Note on Weather Computing and the so-called 24-dimensional Model

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Abstract

The problem of setting up a system of equations by means of which the future large-scale flow pattern might be computed, using an electronic calculating machine, in a reasonable time and with maximum accuracy and efficiency, is discussed. Equations determining the development of a two-parameter, two-dimensional "model", capable of representing the major features of motion of a ballistic atmosphere, are derived. Tests of the model are made on some representative problems, the accurate three-dimensional solutions of which are known.

Now that actual computation of certain aspects of the weather forecast has become a practical possibility, interest in certain simplified "models" of atmospheric motion has increased. It seems at first paradoxical that this should be so. For although one necessary prerequisite for weather computing, an adequate network of upper-air data, has only recently become available, the main issue of optimism regarding the possibility of actually carrying out the necessary calculations in this case available has been the development of large-memory high-speed electronic computing machines. These machines work so very much faster than human computers that it is natural to suppose that, at last we can forget about crude "models" and compute changes in the "actual" atmosphere. Practical experience in attempting to design a computation scheme is disillusioning. It is not merely that certain approximations (equivalent to replacing the "actual" atmosphere by a "mod-

1939: The quasi-geostrophic equations of motion are independent variables. Conf. Publ.
mal and continuity equations determine the rate of change of these parameters and therefore the forecast situation, but since the equations involve details either not represented or inaccurately represented by the parameters some approximations are inevitable. The forecast will therefore contain not only errors of detail due to the use of a finite number of parameters but these parameters themselves will necessarily be to some extent in error. It is important to recognize these two kinds of error which are unavoidable even when the computations are made with complete accuracy. To distinguish them from a third type of error to be discussed later they will be referred to as “physical” or “model” errors. They arise simply and solely because we cannot include all the accurate data. Whether or not such errors are serious depends partly on the number of parameters, partly on the manner in which they are chosen. If the parameters are judiciously chosen it may be possible to represent with sufficient accuracy both the features in which we are most interested and those primarily responsible for changes in them, with only a comparatively small number of parameters. The problem is to discover the most efficient representation, that which enables us to calculate the important features as directly as possible, short-circuiting irrelevant detail. Ideally any significant variation in the initial values of any one or more of the parameters should correspond to a significant difference in the forecast. An advantage of looking at the problems from this point of view is that it suggests the right sense of proportion. Clearly there is no point in computing with more independent well-defined parameters than are determined by the initial data. On the other hand the forecast will contain the same number of parameters and therefore the same amount of detail: it will be as good as one obtained by more elaborate, time-wasting methods.

This simple mathematical approach gives us the general idea and is a good practical guide in the later stages of design of a computing scheme. In the early stages, however, the guidance is too vague and a different physical approach which, though basically equivalent, is more specific, is preferable. Let us commence with an example. It is well known that if we are interested in motion on a grand scale we may usefully replace the “actual” 3-dimensional atmosphere by a 2-dimensional “model”, the “barotropic model”. There is a more or less close relation between the horizontal component of the grand-scale motion at about 500 mb in the atmosphere and the motion computed for the model. Here the model is a physically possible hydro-dynamical system different from the atmosphere but behaving, in many important respects, in a similar manner. We might have set up this model directly from physical reasoning. On the other hand if we had integrated the vorticity equation along the vertical and then neglected certain terms we should have obtained the same final equations and it might be regarded as an accident that these equations correspond to a simple hydrodynamical system. From the point of view of our earlier abstract approach we should have supposed all the parameters representing variation in the vertical. Setting up a model is equivalent to making certain approximations and since all computers have to make approximations there is no inherent defect in the use of models. On the contrary, if approximations correspond to setting up a model we can be sure that the former are at least self-consistent. It does not follow, however, that the most suitable approximations must correspond to a physically possible system. The representation to be described does not accurately correspond to any physical model though it does approximately do so. If the word “model” had not already been used rather vaguely it might be preferable to use some other word but since the representation is the simplicity of physical interpretation and since the epithet “2-dimensional” is already picturesque it is convenient to call this representation a “model”. It represents an improvement on the barotropic model in so far as it contains a very crude representation of variation in the vertical, so crude as to be considered, playfully, as worth only half a dimension!

