A probability model for verifying deterministic forecasts of extreme events

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January 14, 2007

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ABSTRACT

This article proposes a method for verifying deterministic forecasts of rare, extreme events defined by exceedances above a high threshold. A probability model for the joint distribution of forecasts and observations, and based on extreme-value theory, characterises the quality of forecasting systems with two key parameters. This enables verification measures to be estimated for any event rarity and helps to reduce the uncertainty associated with direct estimation. Confidence regions are obtained and the method is used to compare daily precipitation forecasts from two operational numerical weather prediction models.
1 Introduction

Forecasting extreme weather events is a vital task and a focus for international research activities such as the WMO THORPEX programme. This article addresses a key component of this research: how should we assess the quality of forecasts of extreme events?

We focus on events that are both extreme and rare. Such events pose at least three difficulties for forecast verification. First, only a small number of events may be observed. This produces large variation in verification measures between datasets, and therefore large uncertainty about forecast quality. Second, most verification measures necessarily degenerate to trivial values as event rarity increases (Stephenson et al. 2006, manuscript in preparation). This projects misleading impressions of forecast quality and complicates the discrimination between forecasting systems. Third, events may be observed inaccurately due to short space-time scales and non-events may even pass unrecorded. The former penalises good forecasts and the latter leaves some verification measures indeterminate.

We shall address the first two problems, of large uncertainty and degeneracy, with a probability model for the joint distribution of observations and forecasts of extreme events; we disregard the possibilities of inaccurate observations and unrecorded non-events. We consider events that are observed
to occur when a continuous, scalar observation variable $Y$, such as daily precipitation, exceeds an observation threshold $v$, such as 50 mm. This precludes important examples of extreme events such as tornadoes, but includes rare, scalar combinations of multiple variables such as sea surge and wave height. We consider only deterministic forecasts for which the event is forecasted to occur when a continuous, scalar forecast variable $X$ exceeds a forecast threshold $u$. We shall not, therefore, verify probabilistic forecasts, which are desirable for rare events (Murphy 1991), but $X$ itself may be a probabilistic forecast (e.g. Toth et al. 2003), an index derived from a probabilistic forecast (Lalaurette 2003), or simply a forecast for the value of $Y$.

Suppose that we have $n$ pairs of forecast and observation variables, and have chosen thresholds $u$ and $v$. The resulting $n$ binary forecasts and observations are usually summarised in a contingency table such as Table 1. Summaries of Table 1 that describe the correspondence between the forecasts and observations are measures of forecast performance. Three such measures are the hit rate,

$$ H = \frac{a}{a + c}, \quad (1) $$

critical success index,

$$ \text{CSI} = \frac{a}{a + b + c}, \quad (2) $$

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and odds ratio,

\[ \text{OR} = \frac{ad}{bc}. \]  

These and many other measures can be found in Marzban (1998) and Mason (2003).

[Table 1]

We are particularly interested in how performance changes as the observation threshold, \( v \), increases and events become rarer. If a forecast threshold is specified for each \( v \) then we could construct Table 1 and evaluate performance measures for each \( v \), but this direct approach suffers from the two disadvantages mentioned before of uncertainty and degeneracy, as we now illustrate with some precipitation forecasts.

We compare radar observations of daily precipitation totals with two sets of operational forecasts for a single grid point (52.5°N, 3.5°W) in mid-Wales in the UK. The forecasting models are the mesoscale Unified Model (UM) of the UK Met Office and the ALADIN model of Météo-France. Forecasts are initialised at 00 UTC for the subsequent 24 hours, with the first on 1 January 2005 and the last on 11 November 2006. There are 72 days with missing data, leaving 607 pairs of forecast and observation variables for both the UM and ALADIN datasets. The data are plotted in Figs 1 and 2. Both models tend to over-estimate small precipitation totals (below 2 mm) and
slightly under-estimate large totals (above 4 mm). The dependence between forecasts and observations appears strong for both models.

