$, and consider the projected operator $L_{\beta,n} := \Pi_n \circ L_\beta : \Delta_n \to \Delta_n$. The following lemma states that the action of $L_{\beta,n}$ on $\Delta_n$ is described by a matrix $L_{\beta,n,ij}$.

Lemma 11. $L_{\beta,n} \left( \sum_{i=1}^n a_i \chi_{A_i} \right) = \sum_{j=1}^n \left( \sum_{i=1}^n a_i L_{\beta,n,ij} \right) \chi_{A_j}$

where $L_{\beta,n,ij} = \frac{1}{m(A_j)} \int_{A_j \cap T^{-1}A_i} \frac{1}{\left| T^{-1}(y) \right|} dy$.

Proof. Straightforward modification of lemma 2.3 in [12]. □

Let $v_{L_{\beta,n}} = \lambda_{\beta,n} v$, where $\lambda_{\beta,n}$ is the largest eigenvalue of $L_{\beta,n}$. Our idea is that $\lambda_{\beta,n}$ approximates $e^{P(T, \phi_\beta)}$ and the corresponding eigenfunction $h_n = \sum_{i=1}^n v_i \chi_{A_i}$ approximates a
suitably normalized version of the density of the equilibrium state for \((T, \phi_\beta)\). We now state our main result, formalizing these ideas.

**Theorem 12.** Assume that the hypotheses of theorem 9 hold. Let \(\lambda_{\beta,n}\) be the largest magnitude positive eigenvalue of \(L_{\beta,n}\) and \(h_n\) the corresponding eigenfunction. Then

(i) as \(n \to \infty\) the sequence \([h_n]\) converges to \(h\), a multiple of the density of the unique equilibrium state for the pair \((T, \phi_\beta)\)

(ii) \(\lim_{n \to \infty} \lambda_{\beta,n} = \lambda_\beta = e^{P(T, \phi_\beta)}\).

**Proof.** See proofs section. \(\square\)

The remainder of this section outlines the main steps required in the proof of the above theorem. In order to employ a fixed point theorem, we need to consider an approximate version of the normalized operator from section 4. Define \(L'_{\beta,n}: (\Delta_n \cap \{f : f \geq 0, ||f||_1 = 1\}) \circlearrowleft \) by

\[
L'_{\beta,n}f = \frac{L_{\beta,n}f}{||L_{\beta,n}f||_1}.
\]

Analogous to lemma 5 we have the following lemma.

**Lemma 13.** For all \(\beta \in \mathbb{R}\), \(L'_{\beta,n}: (\Delta_n \cap \{f : f \geq 0, ||f||_1 = 1\}) \circlearrowleft \) is

(i) well defined and

(ii) continuous.

**Proof.** Follows immediately from lemma 5, the definition of \(L'_{\beta,n}\) and the fact that for all \(f \in L^1, f \geq 0, \beta \in \mathbb{R}, ||L_{\beta,n}f||_1 = ||L_\beta f||_1\). This latter result is a consequence of the fact that for all \(f \in L^1, f \geq 0, ||\Pi nf||_1 = ||f||_1\) (see [12]). \(\square\)

The variation of functions under the action of our approximate normalized operator is no greater than that of the original normalized operator.

**Lemma 14.** For all \(f \in L^1, f \neq 0, f \geq 0, \beta \in \mathbb{R}, V_I(L'_{\beta,n}f) \leq V_I(L'_{\beta}f)\).

**Proof.** We begin by noting that for all \(f \in L^1, \beta \in \mathbb{R}, V_I(L_{\beta,n}f) \leq V_I(L_\beta f),\) which is a consequence of the fact that for all \(f \in L^1, V_I(\Pi nf) \leq V_I(f)\) (see [12]). This, together with the property that for all \(f \in L^1, f \geq 0, \beta \in \mathbb{R}, ||L_{\beta,n}f||_1 = ||L_\beta f||_1\) yields

\[
V_I(L'_{\beta,n}f) = V_I\left(\frac{L_{\beta,n}f}{||L_{\beta,n}f||_1}\right) = \frac{V_I(L_{\beta,n}f)}{||L_{\beta,n}f||_1} = \frac{V_I(L_\beta f)}{||L_\beta f||_1} = V_I(L'_{\beta}f). \quad \square
\]

We can now establish the existence of a fixed point for our approximate normalized operator in analogy to theorem 9.

**Lemma 15.** For \(k > k(\beta)\), with \(k(\beta)\) defined in (12)–(14), each \(L'_{\beta,n}\) has a fixed point \(h_n \in B'_{k}\).

