VORTICITY TRANSFER AS RELATED TO THE DEVELOPMENT OF THE GENERAL CIRCULATION

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ABSTRACT

The mechanism of the development and maintenance of the zonal circulation of the atmosphere is discussed from the point of view of vorticity transfer by atmospheric disturbances. Because the disturbances with vorticity concentrations are unable to stay in equilibrium in a non-uniform field of absolute vorticity, they are displaced by masses of air from other latitudes. These disturbances seek the latitudes where the absolute vorticity equals their own, resulting in a transfer of vorticity against the absolute vorticity gradient. The meridional gradient of vorticity is increased within the region where the disturbances are active and decreased beyond, respectively creating westerly and easterly currents in these regions. The process is also discussed by using harmonic solutions of the vorticity equation.

The time required for the creation of strong zonal currents is estimated, and it is found to be of the order of three weeks, agreeing with statistical results.

Simple expressions for the distributions of the mean seasonal zonal winds are derived; they fit the observations closely.

1. Introduction

When the motions of the atmosphere are smoothed by taking averages along latitude circles to eliminate the transient irregularities, we obtain zones of westerlies and easterlies, the so-called prevailing zonal winds. Since one effect of ground friction is to destroy the existing flow, the maintenance of these zonal winds, in a steady state, against surface friction requires a transfer of westerly zonal angular momentum from the zones of easterlies into the zones of westerlies, as has been discussed by many meteorologists in recent years (Jeffreys, 1926; Starr, 1948). However, these zonal wind systems are by no means steady. They go through unceasing changes in both intensity and location, more or less in a cyclic manner, fluctuating around certain mean states. Only when a very long period is considered can these systems be taken as steady. During the periods when these zonal circulations are strengthening, more westerly angular momentum must be transferred from the zones of easterlies into the zones of westerlies to overcompensate the frictional losses. The reverse is true when the zonal circulations are weakening. Thus, this transfer of angular momentum is essential for the formation and development of the zonal wind systems.

The problem can also be discussed from the point of view of transfer of vorticity. Since a jet stream of westerlies is characterized by relative cyclonic vorticity to the north and anticyclonic vorticity to the south, a strengthening of the jet requires a change of vorticity in excess of that which is necessary to balance the destruction by friction. Discussion of the maintenance of the mean zonal motions by considering the vorticity balance is merely a different representation of the angular-momentum balance, since vorticity is but an expression of the wind distribution. However, the process can be more readily discussed from the point of view of vorticity transfer, theoretically speaking, because the vertical component of the absolute vorticity, or absolute potential vorticity, of an element of air is conservative for large-scale atmospheric motions, at least as a first approximation.

In a previous paper (Kuo, 1950a), the writer explained this vorticity transfer by motions of atmospheric vortices which result from the existence of a "force" on the vortices, the force arising from the non-uniform distribution of the absolute vorticity. By "atmospheric vortex" we mean a mass of air with a certain amount of uniformly distributed relative vorticity. The dimensions are therefore smaller than those of ordinary cyclones, and cyclones and anticyclones can be taken as aggregations of these vortices. The present paper is an extension of that study, intended to elucidate further the physical process of this vorticity transfer and to derive the mean seasonal wind-distribution in accordance with this process. The vorticity transfer will also be discussed briefly by considering the solution of the simple vorticity equation. The dynamic aspects of the general circulation in relation to the meridional angular-momentum transfer and the change of the kinetic energy of the mean zonal flow have been discussed in another paper (Kuo, 1950b).

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2. Development of zonal winds and the process of meridional vorticity transfer

When the equation of zonal motion is combined with the continuity equation, we obtain the following equation for the zonal angular momentum:

$$a\cos\phi\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial\lambda}+\frac{\partial(\rho uv\cos^2\phi)}{\cos\phi\,\partial\phi}+a\cos\phi\frac{\partial(\rho uw)}{\partial z}-af\rho v\cos\phi=-\frac{\partial p}{\partial\lambda}-aF_x\cos\phi. \quad (1)$$

Here u, v and w are the zonal, meridional and vertical components of the wind velocity, ρ is the density, p the pressure, f the Coriolis parameter, F_x the eastward component of the frictional force per unit volume, a the radius of the earth, λ the longitude and ϕ the latitude. Since we are mainly concerned with the maintenance and development of the mean zonal circulation, we may integrate this equation over the entire column of the atmosphere and also along the whole latitude circle. Then the terms involving the vertical velocity and the pressure gradient disappear. We also assume that there is no net flow of mass across the latitude wall, which must be true when a long period is considered. Thus, (1) gives

$$\frac{\partial}{\partial t}\int_0^\infty\int_0^{2\pi}a^2\rho u\cos^2\phi\,d\lambda\,dz=-a\int_0^\infty\int_0^{2\pi}\frac{\partial(\rho uv\cos^2\phi)}{\partial\phi}d\lambda\,dz-2\pi a^2F_x\cos^2\phi, \quad (2a)$$

which shows that the rate of change of the total zonal angular momentum is equal to the rate of convergence of the meridional transport of angular momentum minus the frictional loss. This equation can also be used to discuss the balance of the total vertical component of the vorticity over the polar cap beyond latitude ϕ, since it is connected with the total zonal momentum by the following relation:

$$\int_V\zeta_1\,dV=\int_V\nabla\times\rho\mathbf{v}_h\,dV=\int_0^\infty\int_0^{2\pi}\int_\phi^{\frac{1}{2}\pi}a[\partial(\rho v)/\partial\lambda-\partial(\rho u\cos\phi)/\partial\phi]\,d\phi\,d\lambda\,dz=\int_0^\infty\int_0^{2\pi}\rho ua\cos\phi\,d\lambda\,dz.$$