Such a crude representation, with only two parameters along each vertical, may be given some preliminary justification. In the first place the barotropic model, with only one parameter along the vertical, has already had some success. Theoretical reasoning alone suggests that we may have a good sense of proportion if we use many more parameters to describe horizontal variation and the variation. For, making much less approximations, we may set up for a 2-dimensional model in the few [1949, p. 47]...

Here $p$ is the pressure deviation from value, $x$ and $y$ are horizontal and $z$ is vertical co-ordinate, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$, $\frac{\partial z}{\partial t}$ potential temperature is the variety $g$ the acceleration of gravity, $f$ the Coriolis factor. Thus $\psi$ is a pure number and if the meridional latitude would be known although it is inaccurate in the sense which are too intense, the equation strongly suggests the scale factor, representing the ratio equally distant in the meridional and vertical respectively. Thus if from ground to tropopause (say dynamically equivalent to a height of about 1,000 km) the different 2 parameters to represent the vertical we should use 2 about 200 km in each horizontal. Since we need to consider the field - thousands of kilometers in the ice - it is clear that most of our parameter must describe horizontal variation. Unions are sufficiently detailed and accurate with some accuracy more parameter per 100 km = 200 km are not justifying in attempting to an accurate model. Although we can without more investigation just 1 parameter at a given (picked) level determined by the data the number greater than the number of radio where, owing to inaccuracy and bad the sounders be much less. Hence it appears that models of the kind to be are not much, if at all, closer that justified by the available data. Any...
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To describe horizontal variation as vertical variation. For, making much less stringent approximations, we may set up the equation for a $n$-dimensional model in the form: [Exley, 1949. P. 47]

$$0 = \frac{\partial}{\partial x} (\frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y} (\frac{\partial \phi}{\partial y}) + \frac{\partial}{\partial z} (\frac{\partial \phi}{\partial z}) + \frac{\partial}{\partial t} \phi$$

Here $\phi$ is the pressure deviation from a standard value, $x$ and $y$ are horizontal and $z$ is the vertical co-ordinate, $\phi$ is the perturbation in the vertical direction.

If we have a model and a set of parameters representing the movement of the air, then we can approximate the changes in the vertical and horizontal respectively. The change in pressure from one pressure surface (say 10 km) to another is dynamically equivalent to a horizontal distance of about 1000 km. If we know the parameters for each pressure surface and the pressure at the other level, we can calculate the change in pressure. This is called the pressure perturbation method. However, there are limitations to this method, as it does not take into account the vertical variation directly.

To overcome this, we can use a $n$-dimensional model. This model takes into account both horizontal and vertical variations. The equation for a $n$-dimensional model is given above. However, it is more complex and requires more computational resources. The model is used to predict weather patterns and can be used for long-term forecasting.

The $n$-dimensional model is useful because it allows us to incorporate both horizontal and vertical variations in the model. This means that it can provide more accurate predictions than a purely horizontal model. However, it can be computationally intensive and requires a large amount of data.

Overall, the $n$-dimensional model is a valuable tool for weather forecasting. It allows us to make more accurate predictions and can be used to improve our understanding of weather patterns.
in some respects rather similar to Sutcliffe's though there are important differences in the way approximations are made, in the ancilliary assumptions and in the presentation of the results. The most important difference is the inclusion of the effect of vertical motion on the $\Phi$-field.

The definition of the $\Phi$ and $\Psi$ fields serves merely to describe the motion succinctly. In order to obtain a working model we have to derive sufficiently (i.e. two) partial differential equations, with boundary conditions, which determine $\Psi$ and $\Phi$ individually, and in terms of quantities which can be computed when $\Psi$ and $\Phi$ are given. In order to do this it will be necessary to make further assumptions or postulate some of which may be regarded as fairly plausible, others somewhat arbitrary. The final two of these assumptions are the constraint with which the model simulates the behaviour of the atmosphere and for this reason a "test" on a problem which has been solved in three-dimensional, with much less stringent assumptions, will be included. Some of the assumptions made are not absolutely necessary, but have the advantage of simplifying the presentation. By making more complicated assumptions it may be possible to improve the fidelity of the model.

Alternatively, a similar result may be obtained if suitable (empirically determined) weighting schemes are included. Actual use of the model will indicate which type of modification is most effective.

