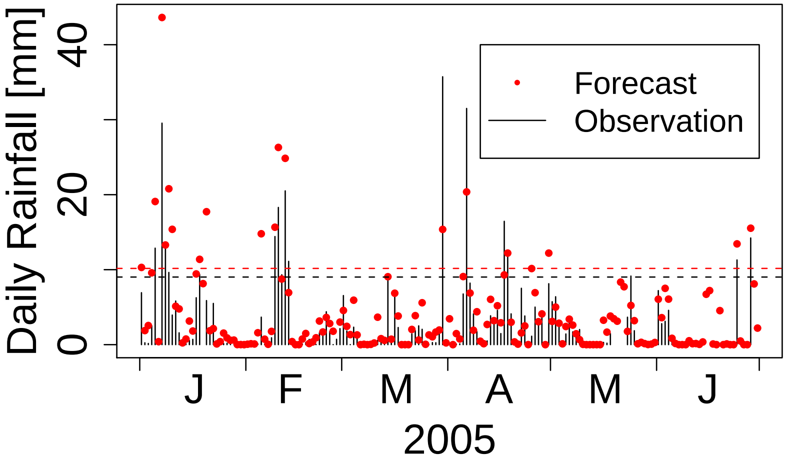
Verifying forecasts of extreme events

Chris Ferro

Walker Institute
Department of Meteorology
University of Reading

UK Extremes Meeting, Sheffield, 17 April 2007

Daily rainfall: mid-Wales, 1 Jan 05 – 11 Nov 06



Thanks to Marion Mittermaier (UKMO)

Verifying binary forecasts

	Observed	Not Observed	
Forecasted	a	b	$a+b$
Not Forecasted	c	d	$c+d$
	$a+c$	$b+d$	n

Summarise with verification measures, e.g.

$$\text{Hit rate} = \frac{a}{a+c}, \qquad \text{Critical success index} = \frac{a}{a+b+c}.$$

Notation and rationale

Event is forecasted if $X > u$ and observed if $Y > v$.

	Observed	Not Obs.	
Forecasted	$\Pr(X > u, Y > v)$	$*$	$\Pr(X > u)$
Not Forecasted	$*$	$*$	$*$
	$\Pr(Y > v)$	$*$	1

- Consider u such that $\Pr(X > u) = \Pr(Y > v) =: p$.
- Extreme-value model for joint probability as function of p.
- Evaluate the table and measures for any small p.

Probability model

Imagine choosing u so that

$$\Pr(X > u) = \Pr(Y > v) =: p \quad \text{(base rate)}.$$

Define
$$\begin{aligned} \tilde{X} &= -\log[1 - F(X)] \quad \text{where} \quad & F(x) &= \Pr(X \le x) \\ \tilde{Y} &= -\log[1 - G(Y)] & G(y) &= \Pr(Y \le y) \end{aligned}$$

and $Z = \min\{\tilde{X}, \tilde{Y}\}$. Then

$$\Pr(X > u, Y > v) = \Pr(Z > -\log p).$$

Model

$$\Pr(Z > z) = \kappa \exp(-z/\eta) \qquad \text{for all } z > w_0$$

i.e.

$$\Pr(X > u, Y > v) = \kappa p^{1/\eta} \qquad \text{for all } p < \exp(-w_0).$$

Probability model

Imagine choosing u so that

$$\Pr(X > u) = \Pr(Y > v) =: p \quad \text{(base rate)}.$$

Define $\tilde{X} = -\log[1 - F(X)]$ where $F(x) = \Pr(X \leq x)$

$\tilde{Y} = -\log[1 - G(Y)]$ $\quad G(y) = \Pr(Y \leq y)$

and $Z = \min\{\tilde{X}, \tilde{Y}\}$. Then

$$\Pr(X > u, Y > v) = \Pr(Z > -\log p).$$

Model

$$\Pr(Z > z) = \kappa \exp(-z/\eta) \qquad \text{for all } z > w_0$$

i.e.

$$\Pr(X > u, Y > v) = \kappa\, p^{1/\eta} \qquad \text{for all } p < \exp(-w_0).$$

Probability model

Imagine choosing u so that

$$\Pr(X > u) = \Pr(Y > v) =: p \quad \text{(base rate)}.$$

Define $\begin{aligned} \tilde{X} &= -\log[1 - F(X)] \\ \tilde{Y} &= -\log[1 - G(Y)] \end{aligned}$ where $\begin{aligned} F(x) &= \Pr(X \leq x) \\ G(y) &= \Pr(Y \leq y) \end{aligned}$

and $Z = \min\{\tilde{X}, \tilde{Y}\}$. Then

$$\Pr(X > u, Y > v) = \Pr(Z > -\log p).$$

Model

$$\Pr(Z > z) \quad = \quad \kappa \exp(-z/\eta) \qquad \text{for all } z > w_0$$

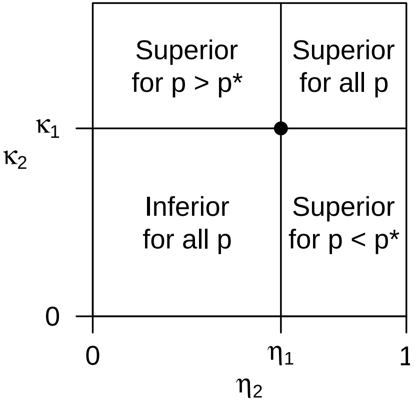
i.e.

