Note on Weather Computing and the so-called $2\frac{1}{2}$-dimensional Model

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Abstract

The problem of setting up a system of equations by means of which the future large-scale flow pattern might be computed, using an electronic calculating machine, in a reasonable time and with maximum accuracy and efficiency, is discussed. Equations determining the development of a two-parameter, two-dimensional 'model', capable of representing the major features of motion of a baroclinic atmosphere, are derived. Tests of the model are made on some representative problems, the accurate three-dimensional solutions of which are known.

Now that actual computation of certain aspects of the weather forecast has become a practical possibility, interest in certain simplified "models" of atmospheric motion has increased. It seems at first paradoxical that this should be so. For although one necessary prerequisite for weather computing, an adequate network of upper-air data, has only recently become available, the main cause of optimism regarding the possibility of actually carrying out the necessary calculations in the time available has been the development of large-memory high-speed electronic computing machines. These machines work so *very* much faster than human computers that it is natural to suppose that at last we can forget about crude "models" and compute changes in the "actual" atmosphere. Practical experience in attempting to design a computation scheme is disillusioning. It is not merely that certain approximations (equivalent to replacing the "actual" atmosphere by a "model" but resulting in very little change in those aspects of the motion related to *weather*) are highly desirable from the point of view of *computational* simplicity, speed and stability. The main difficulty, which appears to result as the combined effect of 4 dimensions (3 space and 1 time) and non-linear equations, is the very rapid increase in computation time with increase in representation of "detail".

It is worth while to attempt to express the problem in general terms. At a given instant the weather situation may be regarded as represented by certain values of a number of parameters which may, but *need not*, be the values of pressure, temperature etc. at points on a 3-dimensional grid. Just as the definition on a television screen increases with increase in the number of spots so the accuracy with which the weather situation may be represented increases with increase in the number of parameters. The equations and boundary conditions of motion together with the thermal and continuity equations determine the rate of change of these parameters and therefore the forecast situation, but since the equations involve details either not represented or inaccurately represented by the parameters some approximations are inevitable. The forecast will therefore contain not only errors of detail due to the use of a finite number of parameters but these parameters themselves will *necessarily* be to some extent in error. It is important to recognize these two kinds of error which are unavoidable even when the *computations* are made with complete accuracy. To distinguish them from a third type of error to be discussed later they will be referred to as "physical" or "model" errors. They arise simply and solely because we *cannot* include all the accurate data. Whether or not such errors are serious depends partly on the number of parameters, partly on the manner in which they are chosen. If the parameters are judiciously chosen it *may* be possible to represent with sufficient accuracy both the features in which we are most interested *and* those primarily responsible for changes in them, with only a comparatively small number of parameters. The problem is to discover the most efficient representation, that which enables us to calculate the important features as directly as possible, short-circuiting irrelevant detail. Ideally any significant variation in the initial values of any one or more of the parameters should correspond to a significant difference in the forecast. An advantage of looking at the problem from this point of view is that it suggests the right sense of proportion. Clearly there is no point in computing with more independent well-defined parameters than are determined by the initial data. On the other hand the forecast will contain the same number of parameters and therefore the same amount of detail: it will be as good as one obtained by more elaborate, time-wasting methods.

This abstract, mathematical approach gives us the general idea and is a good practical guide in the later stages of design of a computing scheme. In the early stages, however, the guidance is too vague and a different, physical approach which, though basically equivalent, is more specific, is preferable. Let us commence with an example. It is well known that if we are interested in motion on a grand scale we may usefully replace the "actual" 3-dimensional atmosphere by a 2-dimensional "model", the "barotropic model". There is a more or less close relation between the horizontal component of the grand-scale motion at about 500 mb in the atmosphere and the motion computed for the model. Here the model is a physically possible hydrodynamic system different from the atmosphere but behaving, in many important respects, in a similar manner. We might have set up this model directly from physical reasoning. On the other hand if we had integrated the vorticity equation along the vertical and then neglected certain terms we should have obtained the same final equations and it might be regarded as an accident that these equations correspond to a simple hydrodynamical system. From the point of view of our earlier abstract approach we should have suppressed all the parameters representing variation in the vertical. Setting up a model is equivalent to making certain approximations and since all computers have to make approximations there is no inherent defect in the use of models. On the contrary, if approximations correspond to setting up a model we can be sure that the former are at least self-consistent. It does not follow, however, that the most suitable approximations must correspond to a physically possible system. The representation to be described does not accurately correspond to any physical model though it does approximately do so. If the word "model" had not already been used rather vaguely it might be preferable to use some other word but since one advantage of the representation is the simplicity of physical interpretation and since the epithet "2½-dimensional" is already picturesque it is convenient to call this representation a "model". It represents an improvement on the barotropic model in so far as it contains a very crude representation of variation in the vertical, so crude as to be considered, playfully, as worth only half a dimension!