[Fig. 1]

[Fig. 2]

To demonstrate the problems with verifying extreme events, we split the UM data randomly into 10 subsets of approximately 60 pairs each. For each subset, we vary the forecast and observation thresholds so that the base rate \((a + c)/n\) is always equal to \((a + b)/n\) and ranges from 0.5 to 0. Matching the two rates yields calibrated forecasts, which we shall focus on later; low base rates correspond to rare events. We plot the hit rate (1) against the base rate for each subset in Fig. 3. The hit rate becomes more variable as events become rarer, and always converges to either 0 or 1 before becoming indeterminate when the thresholds are so high that \(a = c = 0\). This behaviour is typical of most verification measures.

[Fig. 3]

We describe our probability model for reducing these problems in section 2 and discuss its interpretation in section 3. Parameter estimation and other practical issues are addressed in sections 4 and 5. We apply our method to the precipitation forecasts in section 6.
A model for rare-event verification

Approach

If we divide each entry in Table 1 by \( n \) then we obtain the proportion of pairs that falls in each cell. We consider this table of proportions to be an approximation to the ideal table that would be obtained for an infinite number of pairs. The entries in this ideal table are the probabilities that a forecast-observation pair belongs to the different cells and are shown in Table 2. The ideal table embodies the true forecast performance that we wish to estimate; for example, the hit rate (1) is considered an estimate of the equivalent quantity in Table 2, the conditional probability

\[
\Pr(X > u \mid Y > v) = \frac{\Pr(X > u, Y > v)}{\Pr(Y > v)}.
\]

Table 2 must be estimated to determine forecast performance. The table of proportions is one possible estimate, but this leads to the disadvantages illustrated in section 1. We attempt to overcome these disadvantages by applying extreme-value theory (e.g. Beirlant et al. 2004, Coles 2001). Despite relying on only weak assumptions, this theory shows that the probabilities in Table 2 can be modelled as simple functions of \( u, v \), and two other parameters. These parameters are independent of \( u \) and \( v \) when both thresholds are
large, so do not degenerate, but instead measure the fundamental quality of
the forecasting system for extreme events by characterising the relationship
between the forecast and observation variables. Furthermore, the parameters
do not need to be re-estimated for each choice of $u$ and $v$: they are
estimated once only, by fitting the model to a subset of the data. This yields
a model version of Table 2 in which $u$ and $v$ can then be varied to obtain estimates of verification measures for different thresholds. Compared with
the direct, model-free approach of section 1 in which an ever diminishing
number of threshold exceedances is used to re-estimate Table 2 for each pair
of thresholds, restricting attention to a two-parameter model and estimating
the parameters only once helps to reduce sampling uncertainty.

Instead of relying on a forecaster to specify the forecast threshold for each
observation threshold, we shall consider idealised thresholds that satisfy

$$\Pr(X > u) = p = \Pr(Y > v)$$

for specified base rates $p$. In other words, $u$ and $v$ will be theoretical upper
$p$-quantiles of the distributions of $X$ and $Y$, and we shall investigate forecast performance as $p$ decreases to zero over a range of small values. By considering such thresholds, we implicitly calibrate the forecasts by ensuring that the proportion of forecasted events always equals the proportion of observed events. Our analysis will therefore identify the performance that would be
achieved by a set of forecasts were they to be perfectly calibrated. This may be considered as a measure of potential performance that depends on the properties of the forecasting system through the joint distribution of $X$ and $Y$ but not on a forecaster’s particular choice of forecast threshold. Our method says nothing about any bias or other discrepancies, such as those revealed by Fig. 2, between the forecast and observation variables. Extensions of our approach to uncalibrated forecasts are discussed briefly in section 7.