$$\Pr(X > u, Y > v) \quad = \quad \kappa\, p^{1/\eta} \qquad \text{for all } p < \exp(-w_0).$$

Interpretation

	Observed	Not Observed	
Forecasted	$\kappa p^{1/\eta}$	$*$	p
Not Forecasted	$*$	$1 - 2p + \kappa p^{1/\eta}$	$*$
	p	$*$	1

$$\text{Hit rate} = \kappa\, p^{1/\eta - 1}$$



Estimate parameters κ and η

Suppose we have data $Z_t = \min\{\tilde{X}_t, \tilde{Y}_t\}$ for $t = 1, \ldots, n$.

Under *mild conditions*, points (t, Z_t) above a high threshold w_0 are well approximated by a Poisson process and the expected number of points above $z > w_0$ is

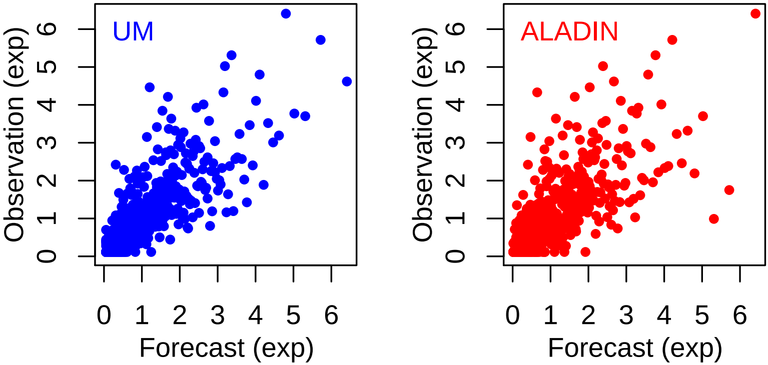
$$\Lambda(z) = n \Pr(Z > z) = n\,\kappa \exp(-z/\eta).$$

Choose w_0 then maximise the likelihood

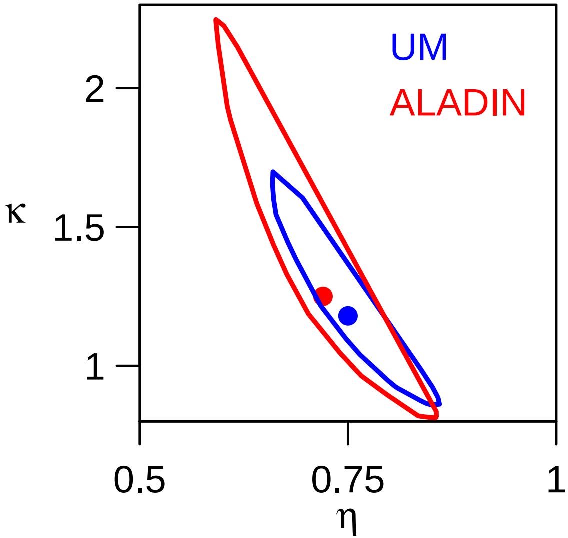
$$\exp[-\Lambda(w_0)] \prod_{Z_t > w_0} \kappa\, \eta^{-1} \exp(-Z_t/\eta).$$

$$\tilde{X}_t = -\log[1 - \hat{F}(X_t)] \quad \text{where} \quad \hat{F}(x) = \frac{1}{n+1} \sum_{t=1}^{n} I(X_t \le x).$$

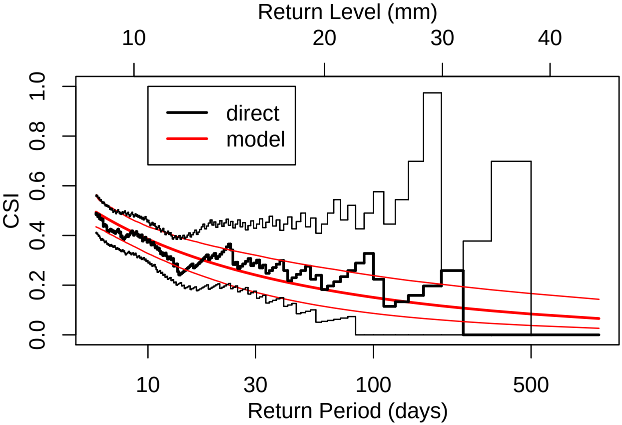
Daily rainfall: mid-Wales, 1 Jan 05 – 11 Nov 06