We shall suppose that the model represents motion on a large scale and that this motion is quasi-geostrophic, that is to say that at all or almost all points the geostrophic formula gives a fairly good approximation to the velocity field and also the curl of the velocity field. The horizontal divergence cannot of course be computed directly but is given with fair accuracy by the vorticity equation, obtained by eliminating the pressure field from the equations of motion. When motion is on a large scale and the Richardson number large compared with unity we may probably ignore the contribution to vorticity change due to overturning in a vertical plane and write:

$$\nabla \psi + \nabla \cdot \Psi = 0 \quad (4.2)$$

As it is well known, a much $\pi$ continuity equation, of similar is obtained by an instead of $\frac{\partial \Psi}{\partial \zeta}$ at the same time replace $\psi$ by $\Psi$ only modification in (4) is the replacement $\frac{\partial \Phi}{\partial \zeta}$ by $\frac{\partial \Psi}{\partial \zeta}$ and in fact the $\Psi$ subsequent analysis may be carried "geostrophic" vertical coordinate, of making our approximations: nuity equation we make them in the "wind" equation. The results as forms and the only difference is interpretation. In the present $\psi$ the means are to be interpreted with means of equal weighting for difference. In the alternative (see they are to be interpreted as for $\Psi$ may be something in extremes gives best results in prac naturally suitable weighting factor to either interpretation) may be empirically. In the subsequent will be assumed that we are using vertical coordinate.

Another approximation will $\Psi$ in equation (4) we shall suppose regions $(\omega_\psi)$ may be neglected condition with $\Psi$ so that:

$$\nabla \cdot \Psi = 0 \quad (4.2)$$

This is certainly not true in the ... non-linear lows. However, for motions
In the atmosphere, the exact proportions of physical properties are uncertain. The atmosphere at a given time is dependent upon the state of the external medium. The atmosphere at a given time may be considered as a three-dimensional medium, but the approximation of a two-dimensional medium is not accurate enough for practical purposes. In the present analysis, the atmosphere is treated as a two-dimensional medium.

The motion of the atmosphere is determined by the balance of forces acting on it. The forces include pressure gradients, Coriolis force, and friction. The equations of motion for the atmosphere can be written as:

\[ \frac{\partial}{\partial t} \left( \rho \mathbf{v} \right) + \nabla \cdot \left( \rho \mathbf{v} \mathbf{v} \right) = \mathbf{F} \]

where \( \mathbf{v} \) is the velocity of the air, \( \rho \) is the density of the air, and \( \mathbf{F} \) is the net force acting on the air. The Coriolis force, which is due to the rotation of the Earth, is given by:

\[ \mathbf{F}_{\text{Coriolis}} = 2 \mathbf{v} \times \mathbf{E} \]

where \( \mathbf{E} \) is the Earth's rotation vector.

The pressure gradient force is given by:

\[ \mathbf{F}_{\text{Pressure Gradient}} = -\nabla P \]

where \( P \) is the pressure.

The friction force is given by:

\[ \mathbf{F}_{\text{Friction}} = -\mathbf{v} \cdot \mathbf{F}_{\text{Friction}} \]

where \( \mathbf{F}_{\text{Friction}} \) is the friction force.

The equations of motion can be simplified for certain conditions, such as steady flow or small changes in the atmosphere. In these cases, the equations can be solved analytically or numerically.

The solutions to these equations can be used to predict the movement of air masses, the formation of weather systems, and other atmospheric phenomena. The solutions can also be used to make weather forecasts, which are essential for planning and safety.

In summary, the study of the atmosphere is a complex and important field that requires a multidisciplinary approach. The understanding of atmospheric processes is crucial for many aspects of human life, including weather forecasting, climate change, and the transportation of pollutants.
effects in the equations but for simplicity this modification will be omitted in the present analysis. Then if we suppose both the base and "top" of the effective part of the atmosphere to be substantially flat we may take the boundary conditions to be \( v_0 = 0 \) at \( z = \pm z_b \). Integrating (9) with respect to \( z \) between these limits:

\[
o = \int_{-z_b}^{z_b} f + \text{curl} \mathbf{u} \, dz \quad \ldots \ldots \ldots \ldots (10)
\]

By definition the velocity field in the model at any level \( z \) is:

\[
v_0 = -\frac{\partial P}{\partial y} \frac{z}{z_b} \Phi, \quad v_r = -\frac{\partial P}{\partial x} \frac{z}{z_b} \Phi
\]

(approximately). This field of motion is strictly non-divergent but a good enough approximation for all purposes except the direct calculation of divergence. In high latitudes \( \Phi \) and \( \Phi \) are very nearly constant multiples of the mean pressure and temperature fields respectively, consistent with the geostrophic approximation and the relatively small value of \( \frac{\partial P}{\partial z} \).