**Proof.** Lemma 14 and \(||L'_{\beta,n}||_1 = 1\) imply that if \(L'_{\beta}\) preserves \(B'_{k}\) then \(L'_{\beta,n}\) also preserves \(B'_{k}\). Thus, by lemma 7, \(L'_{\beta,n}\) preserves \(B'_{k}\) for all \(k \geq k(\beta)\). From lemma 3 we know that \(B'_{k}\) is convex and compact. From lemma 13 we know that \(L'_{\beta,n}: (\Delta_n \cap \{f : f \geq 0, ||f||_1 = 1\}) \circlearrowleft \) is continuous. The result follows by Schauder’s theorem. \(\square\)
Strong convergence of $L'_{\beta,n}$ to $L'_\beta$, as an action on positive $f \in L^1$, is straightforward to establish.

**Lemma 16.** For all $f \in L^1$, $f \neq 0$, $f \geq 0$, and $\beta \in \mathbb{R}$, $||L'_{\beta,n}f - L'_\beta f||_1 \to 0$ as $n \to \infty$.

**Proof.** We first note that because $||f - \Pi_n f||_1 \to 0$ as $n \to \infty$ (see [12]) we have that for all $f \in L^1$, $\beta \in \mathbb{R}$, $||L_\beta f - L_{\beta,n} f||_1 \to 0$ as $n \to \infty$. As $||L_{\beta,n} f||_1 = ||L_\beta f||_1$ for all $f \geq 0$, $\beta \in \mathbb{R}$, one has that for all $f \in L^1$, $f \neq 0$, $f \geq 0$ and $\beta \in \mathbb{R}$

$$||L'_{\beta,n}f - L'_\beta f||_1 = \frac{||L_\beta f - L_{\beta,n} f||_1}{||L_{\beta,n} f||_1} \to 0$$

which goes to zero as $n \to \infty$. □

Lemma 16 together with relative compactness of the sequence of fixed points of $L'_{\beta,n}$ leads to the following lemma.

**Lemma 17.** Let $h_n$ be a fixed point of $L'_{\beta,n}$. Then $h_n \to h$ in $L^1$, as $n \to \infty$, where $h$ is the unique fixed point of $L'_\beta$.

**Proof.** Since $h_n \in B'_{k}$ and $B'_{k}$ is compact in $L^1$, the sequence $\{h_n\}$ is relatively compact in $L^1$. Let $\bar{h}$ be a limit point of this sequence and $\{h_{n_j}\}$ be the corresponding convergent subsequence: $||\bar{h} - h_{n_j}||_1 \to 0$ as $n_j \to \infty$. But

$$||\bar{h} - L'_{\beta,n}h_{n_j}||_1 \leq ||\bar{h} - h_{n_j}||_1 + ||L'_{\beta,n}h_{n_j} - L'_{\beta,n} \bar{h}||_1 + ||L'_{\beta,n} \bar{h} - L'_{\beta} \bar{h}||_1.$$

Because $||L'_{\beta,n}h_{n_j} - L'_{\beta,n} \bar{h}||_1 \leq ||L'_{\beta,n}|| \cdot ||\bar{h} - h_{n_j}||_1 = ||\bar{h} - h_{n_j}||_1$, the second term of equation (16) goes to zero as $n_j$ goes to infinity. Moreover, by lemma 16, $||L'_{\beta,n} \bar{h} - L'_{\beta} \bar{h}||_1 \to 0$ as $n_j \to \infty$. Thus, $L'_{\beta} \bar{h} = \bar{h}$.

Since by lemma 10 we know that $L_\beta$ has a unique eigenfunction $h \in B'_{k}$, $L'_{\beta}$ has a unique fixed point $\bar{h} \in B'_{k}$ and thus $\bar{h}$ must be a multiple of $h$. Thus, the sequence $\{h_n\}$ has only one limit point, which is a multiple of $h$. We therefore must have that

$$\lim_{n \to \infty} h_{\beta} = \bar{h}.$$ □

**7. Discussion**

The rigorous estimation of topological pressure for interval maps is a difficult problem in ergodic theory and thermodynamics. For specific maps, specialized techniques have been developed (e.g. [6, 11, 17, 18, 25]). However, to our knowledge, the results presented here represent the first rigorous numerical approach to estimating pressure for a reasonably broad class of interval maps. We close by remarking that numerical experiments reported in [8] demonstrate that our method is simple to implement, extremely efficient in terms of computing time and is a very practical way to detect phase transitions with respect to the weight functions $\phi_\beta = -\beta \log |T'|$ when they exist. Future work will include the extension of the rigorous results presented here to transformations that exhibit phase transitions.
8. Proofs section