Here V denotes the total volume of the atmosphere north of ϕ, and $\zeta_1=\nabla\times\rho\mathbf{v}_h$ is the vorticity of the horizontal momentum vector $\rho\mathbf{v}_h$. To facilitate the discussion, we expand the right-hand side of (2a) and divide by $a\cos\phi$. This gives

$$\begin{aligned}\frac{\partial}{\partial t}\int_V\zeta_1\,dV&=a\frac{\partial}{\partial t}\int_0^\infty\int_0^{2\pi}\rho u\cos\phi\,d\lambda\,dz\\&=\int_0^\infty\int_0^{2\pi}[v\zeta_1-\rho u\nabla_2\cdot\mathbf{v}_h+(\tfrac{1}{2}v_h^2/a\cos\phi)\,\partial\rho/\partial\lambda]a\cos\phi\,d\lambda\,dz-\int_V\nabla\times\mathbf{F}\,dV\\&=\int_0^\infty\int_0^{2\pi}[\rho v\zeta-u\nabla_2\cdot\rho\mathbf{v}_h+(\tfrac{1}{2}v_h^2/a\cos\phi)\,\partial\rho/\partial\lambda]a\cos\phi\,d\lambda\,dz-\int_V\nabla\times\mathbf{F}\,dV,\end{aligned} \quad (2b)$$

where

$$\zeta=\nabla\times\mathbf{v}_h=\frac{\partial v}{a\cos\phi\,\partial\lambda}-\frac{\partial(u\cos\phi)}{a\cos\phi\,\partial\phi}$$

is the vertical component of the relative vorticity (hereafter simply called "vorticity"). In (2b), the first term in the last member is a flow of vorticity across the boundary, while the other terms may be considered as representing a generation of vorticity along the boundary, since they do not disappear when a rigid wall is placed along the boundary. It is seen that the change of the total vorticity within the region depends only upon the changes on the boundary, which is as should be expected since the total vorticity is merely the circulation around the boundary.

For this particular study, the effect of the variation of density along a latitude circle is small and can safely be neglected. The divergence term can also be neglected, except for the generation of local vorticity concentrations, which occur mainly in a random manner. This approximation is especially desirable for our theoretical discussions, and is permissible since no mean mass divergence can exist for a long period of time and since the magnitude of divergence is much smaller than that of vorticity. Thus, (2b) reduces to

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^\infty\int_0^{2\pi}\rho u\,d\lambda\,dz&=\frac{1}{a\cos\phi}\int_V\frac{\partial\zeta_1}{\partial t}dV\\&=-\int_0^\infty\int_0^{2\pi}\frac{\partial(\rho uv\cos^2\phi)}{a\cos^2\phi\,\partial\phi}d\lambda\,dz-2\pi F_x \quad (2c)\\&\approx\int_0^{p_0}\int_0^{2\pi}\frac{1}{g}v\zeta\,d\lambda\,dp-2\pi F_x,\end{aligned}$$

which shows that the rate of increase of the mean zonal momentum, within the above approximations, is equal to the northward transfer of cyclonic vorticity minus the frictional loss. It may also be noted that the earth's rotation has no direct effect upon this change, although it may affect the momentum or vorticity transfer indirectly.

Each of the velocity components may be broken down into a mean value and a deviation from it, $u = \bar{u} + u'$, $v = \bar{v} + v'$ and $\zeta = \bar{\zeta} + \zeta'$, where the bars denote averages along the latitude circle. Then the northward vorticity transfer is divided into two different processes: an eddy transfer, represented by the quantity $g^{-1}\int_0^{p_0}\int_0^{2\pi} v'\zeta' \, d\lambda \, dp$, and a transfer given by $2\pi g^{-1}\int_0^{p_0}\bar{v}\bar{\zeta} \, dp$, which may be called the transfer by meridional cells. To discuss the vorticity transfer produced by this latter process, let us assume that we have a simple cell with a lower and an upper branch. Since there is no net transfer of mass across the latitude circle, $\bar{v}_1 \Delta p_1 = -\bar{v}_2 \Delta p_2$, where the subscripts denote the values in the lower and the upper branches, respectively. Therefore, the vorticity transfer produced by this meridional cell is given by

$$\int_0^{p_0} \bar{v}\bar{\zeta} \, dp = \bar{v}_2 \Delta p_2 (\bar{\zeta}_2 - \bar{\zeta}_1).$$