Such a crude representation, with only two parameters along each vertical, may be given some preliminary justification. In the first place the barotropic model, with only one parameter along the vertical, has already had some success. Theoretical reasoning also suggests that we may have a good sense of proportion if we use many more parameters to describe horizontal variation than vertical variation. For, making much less stringent approximations, we may set up the equation for a *3-dimensional* model in the form: (EADY, 1949, p. 47)

$$0 = \frac{d}{dt}\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial(\sqrt{b}\,z)^2}\right] + \frac{df}{dy}\frac{\partial p}{\partial x}:$$

$$\sqrt{b} = \frac{\sqrt{g\dfrac{\partial\varphi}{\partial z}}}{f} \quad \ldots\ldots\ldots \quad (1)$$

Here p is the pressure deviation from a standard value, x and y are horizontal and z is the vertical co-ordinate. $\frac{\partial\varphi}{\partial z}$ $\left(\text{or } \frac{1}{\Theta}\frac{\partial\Theta}{\partial z}\right.$, where Θ is potential temperature$\left.\right)$ is the static stability, g the acceleration of gravity, f the coriolis factor. Thus $\sqrt{b}$ is a pure number. A typical value in middle latitudes would be $\sqrt{b} \simeq 100$. Now although (1) is inaccurate for disturbances which are too intense, the form of the equation strongly suggests that $\sqrt{b}$ is a scale factor, representing the ratio of dynamically equivalent distances in the horizontal and vertical respectively. Thus the distance from ground to tropopause (say 10 km) is dynamically equivalent to a horizontal distance of about 1,000 km. If therefore we devote 2 parameters to representation along the vertical we should use 1 parameter in about 500 km in each horizontal direction. Since we need to consider the field over many thousands of kilometers in the horizontal it is clear that most of our parameters should describe horizontal variation. Unless the data are sufficiently detailed and accurate to determine with some accuracy more than one parameter per 500 km × 500 km square we are not justified in attempting to use a more accurate model. Although we cannot say without more investigation just how many parameters at a given (upper) level are well-determined by the data this number cannot be greater than the number of radio-sondes. It may, owing to inaccuracy and bad spacing of the sondes be much less. Hence it would appear that models of the kind to be described are not much, if at all, cruder than the best justified by the available data. Any attempt to represent much more detail would be a waste of time in *weather computing*. (This is not true of course for calculations designed to give theoretical insight.)

Instead of looking at the problem from the point of view of the amount of detail justified by the initial data we may look at it from the point of view of the computation time. Suppose we have an^2 parameters where $a=1$ for the barotropic model and $a=2$ for the present model. n is the number of parameters representing variation in each horizontal direction. Now with linear equations it is frequently possible to choose the parameters in such a manner that the variation of each is independent of all the others. With non-linear equations it is, however, not usually possible to separate the variations in this manner and in general the variation of each parameter depends on *all* the other parameters. It is evident that for accurate calculations the number of arithemetical operations will increase much faster than n^2. It looks as if the number of operations should vary as n^4 (as it does when the relevant differential equations are solved by means of a Green's function) but it is possible with both kinds of model so to arrange the work that the number of operations varies only as n^3. Nevertheless the time required for the computation of time-variations, and therefore for each small time-step in preparing the forecast, increases rapidly as more detail is included. The increase in time taken to calculate the complete forecast is still larger because with more detail smaller time-steps have to be taken. Without going into details it is evident that we must be content with quite moderate values of n and it is important that we should choose an efficient parametric representation. To give an idea of the magnitudes involved it may be mentioned that the writer estimates that, using the most rapid method known to him the value $n = 13$ is about the largest that (corresponding to $a = 2$) a high-speed electronic computer could cope with in order to prepare a forecast within a few hours. It is important to note that this is based on the assumption that the method must not lead to serious errors in the forecast parameters due to *computational* faults. Computational methods and errors associated with them will not be discussed here as they will form the subject of a subsequent paper. It will suffice to note that most of the methods at present in use suffer from very serious computational errors: the number of parameters whose variation, over a reasonably long time interval, is reasonably accurately computed (as compared with the true variation of the *model*) is much smaller than the number of grid points and often very small indeed. Thus whatever method we use the amount of true information is not very large and it is in fact actually harmful to try to include more detail than we can cope with.