Parameter estimates



Verification measures



- Direct estimates degenerate for rare events
- Model estimates change smoothly and are more precise

Conclusion

- Deterministic forecasts of rare, extreme events
- Only two parameters are needed to describe how the quality of calibrated forecasts changes with base rate
- The model gives more precise estimates of forecast quality

Paper and R code available at

www.met.rdg.ac.uk/∼sws02caf

c.a.t.ferro@reading.ac.uk

Univariate extreme-value theory

Let $M_n = \max\{X_1, \ldots, X_n\}$ for i.i.d. X_t.

If there exist $a_n > 0$, b_n and non-degenerate H such that

$$\Pr(M_n \leq a_n x + b_n) \xrightarrow{w} H(x) \qquad \text{as } n \to \infty$$

then H is the generalised extreme-value distribution function

$$H(x) = \exp\left[-\{1 + \gamma(x - \alpha)/\beta\}_+^{-1/\gamma}\right].$$

Point process $(t/n, (X_t - b_n)/a_n)$ converges to Poisson process with intensity measure $-(b - a)\log H(x)$ on $(a, b) \times (x, \infty)$.

Stationary X_t gives $H(x)^\theta$ for $\theta \in (0, 1]$ and compound Poisson.

Bivariate extreme-value theory

$\tilde{X}$ and $\tilde{Y}$ are Exponential with unit means, so

$$\Pr(Z > z) = \begin{cases} \exp(-z) & \text{if } \tilde{X} \equiv \tilde{Y} \\ \exp(-2z) & \text{if } \tilde{X} \perp\!\!\!\perp \tilde{Y} \end{cases}$$

Assume

$$\Pr(Z > z) \sim \mathcal{L}(e^z) \exp(-z/\eta) \qquad \text{as } z \to \infty,$$

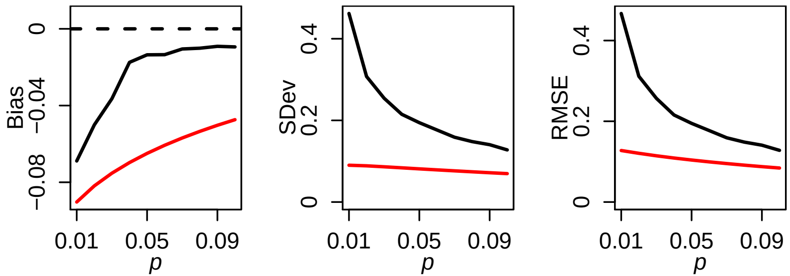
where $0 < \eta \leq 1$ and $\mathcal{L}(rt)/\mathcal{L}(r) \to 1$ as $r \to \infty$ for all $t > 0$.

For a high threshold w_0,

$$\begin{aligned} \Pr(Z > w_0 + z) &\approx \mathcal{L}(e^{w_0+z}) \exp[-(w_0 + z)/\eta] \\ &\approx \mathcal{L}(e^{w_0}) \exp[-(w_0 + z)/\eta]. \end{aligned}$$

Simulation study

- Bivariate Normal data: correlation 0.8
- Direct and model estimates of hit rate



Limiting behaviour of measures

$$\text{Hit rate} = \frac{a}{a+c} \sim \kappa p^{1/\eta - 1} \to \begin{cases} 0 & \text{if } \eta < 1 \\ \kappa & \text{if } \eta = 1 \end{cases}$$

$$\text{PC} = \frac{a+d}{n}, \qquad \text{PSS} = \frac{ad - bc}{(a+c)(b+d)}, \qquad \text{OR} = \frac{ad}{bc}$$

	$\eta < \frac{1}{2}$	$\eta = \frac{1}{2}$	$\eta > \frac{1}{2}$	$\eta = 1$
PC	$1 - 2p \uparrow 1$	$1 - 2p \uparrow 1$	$1 - 2p \uparrow 1$	$1 - 2\bar{\kappa}p \uparrow 1$
PSS	$-p \uparrow 0$	$-\bar{\kappa}p \updownarrow 0$	$\kappa p^{\delta - 1} \downarrow 0$	$\kappa - \bar{\kappa}p \uparrow \kappa$
OR	$\kappa p^{\delta - 2} \downarrow 0$	$\kappa - 2\kappa\bar{\kappa}p \updownarrow \kappa$	$\kappa p^{\delta - 2} \uparrow \infty$	$\kappa / (\bar{\kappa}^2 p) \uparrow \infty$

where $\delta = 1/\eta$ and $\bar{\kappa} = 1 - \kappa$

Limiting behaviour of measures

$$\text{Hit rate} = \frac{a}{a+c} \sim \kappa p^{1/\eta - 1} \to \begin{cases} 0 & \text{if } \eta < 1 \\ \kappa & \text{if } \eta = 1 \end{cases}$$

$$\text{PC} = \frac{a+d}{n}, \qquad \text{PSS} = \frac{ad - bc}{(a+c)(b+d)}, \qquad \text{OR} = \frac{ad}{bc}$$