The marginal probabilities, $\Pr(X > u)$ and $\Pr(Y > v)$, and the joint probability $\Pr(X > u, Y > v)$ determine, by additivity, all other cells in Table 2 and thus any derived verification measure. Our choice of thresholds (4) means that we know $\Pr(X > u)$ and $\Pr(Y > v)$ for any base rate $p$, so the only unknown quantity in Table 2 is $\Pr(X > u, Y > v)$, for which we now develop a model and estimation procedure.

b Joint probability

Let $F(x) = \Pr(X \leq x)$ and $G(y) = \Pr(Y \leq y)$ be the distribution functions of $X$ and $Y$. Then, writing

\[
\tilde{X} = -\log[1 - F(X)],
\]

\[
\tilde{Y} = -\log[1 - G(Y)],
\]

(5)
and
\[ Z = \min\{\tilde{X}, \tilde{Y}\}, \]

we can reduce the joint probability to a one-dimensional quantity:

\[
\Pr(X > u, Y > v) = \Pr\{\tilde{X} > -\log[1 - F(u)], \tilde{Y} > -\log[1 - G(v)]\} = \Pr(\tilde{X} > -\log p, \tilde{Y} > -\log p) = \Pr(Z > -\log p).
\]

Thus, if we can specify a model for the distribution function of \(Z\), which we denote by \(H(z) = \Pr(Z \leq z)\), then we shall have a model for the joint probability. In fact, we need a model for only the upper tail of \(H\) because we are interested in small values of \(p\).

Our data are \(n\) pairs of forecast and observation variables denoted by the time sequence \(\{(X_t, Y_t) : t = 1, \ldots, n\}\), which we shall transform later into realisations \(Z_t = \min\{\tilde{X}_t, \tilde{Y}_t\}\) of \(Z\). Results from extreme-value theory (e.g. Coles 2001, chapter 7) say that, in many situations, the pattern of points formed by exceedances of the \(Z_t\) above a high level \(w_0\) (choice of which is discussed later) is well approximated by a Poisson process. Furthermore, Ledford and Tawn (1996) show that the expected number of points (out of \(n\)) exceeding any high level \(z \geq w_0\) has the form

\[
\exp\left[-\left(\frac{z - \alpha}{\eta}\right)\right],
\]

(6)
where $\alpha$ is a location parameter and $\eta$ is a scale parameter, $0 < \eta \leq 1$, known as the coefficient of tail dependence\textsuperscript{1}. The distribution function of $Z$ is then modelled by the expected proportions

$$H(z) = 1 - \frac{1}{n} \exp \left[ - \left( \frac{z - \alpha}{\eta} \right) \right] \text{ for all } z \geq w_0. \quad (7)$$

Writing $\kappa = n^{-1} \exp(\alpha/\eta)$, we have

$$\Pr(Z > -\log p) = 1 - H(-\log p)$$

$$= \kappa p^{1/\eta} \quad \text{for all } p \leq \exp(-w_0), \quad (8)$$

which defines our model for $\Pr(X > u, Y > v)$, and the complete modelled version of Table 2 is given by Table 3.

[Table 3]

3 Interpretation of the model

The model parameters, $\eta$ and $\kappa$, are independent of the base rate, and also of the marginal distributions of the forecast and observation variables

\textsuperscript{1}Ledford and Tawn (1996) consider the quantity $-1/\log[1 - \exp(-Z)]$ but our expression (6) is obtained after transforming their results to $Z$. See also Ledford and Tawn (1997), Heffernan (2000), and Segers and Vandewalle (2004).
because $Z$ is formed only after standardising $X$ and $Y$ by transformation (5). Rather, $\eta$ and $\kappa$ describe the strength of dependence between $X$ and $Y$ at high levels and so measure the quality of forecasts of extreme events. Moreover, $\eta$ and $\kappa$ determine Table 3, and therefore forecast performance, for all small base rates, $p$. These two parameters are thus key measures of the quality of forecasting systems for extreme events.

Consider two forecasting systems, the first with parameters $(\eta_1, \kappa_1)$ and the second with parameters $(\eta_2, \kappa_2)$. If $\eta_2 > \eta_1$ and $\kappa_2 > \kappa_1$ then

$$\kappa_2 p^{1/\eta_2} > \kappa_1 p^{1/\eta_1}$$

for all $p$. This implies that, for any base rate, the entries on the main diagonal of Table 3 are larger for the second system than for the first, and the off-diagonal entries are smaller. Performance is therefore always superior for the second system. If, on the other hand, $\eta_2 > \eta_1$ but $\kappa_2 < \kappa_1$ then the inequality (9) holds if and only if $p$ is less than

$$p^* = \left( \frac{\kappa_2}{\kappa_1} \right)^{\eta_1/\eta_2}.$$

This implies that the second system is superior only for base rates below $p^*$.