In barotropic motion (or, more generally, in motion without horizontal temperature gradients) there is no variation of motion with height so that only two space dimensions are involved. In baroclinic motion not only is there vertical motion but even the horizontal motion is a function of three dimensions. However, there is considerable synoptic evidence that for large-scale motion the major variations in the horizontal motion are described by supposing the thermal gradient independent of height. Hence we can construct a model of the horizontal part of the motion with two 2-dimensional fields, the field of the mean motion (averaged with regard to height) and the field of relative motion, the mean thermal wind (also averaged with regard to height). Alternatively by adding and subtracting a suitable multiple (either constant or at least independent of the fields of motion) of the relative motion to the mean motion we can express the model in terms of the motion at two representative levels. Now in horizontal barotropic motion the motion is (to a very close approximation) non-divergent so that in this case the mean motion (here the actual motion at any level) may be represented by a (scalar) stream function Ψ. In the case of the model with both mean and relative motion we shall find that each of these may be regarded as approximately non-divergent so that now in addition to the stream function for the mean-motion, Ψ, we have the stream-function for the relative motion, Φ. These will be regarded as (at a given time) independent, though of course their changes are interrelated. The idea of using a model of this type appears to be due to SUTCLIFFE (1947) who has described one in which there are two representative levels. The method of derivation of the equations given below is in some respects rather similar to Sutcliffe's though there are important differences in the way approximations are made, in the ancilliary assumptions and in the presentation of the results. The most important difference is the inclusion of the effect of vertical motion on the Ψ-field.

The definition of the Ψ and Φ fields serves merely to describe the motion succintly. In order to obtain a working model we have to derive sufficient (i. e. two) partial differential equations, with boundary conditions, which determine $\frac{\partial \Psi}{\partial t}, \frac{\partial \Phi}{\partial t}$ in terms of Ψ and Φ only, (or in terms of quantities which can be computed when Ψ and Φ are given). In order to do this it will be necessary to make further assumptions or postulates some of which may be regarded as fairly plausible, others somewhat arbitrary. The final test of these assumptions is the closeness with which the model simulates the behaviour of the atmosphere and for this reason a "test" on a problem which has been solved in three-dimensions, with much less stringent assumptions, will be included. Some of the assumptions made are not absolutely necessary but have the advantage of simplifying the presentation. By making more complicated assumptions it may be possible to improve the fidelity of the model. Alternatively a similar result may be obtainable if suitable (empirically determined) weighting factors are included. Actual use of the model will indicate what type of modification is most effective.

We shall suppose that the model represents motion on a large scale and that this motion is quasi-geostrophic, that is to say that at all or almost all points the geostrophic formula gives a fairly good approximation to the velocity field and also the curl of the velocity field. The horizontal divergence cannot of course be computed directly but is given with fair accuracy by the vorticity equation, obtained by eliminating the pressure field from the equations of motion. When motion is on a large scale and the Richardson number large compared with unity we may probably ignore the contribution to vorticity change due to overturning in a vertical plane and write:

$$\mathrm{div}_H\, v\, (f + curl_H\, v) + \frac{d}{dt}(f + curl_H\, v) = 0 \quad (2)$$

If at the same time we use (in place of the "actual" atmosphere) an incompressible model with the same static stability (but little variation of mean density with height) then we have for the continuity equation

$$\mathrm{div}_H\, v + \frac{\partial v_z}{\partial z} = 0 \quad \ldots\ldots\ldots \quad (3)$$

whence:

$$(f + curl_H\, v)\frac{\partial v_z}{\partial z} = \frac{d}{dt}(f + curl_H\, v) \quad \ldots \quad (4)$$

As is well known, a much more accurate continuity equation, of similar form to (3) is obtained when instead of z we use for a vertical coordinate the mean pressure p and at the same time replace v_z by $v_p = \frac{dp}{dt}$. The only modification in (4) is the replacement of $\frac{\partial v_z}{\partial z}$ by $\frac{\partial v_p}{\partial p}$ and in fact the whole of the subsequent analysis may be carried out using a "pressure" vertical coordinate. Then instead of making our approximations in the continuity equation we make them in the "thermal wind" equation. The results are similar in form and the only difference is in physical interpretation. In the present (spatial) model the means are to be interpreted as height-means with equal weighting for equal height difference. In the alternative (pressure) model they are to be interpreted as pressure-means. It may be that something between these extremes gives best results in practice or alternatively suitable weighting factors (relative to either interpretation) may be developed empirically. In the subsequent analysis it will be assumed that we are using a spatial vertical coordinate.

Another approximation will now be made. In equation (4) we shall suppose that in most regions $|curl_H\, v|$ may be neglected in comparison with f so that:

$$f\frac{\partial v_z}{\partial z} = \frac{d}{dt}(f + curl_H\, v) \quad \ldots\ldots\ldots \quad (5)$$

This is certainly not true in the vicinity of a "jet" and the relative error may also be large near intense depressions or anticyclones. In general the errors are larger at high than at low levels. However, for motion on a *large* scale we may expect to find the approximation not too bad in most regions. From the practical point of view the seriousness of the error depends on the extent to which it can be compensated for by the introduction of (readily computed) "weighting factors". Here we may note that the error may be regarded as associated with incorrect "weighting" of $\frac{\partial v_z}{\partial z}$ in respect to height. Now we have seen that the appropriate weighting of other quantities is to some extent uncertain due to the incompatible requirements for simplicity of the thermal wind and continuity relationships. Moreover we shall subsequently need to adopt a rather crude and schematic distribution for v_z since our data are insufficient to determine its true variation. The total effect of all such errors cannot easily be estimated a priori. Hence the method will be to develop the simplest possible system of equations as a basis for experiment and subsequent critical re-examination.