8.1. Proof of lemma 1

(i) is obvious from the $L_\beta$ definition—see equation (6). To prove (ii) we consider the following cases:

When $\beta \geq 1$, for all $f \in L^1$,

$$\|L_\beta f\|_1 \leq \sum_a \int_{T(L_a)} \left| \frac{f \circ T_a^{-1}}{|T_a \circ T_a^{-1}|} \right| \mathcal{X}(T(L_a)) \leq \sum_a \int_{T(L_a)} \left| \frac{1}{|T_a \circ T_a^{-1}|} \right| \beta f \circ T_a^{-1} \leq \left( \frac{1}{\alpha} \right)^{1-\beta} \sum_a \int_{L_a} |f| = \left( \frac{1}{\alpha} \right)^{1-\beta} \|f\|_1.$$ \hspace{1cm}

When $\beta < 1$ we recall that $T$ is covering. Let $\mathcal{C}_N(0)$ be given as in (5). The mean value theorem together with equation (3) gives

$$\frac{1}{|T^{N(0)} \circ T_a^{N(0)}(y)|} \geq \frac{m(I_a^{N(0)}))}{D} \geq \frac{\mathcal{C}_N(0)}{D}, \forall y \in I_a^{N(0)}. \hspace{1cm} (17)$$

Now for the class of transformation considered here

$$\frac{1}{|T(x)|} \geq \frac{1}{|T^{N(0)}(y)|}, \forall x \in I_a, \forall y \in I_a^{N(0)},$$

which together with (17) implies

$$\frac{1}{|T(x)|} \geq \frac{m(I_a^{(0)}(y))}{D} \geq \frac{\mathcal{C}_N(0)}{D}, \forall x \in I_a, \forall y \in I_a^{(0)}. \hspace{1cm} (18)$$

Raising (18) to $\beta - 1$ (which is negative, since $\beta < 1$) implies

$$\frac{1}{|T_a \circ T_a^{-1}(x)|} \leq \frac{1}{|T^{N(0)} \circ T_a^{N(0)}(x)|} \beta^{-1} \leq \left( \frac{D}{\mathcal{C}_N(0)} \right)^{1-\beta}, \forall x \in I_a, \forall y \in I_a^{N(0)}. \hspace{1cm} (19)$$

Therefore, when $\beta < 1$, for all $f \in L^1$ we have (similarly to the $\beta \geq 1$ case)

$$\|L_\beta f\|_1 \leq \sum_a \int_{T(L_a)} \left| \frac{1}{|T_a \circ T_a^{-1}|} \right| \beta f \circ T_a^{-1} \leq \left( \frac{D}{\mathcal{C}_N(0)} \right)^{1-\beta} \sum_a \int_{L_a} |f| = \left( \frac{D}{\mathcal{C}_N(0)} \right)^{1-\beta} \|f\|_1.$$ \hspace{1cm}

8.2. Proof of lemma 2

Because $L_\beta f \in BV(I), \forall f \in B_k \subset BV(I)$, we may write

$$V_\beta(L_\beta f) = \int_I d(L_\beta f) := \sup \left\{ \int_I L_\beta f \cdot g : g \in C^1(I), \|g\|_{\infty} \leq 1 \right\},$$

where $d(L_\beta f)$ is the generalized derivative (see e.g. [9]). Thus

$$V_\beta(L_\beta f) = \int_I d \left( \sum_a \frac{f \circ T_a^{-1}}{|T_a \circ T_a^{-1}|} \right) \leq \int_I \sum_a \left| \frac{f \circ T_a^{-1}}{|T_a \circ T_a^{-1}|} \right| = \sum_a \int_I \left| \frac{f \circ T_a^{-1}}{|T_a \circ T_a^{-1}|} \right|. \hspace{1cm} (20)$$
Let $T(I_a) = [b_a, b_a']$ and recall from equation (4) that $|b_a - b_a'| = m(T(I_a)) \geq c'$. A straightforward modification of the proof of lemma 3.1.2 in [15] implies that for all $\beta$:

$$
\int_{I_a} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| \leq 2 \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| + 2 \beta \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|.
$$

The above inequality together with (20) leads to

$$
V_I(L_\beta f) \leq 2 \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| + 2 \beta \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|. \tag{21}
$$