These relationships can be represented graphically by plotting $\kappa$ against $\eta$. If $(\eta_2, \kappa_2)$ lies in the quadrant to the top-right of $(\eta_1, \kappa_1)$ then the second system is superior for all base rates; if $(\eta_2, \kappa_2)$ lies to the top-left of $(\eta_1, \kappa_1)$ then the second system is superior only for base rates above $p^*$; if $(\eta_2, \kappa_2)$
lies to the bottom-right of \((\eta_1, \kappa_1)\) then the second system is superior only for base rates below \(p^*\); and if \((\eta_2, \kappa_2)\) lies to the bottom-left of \((\eta_1, \kappa_1)\) then the second system is inferior for all base rates. These quadrants are shown in Fig. 4 for \((\eta_1, \kappa_1) = (0.5, 1)\), the parameter values that correspond to random forecasts.

[Fig. 4]

If a specific base rate, \(p\), is of interest then all systems with parameters \((\eta, \kappa)\) satisfying the equality

\[\kappa p^{1/\eta} = \kappa_1 p^{1/\eta}\]

have the same performance as the system with parameters \((\eta_1, \kappa_1)\). The locus of points satisfying this equality corresponds to a curve in the \(\kappa-\eta\) diagram, and is shown in Fig. 4 for \(p = 0.1\). All points to the right of the curve correspond to systems with superior performance for this base rate; all points to the left correspond to systems with inferior performance for this base rate.

4 Parameter estimation

In order to estimate parameters \(\eta\) and \(\kappa\) we need to construct realisations \(Z_t = \min\{\tilde{X}_t, \tilde{Y}_t\}\) of \(Z\) from our data \(\{(X_t, Y_t) : t = 1, \ldots, n\}\). This involves
the transformation (5) from \((X_t, Y_t)\) to \((\tilde{X}_t, \tilde{Y}_t)\) via the unknown \(F\) and \(G\), which we must therefore estimate. Simple estimators for \(F\) and \(G\) are the empirical distribution functions

\[
\hat{F}(x) = \frac{1}{n+1} \sum_{t=1}^{n} I(X_t \leq x),
\]
\[
\hat{G}(y) = \frac{1}{n+1} \sum_{t=1}^{n} I(Y_t \leq y),
\]

where \(I(A) = 1\) if \(A\) is true and \(I(A) = 0\) otherwise. The denominator \(n + 1\) is used to ensure that the transformed data,

\[
\tilde{X}_t = -\log[1 - \hat{F}(X_t)],
\]
\[
\tilde{Y}_t = -\log[1 - \hat{G}(Y_t)],
\]

are all finite: denominator \(n\) would transform the largest \(X_t\) and \(Y_t\) to \(-\log 0\).

Given our \(Z_t\) and a high level, \(w_0\), above which our model (7) for the distribution function of \(Z\) holds, the parameters \(\alpha\) and \(\eta\) can be estimated by maximising the likelihood

\[
\exp \left\{ -\exp \left[ -\left( \frac{w_0 - \alpha}{\eta} \right) \right] \right\} \prod_{t:Z_t>w_0} \frac{1}{\eta} \exp \left[ -\left( \frac{Z_t - \alpha}{\eta} \right) \right]
\]

subject to the constraint \(0 < \eta \leq 1\). This maximisation can be performed analytically, ignoring the singularity at \(\eta = 0\), to obtain the estimators

\[
\hat{\eta} = \min \left\{ 1, \frac{1}{m} \sum_{t:Z_t>w_0} (Z_t - w_0) \right\}
\]

and \(\hat{\alpha} = w_0 + \hat{\eta} \log m\), where \(m\) is the number of \(Z_t\) exceeding \(w_0\). The
estimator for $\kappa$ is

$$\hat{\kappa} = \frac{1}{n} \exp \left( \frac{\hat{\alpha}}{\hat{\eta}} \right) = \frac{m}{n} \exp \left( \frac{w_0}{\eta} \right).$$

Estimates $\hat{\eta}$ and $\hat{\kappa}$ can then be substituted into Table 3 and estimates of verification measures obtained for any base rate $p \leq \exp(-w_0)$. For example, the hit rate (1) would be $\hat{\kappa}p^{1/\hat{\eta}-1}$.