In low latitudes \( \Psi \) and \( \Phi \) are better determined directly from the wind data. Some slight adjustments will in practice be needed to fit the data in middle latitudes and obtain the best \( \Psi \) and \( \Phi \) representation.

Writing:

\[
v_0 = -\frac{\partial P}{\partial y} \frac{z}{z_b} \Phi, \quad v_r = -\frac{\partial P}{\partial x} \frac{z}{z_b} \Phi
\]

and also:

\[
\omega = \text{curl} \mathbf{u} = -\frac{\partial P}{\partial y} \frac{z}{z_b} \Psi, \quad \frac{\partial}{\partial x} \Phi = \frac{\partial}{\partial y} \Phi
\]

we have on substituting in (9):

\[
o = \int_{-z_b}^{z_b} \left( \frac{v_0 - z}{z_b} \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \Phi \right] \right) \, dz + \int_{-z_b}^{z_b} \left( \frac{v_0 - z}{z_b} \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} \Phi \right] \right) \, dz
\]

for the Jacobian of any two quantities \( A, B \) with respect to \( x, y \). Then for computational purposes (13) is conveniently written:

\[
\phi = \int_{-z_b}^{z_b} \left( \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \Phi \right] \right) \, dz + \int_{-z_b}^{z_b} \left( \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} \Phi \right] \right) \, dz
\]

where the contribution from

\[
\int_{-z_b}^{z_b} \frac{v_0 - z}{z_b} \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \Phi \right] \, dz
\]

has been neglected. Perturbation theory suggests that the contribution from this term is small, compared with that of the remaining terms, when the Richardson number is large, as may be verified by substituting the true values of the velocities corresponding to the development of a baroclinic wave (see Eady 1949). We shall therefore assume that the contribution from this term corresponding to vertical advection of vorticity may be neglected, in more general conditions, over most of the region concerned. The integrand in (10) is, in any case, so small in comparison with \( \Phi \) dependent, a quadric form and odd powers of \( \Phi \) integrable to zero. Hence we obtain:

\[
\frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} \Phi \right] + \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \Phi \right] \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right) \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right) \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right) \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)
\]

and if we write:

\[
D = \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} \Phi \right] + \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \Phi \right] \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right) \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)
\]

corresponding to Langrangian differentiation following the mean motion, then

\[
D \frac{d}{d\tau} + \epsilon = \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right) \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)
\]

The term on the right hand side may be called the development term. It represents the change in absolute vorticity of a vertical column moving with the mean motion (but not of course consisting of the same air particles) and expresses the difference between baroclinic motion and the motion of an "equivalent" barotropic model. We shall use the notation:

\[
\epsilon = \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right) \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)
\]

for the Jacobian of any two quantities \( A, B \) with respect to \( x, y \). Then for computational purposes (13) is conveniently written:

\[
\phi = \int_{-z_b}^{z_b} \left( \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \Phi \right] \right) \, dz + \int_{-z_b}^{z_b} \left( \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} \Phi \right] \right) \, dz
\]

where \( \phi \) denotes a means with respect to \( x, y \) whereas \( \Phi = \Phi (x, y) \) by definition:

\[
\phi = \frac{\epsilon}{\phi} \frac{\partial}{\partial y} \Phi \frac{\partial}{\partial x} \Phi
\]