Now $\bar{\zeta}_2 - \bar{\zeta}_1$ has opposite signs on the two sides of a jet and is zero at the latitude where the zonal winds are strongest. Since the frictional loss of the zonal momentum is highest within the region of strongest zonal winds and is of the same sign on the two sides of a jet, it seems quite difficult to explain the maintenance of zonal circulations by the presence of any simple meridional cells. Even if it is assumed that there are two meridional cells, one on each side of the jet, and that they are working in opposite directions, so that each one gives a positive contribution, the effect is still zero or small in the region of strongest zonal winds. The negative effect of the transfer produced by the direct meridional cell to the south of the jet of westerlies is also shown in the computations by Yeh (1951). The order of magnitude of the divergence effect in the meridional cells can also be estimated from the values of the meridional velocity given by Yeh: $d\bar{v}/dy \approx 0.36 \times 10^{-6}$ sec^{-1} at 35°N. If $\bar{u}_2 - \bar{u} = 10$ m/sec and $\Delta p_1 = 100$ mb, the zonal acceleration produced by this term is $+3.5$ cm sec^{-1} day^{-1}, which is much smaller than that given by the vorticity transfer term. Therefore, the mean divergence can safely be neglected. From these considerations, it seems that the large-scale eddy motions must be mainly responsible for the required momentum or vorticity transfer, as is also suggested by the actual computations (Starr, 1951; Starr and White, 1951).

3. Meridional vorticity transfer by atmospheric vortices

Although a strictly theoretical study of the eddy vorticity transfer requires solution of the dynamic equations, the tendency of this transfer can be discussed more readily. The generation of vorticity is governed by the general vorticity equation, which can be written

$$\frac{\partial \zeta}{\partial t} + \mathbf{v}_h \cdot \nabla Z + Z \nabla_2 \cdot \mathbf{v}_h + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} = S + \nu_1 \nabla_h^2 \zeta + \nu_2 \frac{\partial^2 \zeta}{\partial z^2}, \tag{3}$$

in which Z is the absolute vorticity, S the solenoid term and ν_1 and ν_2 are the coefficients of virtual viscosity. From (3), it can be seen that the local vorticity can change either by advection or by production due to effects of vertical motion, solenoids, or divergence. It may also be mentioned that, when ν varies from place to place, viscosity can also be a source of vorticity. However, although these factors may be important for the generation of strong vorticity concentrations in some small regions, they occur more or less in a random manner and give no direct contribution to the change of vorticity in large areas, especially when averages over entire latitude circles are involved. It can be shown that, for the large-scale motions in the upper troposphere and stratosphere where the general circulation is clearly defined, the solenoid term and the terms involving the vertical velocity are smaller than the term with horizontal divergence; this horizontal divergence term is itself at least one order of magnitude smaller than the term representing the horizontal advection of absolute vorticity and therefore can be neglected, as we are not concerned with the local generation of vorticity. It may also be mentioned that, if the mean motion of the entire column of the atmosphere is considered, the terms involving vertical velocity can be eliminated by using the equation of the integrated vorticity of momentum ζ_1. Thus, the tendency of the vorticity transfer for the large-scale motions in the upper atmosphere can be studied by employing the simple vorticity equation, neglecting all the other terms except the horizontal advection,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (u\zeta)}{\partial x} + \frac{\partial (v\zeta)}{\partial y} + v\frac{dZ_0}{dy} = 0, \tag{4}$$

where Z_0 is the absolute vorticity of the basic zonal flow. This equation, multiplied by $v \cos \phi$ and then integrated over the area bounded by two latitude circles along which the meridional velocity vanishes, gives, after neglect of higher-order small terms,

$$\Gamma \overline{\frac{dv}{dt}} \equiv \int\int_A \zeta \cos\phi \frac{dv}{dt} dA = \int\int_A v^2 \frac{dZ_0}{a\, d\phi} \cos\phi \, dA. \tag{5}$$