5 Implementation issues

a Model assumptions

Our model is inappropriate if either of the forecast or observation variables takes only discrete values. We also made three implicit assumptions about the data in section 2: stationarity, serial independence, and some further, extreme-value conditions. We comment on these assumptions and their implications for our method in the remainder of this section.

b Stationarity

We assumed in section 2 that the probability distribution of the pairs $(X_t, Y_t)$ and, therefore, the entries in Table 3 do not change with time. If the data are instead non-stationary then naive application of our model (8) may yield a poor fit to the data. In this case, the model is an unreliable
source of information about forecast quality. If the model fits well despite
the presence of non-stationarity then results still provide a gross assessment
of forecast quality, although part of the forecast performance may be due
solely to the successful representation of cycles or trends in the forecasting
model. This caveat also applies to the direct method described in section 1.
One way to overcome the problems with non-stationarity is to focus attention
on time periods over which both the climate and the forecasting system are
approximately stable. Restricting to a particular season, for example, can
reduce the effects of an annual cycle. This remedy is applicable for both
our model and the direct approach. An alternative remedy is to extend the
model of section 2 to accommodate non-stationarity. This might involve
time-varying thresholds and marginal distributions, and parametric models
for temporal variations in $\eta$ and $\kappa$. In practice, this requires careful statistical
modelling for each new dataset to ensure that non-stationarity is represented
accurately and to avoid erroneous identification of changes in forecast quality.
Despite the presence of a weak annual cycle in our precipitation data, we shall
ignore non-stationarity in this paper in order to demonstrate the basic model
of section 2.
c Serial independence

The modelling approach outlined in section 2 is appropriate if the exceedances of $w_0$ are serially independent, and so tend to occur singly rather than in clusters. Precisely the same approach may be used, however, even if exceedances cluster. Although such exceedances will deviate from a simple Poisson process, the expected number of exceedances still has the same form (6) and the same likelihood (11) may be used to estimate the parameters (e.g. Ledford and Tawn 2003). Serial dependence should be accounted for when estimating standard errors and constructing confidence intervals, however, and this is discussed in the appendix. Two other approaches for clustered exceedances, which are not pursued here, are to base estimation on cluster maxima (e.g. Ferro and Segers 2003, 2004) or to model the dependence inside clusters (Smith et al. 1997).

d Level choice and model fit

To arrive at Table 3, we must choose a level, $w_0$, above which our model (7) is assumed to hold. Three factors should influence this choice. First, if we wish to use the model to estimate forecast performance at a base rate $p$ then we must choose $w_0 \leq -\log p$. Second, the model parameters are estimated using exceedances of $w_0$, so lower levels admit more data, increasing
precision of estimates. Third, the extreme-value theory that motivates the model is more accurate at higher levels, so using higher \( w_0 \) yields a more accurate description of forecast performance. This trade-off suggests that we should select the lowest level at which the extreme-value approximations are acceptable.

The extreme-value theory behind our model (7) assumes a particular form of decay for the upper tail of the joint distribution function of \( X \) and \( Y \) (Ledford and Tawn 1996). Distinguishing whether or not this condition is satisfied from knowledge of the physical processes generating the data is probably infeasible because the condition admits a wide class of behaviours. Most probability distributions commonly fitted to meteorological variables satisfy the condition for example. Checking that our model fits the data is nevertheless an important part of the analysis, and model fit should guide selection of \( w_0 \). We recommend assessing two features: quality of fit and model stability (Davison and Smith 1990).

*Quality of fit.* If the model (7) holds above \( w_0 \) then any excesses \( Z_t - w_0 \) have distribution function

\[
\Pr(Z_t - w_0 \leq z \mid Z_t > w_0) = \frac{H(w_0 + z) - H(w_0)}{1 - H(w_0)}
\]

\[
= 1 - \exp(-z/\eta),
\]

where \( z \geq 0 \). Consequently, the transformed excesses \( (Z_t - w_0)/\eta \) have an
Exponential distribution with mean 1, and the model fit can be assessed with probability-probability and quantile-quantile plots. Formal hypothesis tests of fit, such as the Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises tests, are also available (e.g. Coelho et al. 2006, manuscript submitted to J. Climate).