The stream functions Ψ and Φ are defined as mean values over a certain depth of atmosphere, the boundaries of which will be taken to be $+z_0$, $-z_0$. Thus $-z_0$ corresponds to the surface of the earth and $+z_0$ the effective "top" of the atmosphere. In fact, of course, there is no top to the atmosphere but there is some synoptic and also theoretical evidence (based on perturbation theory — cf. EADY 1949) that large scale disturbances are mainly confined to the troposphere and extend only a short distance into the stratosphere. Hence the effective depth $2z_0$ will be supposed to correspond to a little more than the height of the tropopause but once again some modification may be desirable in conjunction with the use of weighting factors. z_0 is not a constant in fact but unless a simple approximate relation to the Ψ and Φ fields can be found it may be necessary to use a mean value independent of the field of motion. This mean value certainly decreases slowly with latitude but the exact effect of this variation will not be included in the present analysis. The effective depth also varies because of unevenness of the earth's surface and it is known that this is responsible for the existence of orographic stationary long waves. Since this particular variation of z_0 is well defined beforehand it is easy to include its effects in the equations but for simplicity this modification will be omitted in the present treatment. Then if we suppose both the base and "top" of the effective part of the atmosphere as substantially flat we may take the boundary conditions to be $v_z = 0$ at $z = \pm z_0$. Integrating (5) with respect to z between these limits:

$$0 = \int_{-z_0}^{+z_0} \frac{d}{dt}(f + curl_H v)\, dz \quad \ldots . (6)$$

By definition the velocity field in the model at any level z is:

$$\left.\begin{aligned} v_x &= -\frac{\partial \Psi}{\partial y} - \frac{z}{z_0}\frac{\partial \Phi}{\partial y} \\ v_y &= \frac{\partial \Psi}{\partial x} + \frac{z}{z_0}\frac{\partial \Phi}{\partial x} \end{aligned}\right\} \ldots\ldots (7)$$

approximately. This field of motion is strictly non-divergent but a good enough approximation for all purposes except the *direct* calculation of divergence. In high latitudes Ψ and Φ are very nearly constant multiples of the mean pressure and temperature fields respectively, consistent with the geostrophic approximation and the relatively small value of $\frac{df}{dy}$.

In low latitudes Ψ and Φ are better determined directly from the wind data. Some slight adjustments will in practice be needed to fit the data in middle latitudes and obtain the best Ψ and Φ representation.

Writing:

$$v_{x_0} = -\frac{\partial \Psi}{\partial y}; \quad v_{y_0} = \frac{\partial \Psi}{\partial x};$$

$$v_{x_1} = -\frac{\partial \Phi}{\partial y}; \quad v_{y_1} = \frac{\partial \Phi}{\partial x} \quad \ldots\ldots (8)$$

and also:

$$\zeta_0 = curl_H v_0 = \nabla_H^2 \Psi; \; \zeta_1 = curl_H v_1 = \nabla_H^2 \Phi \; (9)$$

we have on substituting in (6):

$$0 = \int_{-z_0}^{+z_0} \left\{ \left(v_{y_0} + \frac{z}{z_0} v_{y_1}\right)\frac{df}{dy} + \left[\left(v_{x_0} + \frac{z}{z_0} v_{x_1}\right)\frac{\partial}{\partial x} + \right.\right.$$

$$\left.\left. + \left(v_{y_0} + \frac{z}{z_0} v_{y_1}\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right]\left(\zeta_0 + \frac{z}{z_0}\zeta_1\right)\right\} dz \; (10)$$

where the contribution from $\int_{-z_0}^{+z_0} v_z \frac{\partial \zeta}{\partial z}\, dz$

has been neglected. Perturbation theory suggests that the contribution from this term is small, compared with that of the remaining terms, when the Richardson number is large, as may be verified by substituting the true values of the velocities corresponding to the development of a baroclinic wave (see EADY 1949). We shall tentatively assume that the contribution from this term corresponding to vertical advection of vorticity may be neglected, in more general conditions, over most of the region concerned. The integrand in (10) is, so far as variation with z is concerned, a quadratic form and odd powers of z integrate to zero. Hence we obtain:

$$v_{y_0}\frac{df}{dy} + \left(v_{x_0}\frac{\partial}{\partial x} + v_{y_0}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)\zeta_0 +$$

$$+ \frac{1}{3}\left(v_{x_1}\frac{\partial}{\partial x} + v_{y_1}\frac{\partial}{\partial y}\right)\zeta_1 = 0 \quad \ldots . (11)$$

and if we write:

$$\frac{D}{Dt} \equiv \left(v_{x_0}\frac{\partial}{\partial x} + v_{y_0}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right) \; .. (12)$$

corresponding to Langrangian differentiation following the *mean* motion, then

$$\frac{D}{Dt}(f + \zeta_0) = -\frac{1}{3}\left(\frac{\partial \Phi}{\partial x}\frac{\partial \zeta_1}{\partial y} - \frac{\partial \Phi}{\partial y}\frac{\partial \zeta_1}{\partial x}\right) \quad (13)$$

The term on the right hand side may be called the *development* term. It represents the change in absolute vorticity of a vertical column moving with the mean motion (but not of course consisting always of the same air particles) and expresses the difference between baroclinic motion and the motion of an "equivalent" barotropic model. We shall use the notation:

$$\left(\frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}\right) \equiv \frac{\partial (A, B)}{\partial (x, y)} \equiv \{A, B\} \; (14)$$

for the Jacobian of any two quantities A, B, with respect to x, y. Then for computational purposes (13) is conveniently written:

$$-\nabla_H \left(\frac{\partial \Psi}{\partial t}\right) = \frac{df}{dy}\frac{\partial \Psi}{\partial x} + \left\{\Psi, \nabla_H^2 \Psi\right\} + \\ + \frac{1}{3}\left\{\Phi, \nabla_H^2 \Phi\right\} \quad \ldots\ldots\ldots\ldots (15)$$

This gives one of the required equations, relating $\frac{\partial \Psi}{\partial t}$ to Ψ and Φ. We now need an equation to determine $\frac{\partial \Phi}{\partial t}$. Now from the geostrophic relations:

$$\frac{\partial v_x}{\partial z} = -\frac{g}{f}\frac{\partial \varphi}{\partial y}; \quad \frac{\partial v_y}{\partial z} = \frac{g}{f}\frac{\partial \varphi}{\partial x} \quad \ldots (16)$$

where φ is the logarithm of potential temperature, we obtain on comparison with (7):

$$\frac{\partial \Phi}{\partial x} = \frac{g z_0}{f}\frac{\partial \varphi}{\partial x}; \quad \frac{\partial \Phi}{\partial y} = \frac{g z_0}{f}\frac{\partial \varphi}{\partial y} \quad \ldots (17)$$

If we ignore the slow variation of $\frac{z_0}{f}$ with latitude and remember that φ is a function of (x, y, z) whereas $\Phi = \Phi\,(x, y)$ by definition, then

$$\Phi = \frac{g z_0}{f} \cdot \overline{\varphi} \quad \ldots\ldots\ldots (18)$$

where φ denotes a mean with respect to z. Since φ varies also with z we shall write:

$$\varphi = \overline{\varphi} + Bz = \frac{f}{g z_0}\Phi + Bz \ldots (19)$$

where $B = \overline{\frac{\partial \varphi}{\partial z}}$ is the mean static stability. The relative variations of B are not usually very large and they will be ignored in subsequent calculations. From (19) it follows, if we ignore the slow variation of $\frac{f}{z_0}$, that:

$$\frac{d\varphi}{dt} = \frac{f}{g z_0}\frac{D\Phi}{Dt} + Bv_z \quad \ldots\ldots (20)$$

If for the present we suppose the adiabatic approximation to be sufficiently accurate we have:

$$0 = \frac{f}{g z_0}\frac{D\Phi}{Dt} + Bv_z \quad \ldots\ldots\ldots (21)$$

and the change in Φ is apparently a function of z. However, our model is incapable of representing variation of this kind and all we require is an estimate of the *mean* change. Integrating between $-z_0$ and $+z_0$ we get:

$$\frac{D\Phi}{Dt} = -\frac{gB}{2f}\int_{-z_0}^{+z_0} v_z\, dz \quad \ldots\ldots (22)$$

which would be the required equation if we could express the mean value of v_z in terms of Φ and Ψ.