This equation is exact if the motion is nondivergent and horizontal, and if the earth is treated as a plane or cylinder. It shows that the sign of $\Gamma(\overline{dv/dt})$ is that of the gradient of the absolute vorticity of the basic flow; this term has been interpreted as a "force" on atmospheric vortices which drives cyclonic vortices toward regions of higher absolute vorticity and anticyclonic vortices toward regions of lower absolute vorticity. It can also be interpreted simply as the correlation between ζ and dv/dt for the region; therefore, it represents a tendency of vorticity transfer in the direction opposite to the gradient of the absolute vorticity. Thus, in an atmosphere in which the absolute vorticity increases with latitude, if concentrations of vorticity are produced there will be a tendency to produce a northward transfer of vorticity. This will result in a redistribution of the existing vorticity and a corresponding change of the zonal winds. However, it should be mentioned that, if the nondivergent approximation is to hold exactly, the integral $\iint v\zeta \cos\phi\, dA$ vanishes when integrated over the whole area. But, according to (5), this requires the presence of extreme values of Z_0 somewhere in the region, so that $dZ_0/d\phi$ changes sign at some latitude. From this consideration, it may be inferred that, if the atmosphere is to start from relative rest or a weak zonal motion (so that the absolute vorticity increases with latitude at every point), disturbances produced with vorticity concentrations must be associated with divergence. Only when extreme values of the absolute vorticity already have been produced can the vorticity transfer proceed exactly nondivergently. Thus, for the case we have been discussing, the disturbance will not only produce a northward vorticity transfer but will also create a maximum and minimum of absolute vorticity along some northern and southern latitude; this results in the creation of a westerly current between, and easterly currents outside, these latitudes. The importance of the existence of these extreme values of the absolute vorticity can also be seen from (2c), which shows that to increase or maintain the equatorial easterly current against ground friction requires a southward transfer of vorticity across the equator; this is only possible when the absolute vorticity has a minimum north of the equator and a maximum to the south, if the transfer is governed by (5). The mean velocity profile near the tropopause level actually shows this kind of vorticity distribution, which suggests that the required transport of vorticity is mainly accomplished in the upper levels.

Since this vorticity transfer is of fundamental importance in the development and maintenance of the zonal wind systems, let us consider an example through which it can be accomplished. To simplify considerations, we assume that the atmosphere is at relative rest and that disturbances with vorticity concentrations are produced through a divergence effect in a zonal belt. Since the mass distribution does not change, the total divergence along any latitude circle must be zero. Therefore, equal amounts of cyclonic and anticyclonic vorticity will be produced along the same latitude circle, which excludes the possibility of having systematic sources of vorticity. Thus, the subsequent changes of the meridional distribution of vorticity can be produced only through some sorting process that accumulates cyclonic and anticyclonic vortices along different latitudes, as is suggested by (5).

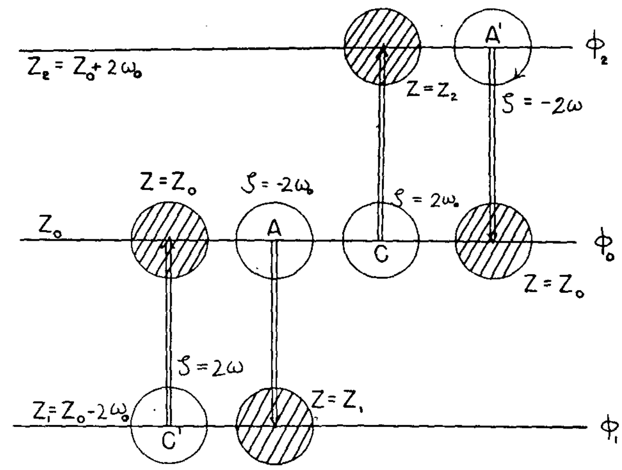

FIG. 1. Schematic displacement of vortices, giving northward transport of vorticity.

To facilitate the discussion, we assume the disturbances are restricted to a belt from ϕ_1 to ϕ_2. However, since these vortices are not able to stay in equilibrium as they are produced in the non-uniform field of absolute vorticity, they tend to seek latitudes where the absolute vorticity of the environment equals their own. Let us assume that an anticyclone is produced along the northern latitude ϕ_2 and a cyclone along the southern latitude ϕ_1, as indicated by A′ and C′ in fig. 1. This is possible if these vortices take their relative deficit and excess of vorticity from the entire mass of fluid along the latitude circle, or from a large segment of it, without producing another compensating cyclone along ϕ_2 and anticyclone along ϕ_1. Therefore, the vorticity of the surrounding fluid along ϕ_2 must have increased and that along ϕ_1 decreased, although probably only slightly, since the total vorticity along the entire latitude circle was not changed by the divergence effect. For simplicity, we also assume that a cyclonic vortex C and an anticyclonic vortex A (of equal mass and equal but opposite relative vorticity $\pm 2\omega_0$) are produced along latitude ϕ_0, halfway between ϕ_1 and ϕ_2, while the vorticity of the surroundings does not change. Latitude ϕ_0 may be taken as the latitude around which the disturbances have been most active.

After these vortices have been produced, they are driven by a "force" toward the latitudes where the absolute vorticity of the new surroundings equals their own, and they will remain there. Suppose A′ and C′ are able to stay at ϕ_0, when they have been displaced to this latitude, with C at ϕ_2 and A at ϕ_1. The mass distribution is preserved if it is assumed that all these vortices have the same mass. After these displacements, equilibrium has been restored, with a resulting transfer of vorticity from ϕ_1 to ϕ_2. An increased vorticity gradient occurs within this region of active disturbance and a decreased vorticity gradient beyond. Corresponding to this change of the vorticity distribution, a mean zonal westerly current will develop between ϕ_1 and ϕ_2, and easterly currents beyond. Thus, the motion of these vortices has the effect of transferring zonal momentum into this belt, as is required for maintenance and development of the westerly wind.