(i) $\beta \geq 1$. Since (21) holds for every $\beta$, to prove case (i) of lemma 2, we only need to analyse each term on the right-hand side of the inequality (21) for $\beta \geq 1$. With respect to the first term we have

$$
2 \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| \leq 2 \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| + 2 \beta \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|
\leq \frac{2 \alpha^\beta}{\beta} \sum_a \int_{I_a} |f| + 2 \beta \sum_a \int_{I_a} |f|
= \frac{2 \alpha^\beta}{\beta} \int_I |f| + 2 \beta \sum_a \int_{I_a} |f|.
$$

With respect to the second term of the right-hand side of the inequality (21) we have

$$
2 \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| \leq \frac{2}{\alpha^\beta} \beta \int_I |f|.
$$

Then the result follows from (21) and the last two inequalities.

(ii) $0 \leq \beta < 1$. Proceeding as in the proof of (i), since (21) holds for every $\beta$, we analyse each term on the right-hand side of the inequality (21), this time for $0 < \beta < 1$. With respect to the first term we write

$$
2 \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| \leq \frac{2 \alpha^\beta}{\beta} \sum_a \int_{I_a} |f| + 2 \beta \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|
= \frac{2 \alpha^\beta}{\beta} \int_I |f| + 2 \beta \sum_a \int_{I_a} |f| \tag{22}
$$

Now we need to look at $\sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|$. Using equation (19) we write

$$
\sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| \leq \left( \frac{D}{\epsilon_{N(0)}} \right)^{1-\beta} \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|
= \left( \frac{D}{\epsilon_{N(0)}} \right)^{1-\beta} \sum_a \int_{I_a} |f|. \tag{23}
$$

From (21) to (23) we have

$$
V_I(L_\beta f) \leq \frac{2 \alpha^\beta}{\beta} \int_I |f| + 2 \left( \frac{D}{\epsilon_{N(0)}} \right)^{1-\beta} \left( \beta + \frac{1}{c'} \right) \int_I |f|
$$

and we are done with the proof of (ii).
(iii) $\beta < 0$. We first observe that by raising (18) to $\beta$ (which is negative in this case) we obtain an upper bound for $1/|T_a \circ T_a^{-1}|^{\beta}$ as
\[
\frac{1}{|T_a \circ T_a^{-1}(x)|^{\beta}} \leq \frac{1}{|T^{N(0)} \circ T_a^{-N(0)}(x)|^{\beta}} \leq \left( \frac{CN(0)}{D} \right)^{\beta}, \quad \forall x \in I_a, \forall y \in I_{a^{(0)}}.
\]

Thus
\[
2 \sum_a \int_{T(I_a)} \left| \frac{d}{|T_a \circ T_a^{-1}|^{\beta}} \right| \leq 2 \left( \frac{CN(0)}{D} \right)^{\beta} \sum_a \int_{T(I_a)} |d|d + 2\varepsilon | \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a \circ T_a^{-1}|^{\beta}} \right|.
\]

Then the proof of (iii) goes exactly the same as the proof of (ii).

8.3. Proofs of lemma 4 and lemma 8

Proof of lemma 4. When $\beta \geq 1$, by raising equation (18) to $\beta - 1$ we have that $\forall x \in I_a, \forall y \in I_{a^{(0)}}$
\[
\left( \frac{1}{|T(x)|} \right)^{\beta-1} \geq \left( \frac{1}{|T^{N(0)}(y)|} \right)^{\beta-1} \geq \left( \frac{m(I_{a^{(0)}})}{D} \right)^{\beta-1} \geq \left( \frac{CN(0)}{D} \right)^{\beta-1}.
\]

Thus, since $f \geq 0$,
\[
||L_\beta f||_1 = \sum_a \int_{T(I_a)} \left| \frac{\frac{1}{T_a \circ T_a^{-1}}}{|T_a \circ T_a^{-1}|^{\beta}} \right| f \chi_{T(I_a)}
\]
\[
= \sum_a \int_{T(I_a)} \left| \frac{1}{|T_a \circ T_a^{-1}|^{\beta-1}} \right| f \chi_{T(I_a)}
\]
\[
\geq \left( \frac{CN(0)}{D} \right)^{\beta-1} \sum_a \int_{T(I_a)} |f| = \left( \frac{CN(0)}{D} \right)^{\beta-1} ||f||_1.
\]

and (i) follows under the assumption that $f \neq 0$.