To do this we once again integrate equation (5) but instead of adding the contributions from each half of the atmosphere ($z = -z_0$ to $z = 0$ and $z = 0$ to $z = +z_0$) we subtract them. Then if $v_z\,(0)$ is the vertical velocity at the middle level:

$$2f\, v_z(0) = \int_{-z_0}^{0} - \int_{0}^{+z_0} \left\{ \left(v_{y_0} + \frac{z}{z_0} v_{y_1}\right)\frac{df}{dy} + \right. \\ + \left[\left(v_{x_0} + \frac{z}{z_0} v_{x_1}\right)\frac{\partial}{\partial x} + \right. \\ \left. \left. + \left(v_{y_0} + \frac{z}{z_0} v_{y_1}\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right]\left(\zeta_0 + \frac{z}{z_0}\zeta_1\right)\right\} dz \quad (23)$$

if once again the vertical advection of vorticity is neglected. On evaluation the even powers of z integrate to zero and we obtain:

$$-\frac{2f}{z_0} v_z(0) = v_{y_1}\frac{df}{dy} + \left(v_{x_1}\frac{\partial}{\partial x} + v_{y_1}\frac{\partial}{\partial y}\right)\zeta_0 + \\ + \left(v_{x_0}\frac{\partial}{\partial x} + v_{y_0}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)\zeta_1 \ldots (24)$$

There is, of course, no necessary relation between the mean value of v_z and the value of $v_z\,(0)$. However, with large scale disturbances we may expect to find that v_z is rather simply distributed with, in *most* regions, the same sign for all z, a simple maximum near the middle level ($z = 0$) and of course zero values at $z = \pm z_0$. The simplest distribution which satisfies these requirements is the parabolic one in which $v_z \propto (z_0 - z)(z_0 + z)$. Then we have:

$$\frac{1}{2z_0}\int_{-z_0}^{+z_0} v_z\, dz = \frac{2}{3} v_z(0) \quad \ldots\ldots (25)$$

As a check we may note that the theoretical distribution for a growing baroclinic disturbance (EADY 1949) does not differ very greatly from this assumed distribution. The principal error arises from the fact that our model is not able to represent the (comparatively small) *phase* change of v_z with height: in this example we know that the distribution of v_z with z is not independent of x, y. However, this feature is associated with the fact that the disturbance is rapidly intensifying and it may be less marked in average conditions. We may note that an error of the same kind is involved in the assumption of constant thermal wind: in a growing disturbance there is a (comparatively small) phase change with height in the φ-field.

From (25) and (24) we obtain an expression for the mean vertical velocity. Substitution in (22) gives:

$$\frac{D\Phi}{Dt}=\frac{1}{3}\frac{gB}{f^2}\cdot z_0^2\left[v_{y_1}\frac{df}{dy}+\left(v_{x_1}\frac{\partial}{\partial x}+v_{y_1}\frac{\partial}{\partial y}\right)\zeta_0+\right.$$
$$\left.+\left(v_{x_0}\frac{\partial}{\partial x}+v_{y_0}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)\zeta_1\right]\quad(26)$$

If, as in equation (1), we write $b=\frac{gB}{f^2}$, which as we have seen is a pure number to be interpreted as the square of the horizontal-vertical scale factor, then (26) may be written:

$$\frac{3}{bz_0^2}\frac{D\Phi}{Dt}=\frac{D}{Dt}\nabla_H^2\Phi+\frac{df}{dy}\frac{\partial\Phi}{\partial x}+\left\{\Phi,\nabla_H^2\Psi\right\}\quad(27)$$

where the notation of (14) has been used for the Jacobian. For computational purposes this result is conveniently rewritten:

$$\left(\frac{3}{bz_0^2}-\nabla_H^2\right)\left(\frac{\partial\Phi}{\partial t}\right)=\frac{df}{dy}\frac{\partial\Phi}{\partial x}+\left\{\Phi,\nabla_H^2\Psi\right\}+$$
$$+\left\{\Psi,\nabla_H^2\Phi\right\}-\frac{3}{bz_0^2}\left\{\Psi,\Phi\right\}\quad\ldots(28)$$

We now have a pair of equations (15) and (28) for determing $\frac{\partial\Psi}{\partial t}$ and $\frac{\partial\Phi}{\partial t}$. The right-hand sides of these equations are known functions so that to determine $\frac{\partial\Psi}{\partial t}$ we have a Poisson differential equation while $\frac{\partial\Phi}{\partial t}$ is determined by a Helmholtz equation. The sign of the constant $\frac{3}{bz_0^2}$ in the Helmholtz equation is such as to ensure a unique readily computed solution. To solve the equations we need to apply suitable boundary conditions. If Ψ and Φ are given over the whole of the earth's surface this is a simple matter. The "boundary" condition is that $\frac{\partial\Psi}{\partial t}$ and $\frac{\partial\Phi}{\partial t}$ must not have singularities anywhere and this determines the functions uniquely. If the data are given over a hemisphere we may suppose that both functions vanish on the equator. With data over more limited regions the appropriate boundary conditions are less obvious and this question will be discussed in a subsequent paper. We may note that the equations (15) and (28) are easily adapted to computations over a spherical surface. Whether it is necessary or convenient to do this is another question the discussion of which is postponed.