To compute the change of the vorticity distribution produced by this process, we suppose the mass of each vortex is m while the total mass in the belt occupied by the vortex is $(n+1)m$. The relative vorticity of the vortices A′ and C′ are -2ω and $+2\omega$, and those of C and A are $2\omega_0$ and $-2\omega_0$, respectively. Thus, the absolute vorticity of A′ is $Z_2 - 2\omega$, where Z_2 is the initial absolute vorticity at ϕ_2. Since A' is able to stay in equilibrium with its new surroundings when it has been displaced toward ϕ_0, where the absolute vorticity remains Z_0, we must have

$$Z_2 - 2\omega = Z_0 + (dZ/dy)L - 2\omega = Z_0,$$

where dZ/dy is the initial gradient of absolute vorticity and L the distance between ϕ_0 and ϕ_2. Thus, we have

$$(dZ/dy)L = 2\omega. \qquad (6)$$

It has been assumed that each of the vortices A′ and C′ takes its respective deficit and excess of vorticity from the entire surrounding fluid along the latitude circles; therefore, the vorticity of the remaining fluid along ϕ_2 must have increased by $2\omega/n$ and that along ϕ_1 decreased by $2\omega/n$, leaving no change along ϕ_0. Since C is to be displaced to ϕ_2 and stay there, we must have

$$2\omega_0 = (dZ/dy)L + 2\omega/n = (1 + n^{-1})2\omega. \qquad (7)$$

Thus, after completion of this exchanging process, the vorticity gradient between ϕ_0 and ϕ_2 has increased, and the resulting gradient of absolute vorticity is given by

$$\left(\frac{dZ}{dy}\right)_f = \frac{dZ}{dy} + \frac{2\omega}{nL} = \left(1 + \frac{1}{n}\right)\frac{dZ}{dy}. \qquad (8)$$

The increase of vorticity along ϕ_2 also results in a decrease of the vorticity gradient between ϕ_2 and the pole in the amount $(2\omega/na)(\frac{1}{2}\pi - \phi_2)^{-1}$, if we assume the decrease is gradual. A similar decrease also takes place between ϕ_1 and the equator.

Although the changes in the vorticity gradients produced by a single exchange will be small when n is large, a large change can result if the process continues; and a maximum of absolute vorticity can be created along ϕ_2 and a minimum along ϕ_1, with a strong vorticity gradient between them and a reversed gradient beyond.

The intensity of the resulting changes depends upon the intensity, size and number of vortices that have been produced. However, there must be an upper limit of vortex size and intensity; above this limit, the assumption that the vortices A′ and C′ take their deficit and excess of vorticity from the entire latitude circle will not be valid. That is to say, when the intensity of A′ is too high, the increase of vorticity must also be very high in the immediate surroundings; therefore, the surrounding fluid will also tend to move away without resulting in an increase of mean vorticity along ϕ_2.

After this maximum and minimum of absolute vorticity has been created along the latitudes ϕ_2 and ϕ_1, further perturbations will enhance these extreme values. Actually, there would exist a tendency for the vortices to oscillate about their equilibrium positions, because they arrive at these points with finite velocities. However, the steep absolute vorticity gradient between ϕ_1 and ϕ_2 prevents these vortices from crossing the latitude ϕ_0, unless the meridional velocity they have achieved is high enough. Therefore, comparatively thorough mixing will take place in the regions to the north and south of ϕ_0, resulting in a higher temperature gradient and a correspondingly large vertical wind shear in the vicinity of this latitude. Since the sign of the mean meridional temperature gradient reverses above the tropopause, the maximum zonal wind will occur at the tropopause level. This, together with the absolute vorticity gradient around latitude ϕ_0, gives a zonal-wind profile of the jet stream type.

When a maximum and a minimum of absolute vorticity exist along latitudes ϕ_2 and ϕ_1, a southward transport of vorticity in the regions north of ϕ_2 and south of ϕ_1 is possible, and the maintenance of the equatorial easterlies can also be explained.

It should be pointed out that the presence of concentrations of perturbation vorticity does not necessarily require the presence of closed streamlines, since the stream pattern is determined to a large extent by the basic current. Thus, wave disturbances may also have vorticity concentrations and, therefore, may produce a meridional vorticity transfer if the troughs and ridges are properly tilted, as will be discussed briefly in the following section by employing the solutions of the linearized vorticity equation.