If we ignore the slow variation of latitude and remember that \( \phi \) is a constant of (\( x, y, \Phi \)) where \( \Phi = \Phi (x, y) \) by definition:

\[
\phi = \frac{\epsilon}{\phi} \frac{\partial}{\partial y} \Phi \frac{\partial}{\partial x} \Phi
\]

where \( \epsilon \) is the mean state linearized relative variations of \( \Phi = \Phi (x, y) \). Then we can calculate it by using the following:

\[
\epsilon = \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)^2 \frac{1}{3} \left( \frac{\partial}{\partial y} \Phi \right) \frac{1}{3} \left( \frac{\partial}{\partial x} \Phi \right)
\]

for the Jacobian of any two quantities \( A, B \) with respect to \( x, y \). Then for computational purposes (13) is conveniently written:

\[
\phi = \int_{-z_b}^{z_b} \left( \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \Phi \right] \right) \, dz + \int_{-z_b}^{z_b} \left( \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial y} \Phi \right] \right) \, dz
\]
and the change in $\Phi$ is approximately a function of $x$. However, our model is incapable of representing variation of this kind and all we require is an estimate of the mean change. Integrating between $-z_a$ and $+z_a$ we get:

$$\frac{D\Phi}{Dx} = \frac{\rho}{\rho_0} \int_{-z_a}^{z_a} \phi dx$$

which would be the required equation if we could express the mean value of $\phi$ in terms of $\Phi$ and $\psi$.

To do this we once again integrate equation (9) but instead of adding the contributions from each half of the atmosphere ($x = -z_a$ to $x = 0$ and $x = 0$ to $x = z_a$) we subtract them. Then if $v_z(0)$ is the vertical velocity at the middle level:

$$v_z(0) = \int_{-z_a}^{z_a} \left( v_n + \frac{z_a}{2} \frac{\partial \phi}{\partial y} \right) dy + \frac{1}{2} \frac{\partial \phi}{\partial y}$$

where $\phi$ denotes a mean with respect to $z$.

The right-hand side may be thought of as representing the mean motion of the atmosphere.

The free variation of $\Phi$ is not usually very large and they will be ignored in subsequent calculations. From (9) it follows, if we ignore the free variation of $\phi_a$, that:

$$\phi = \frac{D\Phi}{E_x} + B_n dx$$

for the present we suppose the adiabatic approximation to be sufficiently accurate, we have:

$$\phi = \frac{D\Phi}{E_x}$$

assuming any two quantities $A$, $B_n$, $x$, $y$, then for computational purposes this is conveniently written:

$$(A, B_n) = \left( \frac{D\Phi}{E_x}, B_n \right)$$

and the change in $\Phi$ is approximately a function of $z$. However, our model is incapable of representing variation of this kind and all we require is an estimate of the mean change. Integrating between $-z_a$ and $+z_a$ we get:

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for the present we suppose the adiabatic approximation to be sufficiently accurate, we have:

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assuming any two quantities $A$, $B_n$, $x$, $y$, then for computational purposes this is conveniently written:
As a check we may note that the theoretical distribution for a growing baroclinic disturbance (Eady 1949) does not differ very greatly from the assumed distribution. The principal error arises from the fact that our model is not able to represent the (comparatively small) phase change of \( v_r \) with height: in this example we know that the distribution of \( v_r \) with \( z \) is not independent of \( \lambda, \mu \). However, this feature is associated with the fact that the disturbance is rapidly intensifying and it may be less marked in average conditions. We may note that an error of this kind is involved in the assumption of constant thermal wind: in a growing disturbance there is a (comparatively small) phase change with height in the \( \theta \)-field.

From (34) and (35a) we obtain an expression for the mean vertical velocity. Substitution in (32) gives:

\[
\frac{\partial^2 \Phi}{\partial z^2} \Phi = -\frac{1}{f} \frac{\partial^2 \Phi}{\partial t \partial \lambda} \left[ \frac{\partial \theta}{\partial z} \right] + \left[ \frac{\partial^2 \Phi}{\partial \theta \partial \mu} \right] \frac{\partial \theta}{\partial z} + \left[ \frac{\partial^2 \Phi}{\partial \mu^2} \right] \frac{\partial \mu}{\partial z} + \left[ \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} \right] \frac{\partial \lambda}{\partial z} \ldots (60)
\]