**Model stability.** If our model (7) holds above $w_0$ then it should also hold above all $w > w_0$. This stability can be checked by plotting against $w$ the parameter estimates obtained by fitting the model to exceedances of $w$. Any level above which these graphs are approximately constant is a candidate for $w_0$. Adding confidence intervals to these graphs helps to visualise sampling uncertainty. Another useful tool is a plot of the sample mean of excesses $Z_t - w$ against $w$. The mean excess from the Exponential distribution (12) is $\eta$, so the graph should be approximately horizontal for all sufficiently high $w$.

In addition to these empirical checks, knowledge of the physical processes governing the observations and of the forecasting system should be used to challenge the model assumptions. Although arguing for particular distributional forms is hard, we may still address the more basic assumption that the extreme values to which the model is fitted represent a single population of forecast and observation variables with a common joint distribution function. Tabony (1983) gives several examples for which a set of extreme
values represents more than one population, each associated with a different dominant mechanism: hurricane and non-hurricane winds for example. Multiple populations can also derive from critical values of the measured variable above which either the physical or measurement processes or the forecasting system exhibits qualitatively different behaviour: surface temperatures above freezing for example. Such inhomogeneities may be revealed during model checking by systematic departures from the theoretical patterns of fit and stability. If we aim to extrapolate to very rare events, however, then we must be confident that no new factors will become influential at those higher levels.

\textit{e} Summary

Our verification procedure is as follows.

1. Convert the data \((X_t, Y_t)\) to \(Z_t = \min\{\tilde{X}_t, \tilde{Y}_t\}\) by transformation (10).

2. Estimate the parameters of the model (8) by maximising the likelihood (11) over a range of candidate values for \(w_0\).

3. Examine the quality of fit and model stability over this range and select the lowest acceptable value for \(w_0\).

4. Using the \(\hat{\eta}\) and \(\hat{\kappa}\) obtained for the selected \(w_0\), construct Table 3.
for each base rate \( p \leq \exp(-w_0) \) of interest and compute verification measures.

In the following section we apply this procedure to the precipitation forecasts introduced in section 1. There, we also report confidence regions for the model parameters and verification measures, which are constructed using the bootstrap methodology documented in the appendix.

6 Application to precipitation forecasts

We apply our verification procedure to both the UM and ALADIN precipitation forecasts introduced in section 1. Having applied the transformations (10) we need to select a level, \( w_0 \), for each dataset above which our model (7) is used for the distribution of the \( Z_t \). Plots of parameter estimates and of mean excesses against candidate levels are shown in Figs 5 and 6. These indicate that levels around the upper 12% quantiles of the \( Z_t \), equal to 1.72 and 1.69 for the UM and ALADIN forecasts respectively, are feasible. The probability and quantile plots in Figs 7 and 8 show that the transformed excesses above these levels do approximate Exponential distributions with mean 1, with the possible exception of a single outlying value for the ALADIN forecasts. The goodness-of-fit tests mentioned in section 5d all have bootstrap p-values exceeding 0.7 for both datasets, indicating no
evidence that the exceedances above these levels fail to follow the model (7) so we select $w_0 = 1.72$ for the UM forecasts and $w_0 = 1.69$ for the ALADIN forecasts.

[Fig. 5]

[Fig. 6]

[Fig. 7]

[Fig. 8]

The resulting parameter estimates are $\hat{\eta} = 0.75$ and $\hat{\kappa} = 1.18$ for UM, and $\hat{\eta} = 0.72$ and $\hat{\kappa} = 1.25$ for ALADIN. Bootstrapped 90% confidence regions for the parameters are shown in Fig. 9 and we find no significant difference between the parameters for the two forecasting models. We can also say that the forecast quality of both models is significantly different from that of random forecasts at the 10% level of significance since neither confidence region contains the point (0.5, 1). In fact, since both sets of parameters lie in the quadrant to the top-right of (0.5, 1) we conclude that both models outperform random forecasts for all base rates below $\exp(-w_0) \approx 0.18$.