4. Vorticity transfer according to solutions of the vorticity equation

When the motion is assumed to be horizontal and nondivergent, it can be described by the equation expressing the conservation of absolute vorticity of an air element. The motion is further assumed to be a superposition of a small perturbation on a purely zonal basic current U, which depends on y only. For this motion, a perturbation stream-function ψ can be introduced, defined by

$$u' = -\partial\psi/\partial y, \qquad v = \partial\psi/\partial x. \tag{9}$$

Since the perturbation is assumed to be small, the vorticity equation can be linearized and written

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial Z_0}{\partial y} = 0. \tag{10}$$

Without losing generality, we assume that the perturbation stream-function is represented by the sum of a number of harmonic perturbations of the form

$$\psi = \Phi(y)\, e^{ik(x-ct)}, \tag{11}$$

in which k is real and positive, representing the wave number, while c may be complex. The real part of c gives the phase velocity, and the imaginary part gives the amplifying factor for the particular wave. Since (10) is linear, the harmonic perturbations are independent of each other; therefore, their effect can be studied separately. Substitution of (11) into (10) reduces it to an ordinary differential equation in Φ:

$$(U - c)(d^2\Phi/dy^2 - k^2\Phi) + (dZ_0/dy)\Phi = 0. \tag{12}$$

It is also assumed that the meridional velocity vanishes along the two boundaries $y = y_1$ and $y = y_2$, which may be taken as the equator and the pole for this particular study. Thus, every component harmonic solution Φ_j must satisfy (12) and the conditions $\Phi_j = 0$ at the boundaries.

We now find expressions for v' and ζ' in terms of Φ. Since c is complex, we must also take Φ as complex, *i.e.*, $\Phi = \Phi_r + i\Phi_i$. Therefore,

$$\begin{aligned} v' = \partial\psi/\partial x &= ik(\Phi_r + i\Phi_i)\, e^{ik(x-ct)} \\ &= -k[\Phi_r \sin\xi + \Phi_i \cos\xi]e^{kc_it}, \end{aligned} \tag{13}$$

$$\begin{aligned} \zeta' = \nabla^2\psi = [(\Phi_r'' - k^2\Phi_r)\cos\xi \\ - (\Phi_i'' - k^2\Phi_i)\sin\xi]e^{kc_it}, \end{aligned}$$

where the primes denote differentiation with respect to y, and $\xi = k(x - c_rt)$ is the phase angle. The meridional vorticity transfer produced by this harmonic is therefore given by

$$\begin{aligned} \int_0^{2\pi} v'\zeta'\, d\xi &= k\pi[\Phi_i''\Phi_r - \Phi_r''\Phi_i]\, e^{2kc_it} \\ &= k\pi\frac{d}{dy}[\Phi_r\Phi_i' - \Phi_r'\Phi_i]\, e^{2kc_it}. \end{aligned} \tag{14}$$

To determine the sign of this quantity, we multiply (12) by the conjugate function $\Phi^* = \Phi_r - i\Phi_i$; equating the real and the imaginary parts separately to zero, we obtain

$$\begin{aligned} \frac{d}{dy}[\Phi_r\Phi_r' + \Phi_i\Phi_i'] \\ = |\Phi'|^2 + \left[k^2 - \frac{(u - c_r)}{|u - c|^2}\frac{dZ_0}{dy}\right]|\Phi|^2, \end{aligned} \tag{15}$$

$$\frac{d}{dy}[\Phi_r'\Phi_i - \Phi_r\Phi_i'] = \frac{c_i}{|u - c|^2}\frac{dZ_0}{dy}|\Phi|^2. \tag{16}$$

If we integrate (16) from y_1 to y_2 and make use of the boundary conditions, it is seen that, in order to have $c_i \neq 0$, we must have dZ_0/dy change sign in the region (Kuo, 1949). Substituting (16) into (14) we obtain

$$\begin{aligned} \int_0^{2\pi} v'\zeta'\, d\xi &= 2\pi\overline{v'\zeta'} \\ &= -\pi\frac{kc_i}{|u - c|^2}e^{2kc_it}\frac{dZ_0}{dy}|\Phi|^2. \end{aligned} \tag{17}$$

Thus, the sign of $\overline{v'\zeta'}$ is that of $-c_i\, dZ_0/dy$. If we substitute (17) into (2c) and integrate with respect to time, with friction neglected, we find

$$\begin{aligned} \Delta\bar{u} = \bar{u}_t - \bar{u}_0 &= \int_0^t \overline{v'\zeta'}\, dt \\ &= \tfrac{1}{4}(1 - e^{2kc_it})\frac{dZ_0}{dy}\frac{|\Phi|^2}{|u - c|^2}. \end{aligned} \tag{18}$$

These two equations show that the effect of the damped disturbances ($c_i < 0$) is to produce a vorticity transfer in the direction opposite to the absolute vorticity gradient, an increase of the zonal westerlies in the region where the absolute vorticity Z_0 increases with latitude, and a decrease of the zonal westerlies in the regions where Z_0 decreases with latitude. The effect of the amplified disturbances ($c_i > 0$) is just the reverse.