In the above account we have neglected two features which *in the long run* must play an important part in determining atmospheric motion — surface frictional drag and influx and efflux of heat through radiation and convection from the earth's surface. Formally it is a simple matter to include both these features in the model. For example if (to make a crude estimate) we suppose the surface stress proportional to and in the direction opposite to the geostrophic wind at $z=-z_0$, then the torque acting on the column of air above will be proportional to the curl of the wind field at $z=-z_0$, i.e. to $\nabla_H^2(\Psi-\Phi)$. This torque measures an additional rate of change of the vorticity of the column and we have only to add this term to the right-hand side of (15) to represent the effect of friction. The effects of heating and cooling are expressed by adding Q, the rate of change of φ due to non-adiabatic heating or cooling, to the left-hand side of (21). Proceeding in the same way as before we obtain an additional term, $\frac{3f}{Bz_0}\bar{Q}$ where $\bar{Q}$ is the mean value of Q averaged with respect to z, on the right-hand side of (28). Of course, in order to apply this result we must have some simple means of estimating Q when Ψ and Φ are given.

Perhaps the best test of the formulae would be a prolonged series of experimental calculations for comparison with observed behaviour. Alternatively we may test the model for types of motion where the 3-dimensional solutions are known. We have already made some comparisons with the results of perturbation theory. A more elaborate test may be made by seeing to what extent some of the quantitative results of perturbation theory are reproduced by the model. Undisturbed horizontal baroclinic flow may be represented by a mean flow $U = -\left(\frac{\partial \Psi}{\partial y}\right)_0$ and a thermal wind $T = -\left(\frac{\partial \Phi}{\partial y}\right)_0$ where U and T are constants. If now we superpose a small perturbation represented by stream functions Ψ_1, and Φ_1, we obtain, on substituting in (15) and (28) and picking out the first order terms, the perturbation equations:

$$0 = \left(U \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) \nabla_H^2 \Psi_1 + \frac{df}{dy} \frac{\partial}{\partial x} \Psi_1 + + \frac{1}{3} T \frac{\partial}{\partial x} \nabla_H^2 \Phi_1 \quad \ldots\ldots. (29)$$

and:

$$0 = \left(U \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(\nabla_H^2 - \frac{3}{bz_0^2}\right)\Phi_1 + \frac{df}{dy} \frac{\partial}{\partial x} \Phi_1 + + T \frac{\partial}{\partial x}\left(\nabla_H^2 + \frac{3}{bz_0^2}\right)\Psi_1 \quad \ldots. (30)$$

Corresponding to the assumptions made in the 3-dimensional theory (EADY 1949) we suppose that $\sqrt{b}\, z_0 \equiv H$ and $\frac{df}{dy}$ may be taken as effectively constant. Then the simultaneous equations (29) and (30) evidently have solutions of the form:

$$\Psi_1 = E e^{i(\lambda x + \mu y + \vartheta t)} : \Phi_1 = F e^{i(\lambda x + \mu y + \vartheta t)} \quad (31)$$

where E, F, λ, μ, ϑ are constants. Equations (29) and (30) will be satisfied if:

$$\begin{cases} (U_w - U_r)\ E + \frac{1}{3} T F = 0 \\ [U_w (1 + X) - U_r]\ F + T(1 - X)\ E = 0 \end{cases} \quad (32)$$

where we have written:

$$U_w = U + \frac{\vartheta}{\lambda}; \quad U_r = \frac{1}{(\lambda^2 + \mu^2)} \cdot \frac{df}{dy};$$

$$X = \frac{3}{(\lambda^2 + \mu^2) H^2} = \frac{3}{(\lambda^2 + \mu^2) bz_0^2} \quad \ldots (33)$$

Eliminating E/F from equations (32):

$$(1 + X)\left(\frac{U_w}{U_r}\right)^2 - (2 + X)\left(\frac{U_w}{U_r}\right) + 1 - - \frac{T^2}{3 U_r^2}(1 - X) = 0 \quad \ldots\ldots (34)$$

whence:

$$\frac{U_w}{U_r} = \frac{(2 + X) \pm \sqrt{X^2 + \frac{4}{3}\frac{T^2}{U_r^2}(1 - X^2)}}{2(1 + X)} \quad (35)$$