When $\beta < 1$, we only need to observe $|\frac{1}{T(x)}|^{\beta} \geq \frac{1}{T^{N(0)}}$, which implies that
\[
||L_\beta f||_1 = \sum_a \int_{T(I_a)} \left| \frac{1}{|T_a \circ T_a^{-1}|^{\beta-1}} \right| f \chi_{T(I_a)}
\]
\[
\geq \left( \frac{1}{\alpha} \right)^{\beta-1} \sum_a \int_{T(I_a)} |f| = \left( \frac{1}{\alpha} \right)^{\beta-1} ||f||_1.
\]

Thus, (ii) follows under the same assumption $f \neq 0$.

Proof of lemma 8. For $f \in B_k$ let
\[
\tilde{f} = \sum_{\alpha^{(0)}} \left( \text{ess inf}_{I_{\alpha^{(0)}}} f \right) \chi_{I_{\alpha^{(0)}}}.
\]

By lemma 3.2.1 in [15], $||\tilde{f}||_1 \geq ||f||_1(1 - \alpha^{-N(0)}k)$ and thus for all $f \in B_k'$,
\[
||\tilde{f}||_1 \geq (1 - \alpha^{-N(0)}k).
\]
From equation (17) we know that 
\[ \frac{1}{T^{N(0)} \circ T^{-1}^{N(0)}(x)} \geq \frac{m(I_{\alpha}(N(0)))}{D} \geq c_{N(0)} D, \forall x \in I_{\alpha}(N(0)). \]
This together with (25) gives
\[ L_{\beta}^{N(0)} f = \sum_{a(N(0))} f \circ T^{-1}_{a(N(0))} \frac{1}{(T^{N(0)} \circ T^{-1}^{N(0)})^\beta} \]
\[ \geq \sum_{a(N(0))} \left( \inf_{I_{\alpha}(N(0))} f \right) \frac{1}{(T^{N(0)} \circ T^{-1}^{N(0)})^\beta} \frac{1}{(T^{N(0)} \circ T^{-1}^{N(0)})} \]
\[ \geq M \sum_{a(N(0))} \tilde{T}_{a(N(0))} m(I_{\alpha}(N(0)))/D = M \| \tilde{f} \|_1/D, \]
where \( M = (c_{N(0)}/D)^{\beta-1} \cdot (c_{N(0)}/D) \) if \( \beta \geq 1 \) and \( M = (1/\alpha)^{\beta-1} \cdot (c_{N(0)}/D) \) if \( \beta < 1 \). This choice of \( M \) is motivated by equation (24) when \( \beta \geq 1 \) and by the fact that \( \| T \| \geq 1/\alpha \) when \( \beta < 1 \).

To complete choose \( M(k) = M(1 - \alpha^{-N(0)k}) \).

8.4. Proof of theorem 12

**Proof.** Let \( \lambda_{\beta,n} \) be an eigenvalue of \( L_{\beta,n} \) (as defined in lemma 11) and \( h_n \) the corresponding eigenfunction normalized so that \( \| h_n \|_1 = 1 \). By lemma 11 we know that any eigenvalue, eigenfunction pair of \( L_{\beta,n} \) is an eigenvalue, eigenfunction pair of \( L_{\beta,n} \). Since we also know that any normalized eigenfunction of \( L_{\beta,n} \) is a fixed point of \( L_{\beta,n} \), lemma 17 implies that \( \{ h_n \} \) converges to the unique fixed point of \( L_{\beta} \) as \( n \to \infty \). Furthermore, lemma 10 implies that this unique fixed point is a multiple of the density of the unique equilibrium state for the pair \((T, \phi_{\beta})\).

We now prove \((ii)\). Recall that \( \lambda_{\beta,n} = || L_{\beta,n} h_n ||_1 \) and \( \lambda_{\beta} = || L_{\beta} h ||_1 = || L_{\beta} h_n ||_1 \). Thus using the reverse triangle inequality \( |\lambda_{\beta,n} - \lambda_{\beta}| = || L_{\beta,n} h_n ||_1 - || L_{\beta} h ||_1 = || L_{\beta} h_n ||_1 - || L_{\beta} h ||_1 | \leq || L_{\beta} h ||_1 \| h_n - h \|_1 \). From lemma 1 we know that \( || L_{\beta} ||_1 \) is bounded. By lemma 17 we know that \( || h_n - h ||_1 \to 0 \) as \( n \to \infty \). Thus, \( |\lambda_{\beta,n} - \lambda| \to 0 \) as \( n \to \infty \). The desired result now follows by lemma 10. \( \square \)

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References