If, as in equation (1), we write \( \beta = \frac{\partial B}{\partial z} \), which as we have seen is a pure number to be interpreted as the square of the horizontal-vertical scale factor, then (59) may be written:

\[
\frac{1}{k^2} \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial \theta \partial \mu} \left[ \frac{\partial \theta}{\partial z} \Phi \right] + \frac{\partial^2 \Phi}{\partial \mu^2} \frac{\partial \mu}{\partial z} \Phi + \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} \frac{\partial \lambda}{\partial z} \Phi \ldots (77)
\]

where the notation of (36) has been used for the Jacobian. For computational purposes this result is conveniently rewritten:

\[
\begin{align*}
\left( \frac{\partial^2 \Phi}{\partial z^2} \right) \Phi &= \frac{\partial^2 \Phi}{\partial \theta \partial \mu} \left[ \frac{\partial \theta}{\partial z} \right] \Phi + \frac{\partial^2 \Phi}{\partial \mu^2} \frac{\partial \mu}{\partial z} \Phi + \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} \frac{\partial \lambda}{\partial z} \Phi \ldots (84) \\
\end{align*}
\]

We now have a pair of equations (15) and (38) for determining \( \frac{\partial^2 \Phi}{\partial z^2} \) and \( \frac{\partial^2 \Phi}{\partial \lambda^2} \). The right-hand sides of these equations are known functions so that to determine \( \frac{\partial^2 \Phi}{\partial z^2} \) we have a Poison differential equation while \( \frac{\partial^2 \Phi}{\partial \lambda^2} \) is determined by a Helmholtz equation. The sign of the constant \( -1 \) in the Helmholtz equation is such as to ensure a unique readily computed solution. To solve the equations we need to apply suitable boundary conditions. If \( \Psi \) and \( \Phi \) are given over the whole of the earth's surface this is a simple matter. The "boundary" condition is that \( \frac{\partial \Phi}{\partial z} = 0 \) and \( \frac{\partial \Phi}{\partial \lambda} = 0 \), which must not have singularities anywhere and this determines the functions uniquely. If the data are given over a hemisphere we may suppose that both functions vanish on the equator. With data over limited regions the appropriate boundary conditions are less obvious and this question will be discussed in a subsequent paper. We may note that the equations (15) and (38) are easily adapted to computations over a spherical surface. Whether it is necessary or convenient to do this is another question the discussion of which is postponed.

In the above account we have neglected two features which in the long run must play an important part in determining atmospheric motion: surface frictional drag and influx and efflux of heat through radiation and convection from the earth's surface. Formally it is a simple matter to include both these features in the model. For example if (to make a crude estimate) we suppose that the surface stress proportional to and in the direction opposite to the geostrophic wind at \( z = -h \), then the torque acting on the column of air above will be proportional to the curl of the wind field at \( z = -h \) i.e. \( \mathbf{\nabla} \times \mathbf{\nabla} \Phi \) and \( \mathbf{\nabla} \times \mathbf{\nabla} \Psi \).

Corresponding to the assumption of the 3-dimensional theory (Eady 1949) we suppose that \( \mathbf{u} \) is \( \Phi \) as effectively constant. Then the equations (29) and (30) evidently reduce to the form:

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{\nabla} \Phi &= k \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{f} \frac{\partial^2 \Phi}{\partial \theta \partial \mu} \left[ \frac{\partial \theta}{\partial z} \right] + \frac{1}{f} \frac{\partial^2 \Phi}{\partial \mu^2} \frac{\partial \mu}{\partial z} + \frac{1}{f} \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} \frac{\partial \lambda}{\partial z} \ldots (78)
\end{align*}
\]

where \( \alpha, \beta, \mu, \theta \) are constants. (29) and (30) will be satisfied if:

\[
\begin{align*}
\{ \Phi_0 - \Phi \} + \frac{1}{f} \frac{\partial^2 \Phi}{\partial \theta \partial \mu} \left[ \frac{\partial \theta}{\partial z} \right] + \frac{1}{f} \frac{\partial^2 \Phi}{\partial \mu^2} \frac{\partial \mu}{\partial z} + \frac{1}{f} \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} \frac{\partial \lambda}{\partial z} = 0
\end{align*}
\]

in order to apply this result we shall introduce some simple means of estimating \( \Psi \) and \( \Phi \).
in order to apply this result we must have some simple means of estimating Q when \( P \) and \( \phi \) are given.