The boundary conditions in the horizontal directions require that both λ and μ should be real. Hence X and $\left(\frac{T}{U_r}\right)^2$ are necessarily real and positive. The disturbances will be unstable if, and only if, ϑ is complex, i. e. if $\frac{U_w}{U_r}$ is complex. The condition for instability is therefore:

$$X^2 + \frac{4}{3}\frac{T^2}{U_r^2}(1 - X^2) < 0 \quad \ldots (36)$$

Now $\sqrt{X}$ is a number proportional to the ratio of the "effective" wavelength $\frac{2\pi}{\sqrt{\lambda^2 + \mu^2}}$ to the "dynamic depth" 2H. There will be instability for *some* wavelengths if (rearranging (36))

$$\frac{U_r^2}{X^2 T^2} = \left(\frac{H^2}{3T}\frac{df}{dy}\right)^2 < \frac{4}{3}\frac{1}{X^2}\left(1 - \frac{1}{X^2}\right) \quad (37)$$

for any value of $(\lambda^2 + \mu^2)$, i. e. for any positive X. The maximum value of the right-hand side is $1/3$ (when $X^2 = 2$). Hence the condition for instability of the initial flow is:

$$\left|\frac{H^2}{T}\frac{df}{dy}\right| < \sqrt{3} \quad \ldots\ldots.. (38)$$

The condition may also be written in the form:

$$s \equiv \left|\frac{A}{B}\right| > \frac{1}{\sqrt{3}} \frac{z_0}{f} \frac{df}{dy} \quad \ldots\ldots (39)$$

where s is the slope of the (unperturbed) isentropic surfaces. This result is of considerable interest, though it has yet to be checked by three-dimensional calculations. The corresponding wavelength for the disturbances which first become unstable as T is increased is given by $X = \sqrt{2}$. Eliminating H^2 we obtain

$$\frac{1}{(\lambda^2 + \mu^2)} \frac{df}{dy} = \sqrt{\frac{2}{3}} \cdot T \quad \ldots\ldots (40)$$

and this result may be compared with the stability criterion discovered by CHARNEY (1947).

The simplicity of the above calculations illustrates the power of the 2 ½-dimensional model. As a more precise test of its accuracy we shall consider the case when $\frac{df}{dy}$ is neglected. Then $U_r = 0$ and in place of (34) we have:

$$U_w^2 = \frac{T^2}{3} \left(\frac{1 - X}{1 + X}\right) \quad \ldots\ldots (41)$$

In this case there always exist unstable waves, as is evident also from (38). The condition to be satisfied is that $X > 1$ or:

$$\frac{1}{\sqrt{\lambda^2 + \mu^2}} > \frac{1}{\sqrt{3}} \cdot \sqrt{b}\, z_0 \simeq \frac{1}{1.732} \cdot \sqrt{b}\, z_0 \quad (42)$$

This compares with the true value:

$$\frac{1}{\sqrt{\lambda^2 + \mu^2}} > \frac{1}{1.1997} \cdot \sqrt{b}\, z_0 \quad \ldots\ldots (43)$$

(EADY 1949 p. 39). The only error is in the numerical factor and this is clearly of a kind which *could* be eliminated by modifying the constants in the equations (15) and (28).

It is easily verified that the disturbance of maximum growth rate (maximum imaginary part of ϑ) corresponds to $\mu = 0$ and maximum $\frac{U_w^2}{X}$. From (41) we find that $X = 1 + \sqrt{2}$ gives the corresponding wavelength so that:

$$\frac{1}{\lambda} = \frac{\sqrt{1 + \sqrt{2}}}{\sqrt{3}} \cdot \sqrt{b}\, z_0 \simeq \frac{1}{1.115} \cdot \sqrt{b}\, z_0 \quad (44)$$

as compared with the true value:

$$\frac{1}{\lambda} = \frac{1}{0.8031} \cdot \sqrt{b}\, z_0 \quad \ldots\ldots (45)$$

The corresponding value of ϑ is purely imaginary as in the accurate calculations. The numerical magnitude deduced from (44) and (41) is:

$$|\vartheta| = (\sqrt{2} - 1) \cdot \frac{T}{\sqrt{b}\, z_0} \simeq 0.4142 \frac{T}{\sqrt{b}\, z_0} \quad (46)$$

as compared with the true value:

$$|\vartheta| = 0.3098 \frac{T}{\sqrt{b}\, z_0} \quad \ldots\ldots (47)$$

The value of $\left|\frac{\vartheta}{\lambda}\right|$ computed for the model is:

$$\left|\frac{\vartheta}{\lambda}\right| = \frac{1}{\sqrt{3}\sqrt{1 + \sqrt{2}}} \cdot T \simeq 0.372\, T \quad \ldots\ldots (48)$$

which compares favourably with the true value:

$$\left|\frac{\vartheta}{\lambda}\right| = 0.3858\, T \quad \ldots\ldots (49)$$

The above is probably a fairly severe test of the model. For comparison we may note that the assumption that temperature is horizontally advected (neglect of vertical motion) leads to values of $|\vartheta|$ which are very much too large for the short wave-lengths — in fact there is no minimum wave-length for instability. For very long waves the effect of vertical motion is much less but on the other hand it is precisely in these conditions that an equivalent barotropic model can be constructed to reproduce the behaviour of the mean flow with some degree of accuracy.

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