	$\eta < \frac{1}{2}$	$\eta = \frac{1}{2}$	$\eta > \frac{1}{2}$	$\eta = 1$
PC	$1 - 2p \uparrow 1$	$1 - 2p \uparrow 1$	$1 - 2p \uparrow 1$	$1 - 2\bar{\kappa}p \uparrow 1$
PSS	$-p \uparrow 0$	$-\bar{\kappa}p \updownarrow 0$	$\kappa p^{\delta-1} \downarrow 0$	$\kappa - \bar{\kappa}p \uparrow \kappa$
OR	$\kappa p^{\delta-2} \downarrow 0$	$\kappa - 2\kappa\bar{\kappa}p \updownarrow \kappa$	$\kappa p^{\delta-2} \uparrow \infty$	$\kappa/(\bar{\kappa}^2 p) \uparrow \infty$

where $\delta = 1/\eta$ and $\bar{\kappa} = 1 - \kappa$

Limiting behaviour of measures

$$\text{Hit rate} = \frac{a}{a+c} \sim \kappa p^{1/\eta - 1} \to \begin{cases} 0 & \text{if } \eta < 1 \\ \kappa & \text{if } \eta = 1 \end{cases}$$

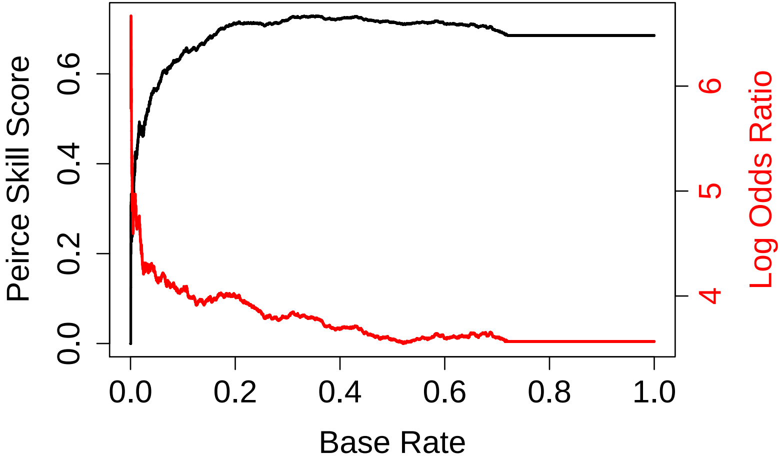
$$\text{PC} = \frac{a+d}{n}, \qquad \text{PSS} = \frac{ad - bc}{(a+c)(b+d)}, \qquad \text{OR} = \frac{ad}{bc}$$

	$\eta < \frac{1}{2}$	$\eta = \frac{1}{2}$	$\eta > \frac{1}{2}$	$\eta = 1$
PC	$1 - 2p \uparrow 1$	$1 - 2p \uparrow 1$	$1 - 2p \uparrow 1$	$1 - 2\bar{\kappa}p \uparrow 1$
PSS	$-p \uparrow 0$	$-\bar{\kappa}p \updownarrow 0$	$\kappa p^{\delta-1} \downarrow 0$	$\kappa - \bar{\kappa}p \uparrow \kappa$
OR	$\kappa p^{\delta-2} \downarrow 0$	$\kappa - 2\kappa\bar{\kappa}p \updownarrow \kappa$	$\kappa p^{\delta-2} \uparrow \infty$	$\kappa/(\bar{\kappa}^2 p) \uparrow \infty$

where $\delta = 1/\eta$ and $\bar{\kappa} = 1 - \kappa$

Contradictory skill scores?

ERA-40 daily rainfall forecasts: $\eta = 0.81$, $\kappa = 1.16$



Probabilistic forecasts

Binary observation $I = I(Y > v)$

Probabilistic forecast $P = m^{-1} \sum_{i=1}^{m} I(X_i > u)$

Brier score $B = n^{-1} \sum_{t=1}^{n} (P_t - I_t)^2$

Assuming stationarity and exchangeable ensemble members,

$$E(B) = \Pr(X_1 > u, X_2 > u) - \Pr(X_1 > u, Y > v) + \Pr(Y > v) + \frac{1}{m} [\Pr(X_1 > u) - \Pr(X_1 > u, X_2 > u)] .$$

Two (η, κ)-models for the joint probabilities, estimated from an independence likelihood.