In an atmosphere in which Z_0 is a monotonically increasing function of y, it can be shown that all disturbances that are propagated with a phase velocity $c_r = U(y_s)$ at some point y_s will be damped out (Kuo, 1950b). Only the very slowly moving disturbances can remain neutral, and no disturbance can be amplified. These damped disturbances must be considered as having been produced by some other process, which is most probably associated with divergence. The damped disturbances have the effect of producing a vorticity transfer against the absolute vorticity gradient, and thereby to increase the westerly current and create extreme values of the mean absolute vorticity at some latitudes. It should be pointed out that (17) and (18) cannot be valid at every point for this case, for the integration along y will give a change of the total zonal momentum. This slight contradiction can be avoided if divergence is taken into consideration in the vicinity of the singular point. If the absolute vorticity of the basic current has extreme values initially, this is not necessary, and amplifying disturbances can also exist. The effect of the amplifying harmonics is to produce a transport of vorticity in the direction of the absolute vorticity gradient, thereby decreasing the vorticity gradient and the intensity of the zonal flow. The resulting change at any moment depends upon the total effect of all the existing harmonic disturbances at that moment. This process has been discussed in another paper (Kuo, 1950b).

It can also be shown that the vorticity transfer

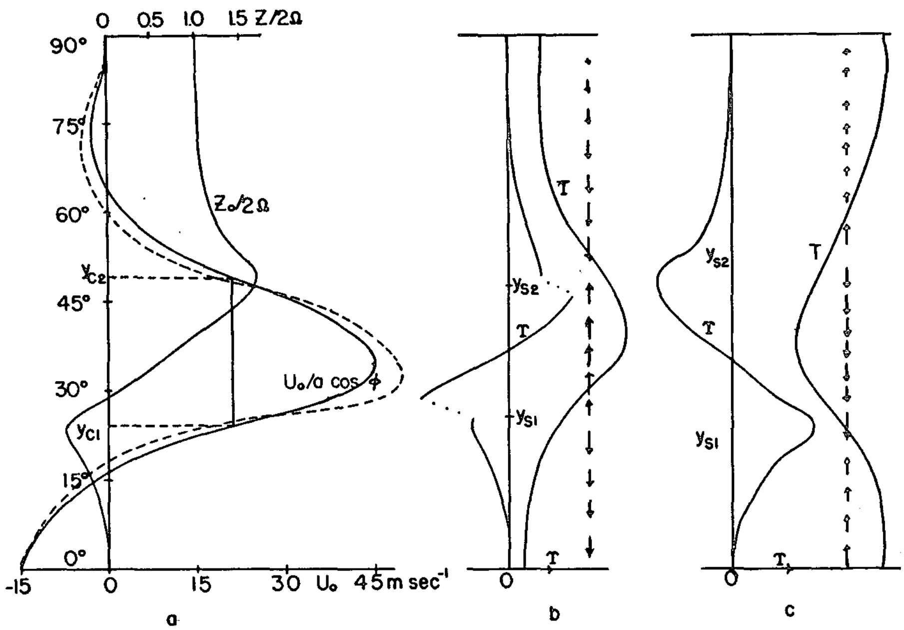

FIG. 2. (a): Distribution of mean zonal current and absolute vorticity at tropopause level; dashed curve represents zonal winds at some later time, produced by damped disturbances. (b): Schematic orientation of troughs and ridges (T), meridional distribution of angular-momentum transfer $-\Upsilon = \overline{\rho u'v'}\cos^2\phi$, and vorticity transport (arrows) produced by damped disturbances. (c): Same as (b) for amplified disturbances.

can be discussed in terms of the direction of the troughs and ridges of the streamlines. The troughs and ridges are curves along which $v' = 0$, or the loci of $\tan\xi = -\Phi_i/\Phi_r$. The angle α which the trough line makes with the meridian is given by

$$\tan\alpha = \frac{d\xi}{dy} = \frac{\Phi_r'\Phi_i - \Phi_r\Phi_i'}{\Phi_r^2\sec^2\xi} = \frac{\Phi_r'\Phi_i - \Phi_r\Phi_i'}{|\Phi|^2}, \quad (19a)$$

which is proportional to the meridional eddy momentum transfer $\overline{u'v'}$ (Kuo, 1949, 1950b). From this, we obtain

$$\frac{d}{dy}[\Phi_r'\Phi_i - \Phi_r\Phi_i'] = |\Phi|^2\frac{d^2\xi}{dy^2} + \frac{d|\Phi|^2}{dy}\frac{d\xi}{dy}. \quad (19b)$$

Substituting into (14), we find

$$\int_0^{2\pi} v'\zeta'\,d\xi = -k\pi\left[|\Phi|^2\frac{d^2\xi}{dy^2} + \frac{d|\Phi|^2}{dy}\frac{d\xi}{dy}\right]. \quad (20)$$

Since the intensity of the perturbation generally has its maximum value in the central part of the region, a positive transfer requires a negative curvature of the trough, or a trough with its convex side toward the east. If the momentum transfer is positive to the south and negative to the north, the second term on the right of (20) is negative; therefore, a still larger negative curvature is required to give a positive vorticity transfer. Fig. 2 shows the tilting of the troughs and directions of the momentum and vorticity transfer produced by the damped and amplifying disturbances when the absolute vorticity has maximum and minimum values.

5. Estimation of time needed to build up zonal currents

The time lag between the introduction of the disturbances with vorticity concentrations and the appearance of a strong westerly current can be estimated from the preceding considerations, since it can be taken as the time required by the new vortices to reach their equilibrium positions. Thus, we wish to evaluate the time required by the vortex to travel the distance L between ϕ_0 and ϕ_2, which may be called the mixing length for the vortex.