[Fig. 9]
We compare estimates of the hit rate (1), critical success index (2), and odds ratio (3) obtained by the direct method discussed in section 1 and by our model for both the UM and ALADIN forecasts. The estimates are plotted in Fig. 10 against return period (reciprocal base rate) to emphasize rare events. For return periods up to about 100 days, the model-based estimates follow the direct estimates closely, lying within the latter’s 90% confidence intervals, demonstrating that our model is able to capture the changes in the measures with event rarity. For longer return periods, the direct estimates of H, CSI, and OR degenerate to 0 or $-\infty$, while the model-based estimates show a smooth continuation. The confidence intervals for the model-based estimates are also much narrower, revealing the extent to which the model helps to reduce uncertainty. All three measures are slightly lower for ALADIN than for UM at all return periods considered here, but the differences are small relative to the 90% confidence intervals. Several other verification measures (not shown) were also examined and yielded qualitatively similar results.

[Fig. 10]

The ALADIN dataset contains an outlying $Z$ value caused by a particularly good forecast of the largest observation (Fig. 1b). Removing this outlier changes the estimates of $\eta$ and $\kappa$ slightly, but they remain within the 90% confidence region shown in Fig. 9. The model-based estimates of H, CSI,
and OR are accordingly changed only slightly, while the direct estimates all become zero for return periods exceeding 70 days (not shown).

7 Discussion

We used a probability model from extreme-value theory to verify forecasts of extreme events. The model helps to reduce uncertainty in verification measures and identifies two key parameters for describing the quality of forecasting systems for extreme events. Bootstrap confidence regions were proposed for quantifying sampling uncertainty and for comparing the quality of different forecasting systems. The model can accommodate serial dependence and was found to describe accurately the quality of two sets of precipitation forecasts. The model can also be extended to handle non-stationarity, but this is left for future work. Although we defined events by exceedances over high thresholds, a similar approach can be used for deficits below low thresholds as long as events remain rare.

The verification model proposed here is explored further by Stephenson et al. (2006, manuscript in preparation). They evaluate several verification measures using Table 3, investigate how \( \eta \) and \( \kappa \) determine their limiting behaviours as the base rate decreases, and discuss the implications for verification studies.
Our analysis estimated the performance of only calibrated forecasts (4) so that the joint probability in Table 2 could be reduced to a one-dimensional quantity (8) that can be modelled with just two parameters. If no restriction is placed on the relationship between $\Pr(X > u)$ and $\Pr(Y > v)$ then the joint probability can not be so reduced and extreme-value theory does not provide a characterisation in terms of a finite number of parameters, making the model considerably more complex. Models do exist, however: see Segers and Vandewalle (2004, section 9.5) and references therein for further details. Alternatively, the performance of uncalibrated forecasts could be estimated via the relative operating characteristics curve along which the odds ratio is constant and takes the value obtained for the calibrated forecasts (e.g. Swets 1986).

Computer code for the procedures presented in this article and written in the statistical programming language R is available from the author.

Acknowledgements. Conversations with Professor I. T. Jolliffe and Drs F. J. Doblas-Reyes, M. Göber, M. Mittermaier, and D. B. Stephenson, plus comments from referees helped to stimulate and clarify this work. I am grateful to the UK Met Office and Météo-France for releasing the forecast data. This work was conducted as part of the NCAS–Climate Programme.
APPENDIX

Bootstrap methodology

We obtain standard errors and confidence intervals for the parameters, \( \eta \) and \( \kappa \), of our model (8) and for the estimated verification measures by employing the following, non-parametric bootstrap resampling scheme. The same approach is used in section 6 to obtain confidence intervals for direct estimates of verification measures. See Davison and Hinkley (1997) for a general introduction to the bootstrap.

1. Resample \( n \) pairs \( \{(X_t^*, Y_t^*) : t = 1, \ldots, n\} \) with replacement from \( \{(X_t, Y_t) : t = 1, \ldots, n\} \).