Perhaps the best test of the formulae would be a prolonged series of experimental calculations for comparison with observed behaviour. Alternatively we may test the model for types of motion where the 3-dimensional solutions are known. We have already made some comparisons with the results of perturbation theory. A more elaborate test may be made by seeing to what extent some of the quantitative results of perturbation theory are reproduced by the model. Undisturbed horizontal baroclinic flow may be represented by a mean flow \( U = -\left( \frac{\partial f}{\partial y} \right) \) and a thermal wind \( T = \left( \frac{\partial f}{\partial x} \right) \) where \( U \) and \( T \) are constants.

If now we suppose a small perturbation represented by stream functions \( \psi_1 \) and \( \psi_2 \), we obtain, on substituting in (6) and (16) and picking out the first order terms, the perturbation equations:

\[
0 = \left( \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial z} \right) \nabla \cdot \psi_1 + \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \psi_1 + \frac{1}{T} \frac{\partial}{\partial x} \nabla \cdot \psi_2, \ldots \ldots \ldots (28)
\]

and:

\[
0 = \left( \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial z} \right) \nabla \cdot \psi_2 + \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \psi_2 + \frac{1}{T} \frac{\partial}{\partial x} \nabla \cdot \psi_1, \ldots \ldots \ldots (29)
\]

Corresponding to the assumptions made in the 3-dimensional theory (Bryan 1949) we suppose that \( \frac{\partial^2 z}{\partial x^2} \) and \( \frac{\partial^2 z}{\partial y^2} \) may be taken as effectively constant. Then the simultaneous equations (28) and (29) evidently have solutions of the form:

\[
\psi_1 = E \exp \left( -ix + iy + k_1 z \right), \quad \psi_2 = E \exp \left( -ix + iy + k_2 z \right)
\]

where \( E \), \( F \), \( k_1 \), \( k_2 \), \( \phi \) and \( \phi \) are constants. Equations (28) and (29) will be satisfied if:

\[
\begin{align*}
(U_a - U) + \frac{1}{T} \frac{\partial}{\partial x} \nabla \cdot \psi_1 &= 0 \\
(U_a - U) + \frac{1}{T} \frac{\partial}{\partial x} \nabla \cdot \psi_2 &= 0
\end{align*}
\]

where we have written:

\[
U_a = U + \frac{\partial f}{\partial y}, \quad U = \frac{1}{(\partial f/\partial y) h \partial z} \frac{\partial f}{\partial y}, \quad X = \frac{2}{(\partial f/\partial y) h \partial z} \frac{\partial f}{\partial y}
\]

Eliminating \( F/E \) from equations (28):

\[
(1 + X) \left( \frac{U_a}{U_0} \right)^2 + (\alpha + X) \left( \frac{U_a}{U_0} \right)^2 + 1 - \frac{T}{X} \frac{U_a}{U_0} (1 - X) = 0 \ldots \ldots (31)
\]

whence:

\[
U_a = \frac{U_0}{(1 + X)^{1/2}} \sqrt{X^2 - \frac{4}{3} \frac{T}{X} \frac{U_a}{U_0} (1 - X)} \ldots \ldots (31)
\]

The boundary conditions in the horizontal directions require that both \( k_1 \) and \( k_2 \) should be real. Hence \( X \) and \( U_a \) are necessarily real and positive. The disturbances will be unstable if, and only if, \( k_1 \) is complex, i.e. if \( U_a \) is complex. The conditions for instability are therefore:

\[
X^2 + \frac{4}{3} \frac{T}{X} \frac{U_a}{U_0} (1 - X) < 0 \ldots \ldots (39)
\]

Now \( \sqrt{X} \) is a number proportional to the ratio of the "effective" wavelength \( \sqrt{X} \) to the "dynamic depth" \( z \). There will be instability for some wavelengths if (rearranging (39))

\[
\frac{\partial^2 E}{\partial x^2} \frac{U_a}{U_0} = \frac{1}{4} \frac{T}{X} \frac{U_a}{U_0} (1 - X) \ldots \ldots (47)
\]

for any value of \( (\partial f/\partial y) \), i.e. for any positive \( X \). The maximum value of the right-hand side is \( 1/4 \) (when \( X^2 = 2 \)). Hence the condition for instability of the initial flow is:

\[
\left| \frac{\partial f}{\partial y} \right| < \frac{1}{4} \sqrt{X} \ldots \ldots (48)
\]
The condition may also be written in the form:
\[
\alpha = \frac{A}{B} > 1 \frac{\beta}{\gamma} \frac{df}{dy} \ldots \ldots (39)
\]
where \( \gamma \) is the slope of the undisturbed isentropic surface. This result is of considerable interest, though \( \alpha \) has yet to be checked by three-dimensional calculations. The corresponding wavelength for the disturbances which first become unstable as \( T \) is increased is given by \( \lambda = \sqrt{2} \). Eliminating \( \beta \), we obtain
\[
1 = \frac{1}{\alpha^2} \frac{df}{dy} \sim \sqrt{\frac{1}{2} \lambda} \cdot T \ldots \ldots (40)
\]
and this result may be compared with the stability criterion discovered by Cattarini (1947).

The similitude of the above calculations illustrates the power of the 2-\( \nu \)-dimensional model. As a more precise test of its accuracy we shall consider the case when \( \frac{df}{dy} \) is neglected.

Then \( U_0 = 0 \) and it is clear from (34) we have:
\[
U_0 = \frac{T}{\alpha} \left( 1 - \frac{X}{1 + X} \right) \ldots \ldots (41)
\]
In this case there always exist unstable waves, as is evident also from (38). The condition to be satisfied is that \( X > 1 \) or:
\[
\frac{1}{\alpha^2} \frac{df}{dy} > 1 \ldots \ldots (42)
\]
\[
\sqrt{\frac{1}{2} \lambda} \cdot T \ldots \ldots (43)
\]
which compares with the true value:
\[
\frac{1}{\alpha^2} \frac{df}{dy} > 1 \ldots \ldots (44)
\]
(Eakin 1962 p. 39). The only error is in the numerical factor and this is clearly of a kind which could be eliminated by modifying the constants in the equations (15) and (16).

It is easily verified that the disturbance of maximum growth rate (maximum imaginary part of \( \phi \)) corresponds to \( \mu = 0 \) and maximum \( U_0 \).

Form (41) we find that \( \lambda = 1 + \sqrt{2} \) gives the corresponding wavelength so that:

The corresponding value of \( \phi \) is purely imaginary as in the accurate calculations. The numerical results deduced from (44) and (45) is:
\[
\phi = (\sqrt{2} - 1) \cdot \frac{T}{\sqrt{2} \cdot \lambda} \approx 0.4142 \cdot \frac{T}{\sqrt{2} \cdot \lambda} \ldots \ldots (46)
\]
as compared with the true value:
\[
\phi = 0.3854 \cdot \frac{T}{\sqrt{2} \cdot \lambda} \ldots \ldots (47)
\]
The value of \( \phi \) computed for the model is:
\[
\phi = \frac{1}{\sqrt{2} \cdot \lambda} \cdot T = 0.372 \cdot T \ldots \ldots (48)
\]
which compares favourably with the true value:
\[
\phi = 0.3854 \cdot T \ldots \ldots (49)
\]
The above is probably a fairly severe test of the model. For comparison we may note that the assumption that temperature is horizontal, tially advected (velocities of vertical motion) leads to values of \( \phi \), which are very much too large for the short wave-length — in fact there is no maximum wavelength for instability. For very long waves the effect of vertical motion is much less but on the other hand it is precisely in these conditions that an equivalent isentropic model can be constructed to reproduce the behaviour of the mean flow with some degree of accuracy.

The writer would like to express his thanks to Professor C.-G. Rousby and members of Business for Meteorology, Stockholm, for freely and constructive criticism, during a recent visit, of the ideas presented here.
\[ \sqrt{\frac{1}{k}} z_a \approx 1.15 \cdot \sqrt{\frac{1}{k}} z_b \quad (44) \]

with the true value:

\[ \sqrt{\frac{1}{k}} z_a \approx 0.341 \cdot \sqrt{\frac{1}{k}} z_b \quad (45) \]

Hence the value of \( \delta \) purely imaginary.

The value deduced from (44) and (45) with the true value:

\[ \frac{T}{\sqrt{\frac{1}{k}} z_a} \quad (46) \]

with the true value:

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