2. Re-estimate the verification model and re-compute any verification measures of interest for these new data.

These steps are repeated \( N \) times to yield estimates \( \{\hat{\phi}_i^* : i = 1, \ldots, N\} \) of quantities of interest, \( \phi \), such as model parameters or verification measures for particular base rates. If \( \hat{\phi} \) is the estimate from the original data then an estimate of the standard error of \( \hat{\phi} \) is the standard deviation of the \( \hat{\phi}_i^* \). A \( P\% \) confidence interval for \( \phi \) is given by the lower and upper \( (50 - P/2)\% \) quantiles of the \( \hat{\phi}_i^* \). Usually, \( N = 100 \) is preferred for standard errors, and \( N = 1000 \) for confidence intervals (Davison and Hinkley 1997, pp. 25 and 202). Block resampling (e.g. Wilks 1997) can be used to account for any serial
dependence in the data, but this made little difference for our precipitation forecasts.

We also construct two-dimensional confidence regions for the vector \((\hat{\eta}, \kappa)\). A \(P\%\) confidence region contains \(P\%\) of the points in the bootstrap sample \(\{(\hat{\eta}_i^*, \hat{\kappa}_i^*) : i = 1, \ldots, N\}\). The following algorithm (Hall 1987) uses the notion of a convex hull, the smallest convex region containing a set of points, to define unique confidence regions.

1. Find the convex hull of the points \(\{(\hat{\eta}_i^*, \hat{\kappa}_i^*) : i = 1, \ldots, N\}\).

2. If the proportion of the original \(N\) points lying strictly inside the convex hull is greater than \(P\%\) then remove from the bootstrap sample those points lying on the boundary of the convex hull.

3. Find the convex hull of the reduced bootstrap sample.

Steps 2 and 3 are repeated until approximately \(P\%\) of the original points lie inside the convex hull, which is then taken to be the \(P\%\) confidence region. Green and Silverman (1979) describe an algorithm for finding convex hulls.
REFERENCES


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10. Direct (solid) and model-based (dashed) estimates (thick) for UM (left) and ALADIN (right) precipitation forecasts of hit rate (top), critical success index (middle), and log odds ratio (bottom) against return period (reciprocal base rate) and return level (corresponding quantile of the observations) with bootstrap 90% confidence intervals (thin).
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Table 1: A contingency table representing the number of $n$ forecast-observation pairs in each cell.

<table>
<thead>
<tr>
<th></th>
<th>Observed</th>
<th>Not Observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forecasted</td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>Not Forecasted</td>
<td>$c$</td>
<td>$d$</td>
</tr>
<tr>
<td></td>
<td>$a+c$</td>
<td>$b+d$</td>
</tr>
</tbody>
</table>
Table 2: The ideal contingency table for an infinite sample size.

<table>
<thead>
<tr>
<th></th>
<th>Observed</th>
<th>Not Observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forecasted</td>
<td>$\Pr(X &gt; u, Y &gt; v)$</td>
<td>$\Pr(X &gt; u, Y \leq v)$</td>
</tr>
<tr>
<td>Not Forecasted</td>
<td>$\Pr(X \leq u, Y &gt; v)$</td>
<td>$\Pr(X \leq u, Y \leq v)$</td>
</tr>
<tr>
<td></td>
<td>$\Pr(Y &gt; v)$</td>
<td>$\Pr(Y \leq v)$</td>
</tr>
</tbody>
</table>
Table 3: The modelled version of Table 2.

<table>
<thead>
<tr>
<th></th>
<th>Observed</th>
<th>Not Observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forecasted</td>
<td>$\kappa p^{1/\eta}$</td>
<td>$p - \kappa p^{1/\eta}$</td>
</tr>
<tr>
<td>Not Forecasted</td>
<td>$p - \kappa p^{1/\eta}$</td>
<td>$1 - 2p + \kappa p^{1/\eta}$</td>
</tr>
<tr>
<td></td>
<td>$p$</td>
<td>$1 - p$</td>
</tr>
</tbody>
</table>