To simplify the estimate, let us neglect the spherical shape of the earth and assume that the surrounding atmosphere is undisturbed. The latitudinal derivative of the Coriolis parameter is taken as constant, and the radius of the vortex is R. The fluid inside the vortex is supposed to be in solid rotation with angular velocity ω, corresponding to the perturbation vorticity 2ω. Under these assumptions, $a^{-1}\,dZ/d\phi = \beta$, $v = \omega x$ and $\Gamma = 2\pi R^2\omega$. Therefore, (5) gives

$$\bar{a}_y = \overline{dv/dt} = \tfrac{1}{8}R^2\omega\beta. \quad (21)$$

From (6), $L = 2\omega/\beta$, while, on the other hand, we also have $L = \frac{1}{2}\bar{a}_y T^2$, where T is the time the vortex requires to travel the distance L. Thus,

$$T = 4\sqrt{2}/R\beta, \quad (22)$$

which is independent of the relative vorticity of the vortex, but is inversely proportional to the radius R. If $R = 200$ km and $\beta = 1.62 \times 10^{-11}$ m^{-1} sec^{-1} ($\phi_0 = 45$ deg), T is roughly 20 days, agreeing with statistical results (Allen and others, 1940; Willett, 1947; Riehl and others, 1950).

6. Seasonal mean zonal currents resulting from vorticity transfer produced by disturbances

As discussed in the preceding sections, disturbances with some concentration of vorticity, whatever their origin, act to transfer vorticity northward under normal conditions in the atmosphere. This vorticity transfer has the effect of increasing the vorticity gradient within the region in which the disturbances are active and decreasing the gradient beyond; it thus tends to produce a maximum of absolute vorticity along ϕ_2 and a minimum along ϕ_1, where the cyclonic and anticyclonic vortices find their respective positions of equilibrium, and to develop a strong westerly current between these latitudes, with easterlies beyond. As the intensity and location of these zonal currents depend upon the intensity and location of the disturbances, they will vary from day to day and from longitude to longitude. Thus, if seasonal and longitudinal means are considered, the maxima and minima of absolute vorticity will be greatly reduced and most probably will disappear; yet, the effect of the vorticity transfer still must be to increase the vorticity gradient within the region in which the disturbances have been active, and to decrease the gradient outside this region.

To derive the zonal wind distribution, we may assume a plausible distribution of perturbations and then find the resulting meridional vorticity distribution when equilibrium is achieved. However, the general features of the resulting vorticity distribution can be anticipated without these laborious computations. Thus, we may start by assuming a plausible vorticity distribution in accordance with the general physical process discussed above. Suppose ϕ_0 is the latitude around which the disturbances with vorticity concentrations have been active during the period, and ϕ_2 and ϕ_1 are the northern and southern boundaries of this belt. This implies that the largest changes of the mean vorticity will occur along the latter latitudes. The magnitude of these changes will be roughly proportional to the number and intensity of the vortices produced; but, as has been mentioned before, there must be an upper limit above which our assumptions will not be valid.

From continuity considerations, we assume that the change decreases gradually from ϕ_2 to the pole. Thus, to the north of ϕ_2 we may take a vorticity distribution

$$\bar{\zeta} = -\frac{\partial(\bar{u}\cos\phi)}{a\cos\phi\,\partial\phi} = B(1 - \sin^3\phi)\Omega, \qquad (23)$$

where B is a constant determined by the number and intensity of the vortices, and Ω is the angular velocity of earth's rotation. The corresponding absolute vorticity gradient is given by

$$dZ/d\phi = 2\Omega\cos\phi(1 - \tfrac{3}{2}B\sin^2\phi). \qquad (24)$$

From (23), the mean zonal velocity $\bar{u}$ for $\phi > \phi_2$ is found to be given by

$$4\bar{u}\cos\phi/a\Omega = B[\sin\phi(\sin^3\phi - 4) + 3]. \qquad (25)$$

Around latitude ϕ_0, where the disturbances are most active, the separation of the cyclonic and anticyclonic vortices into higher and lower latitudes increases the vorticity gradient. We assume that the distance from ϕ_0 to the equator is large enough so that the equator is not affected. We also take the region between ϕ_2 and the equator as a whole, and use the expression

$$Z = 2\Omega\sin\phi + A\Omega\sin\phi\sin(\phi - \phi_0) \qquad (26)$$

for $0 < \phi < \phi_2$, in conformity with the physical process discussed above. This expression gives a vorticity gradient greater than β between ϕ_2 and $\frac{1}{2}\phi_0$ and smaller than β between $\frac{1}{2}\phi_0$ and the equator, while $\bar{\zeta}$ is zero at the equator. The value of the constant A is to be determined by the continuity of Z at ϕ_2, and is found to be $B(1 - \sin^3\phi_2)[\sin\phi_2\sin(\phi_2 - \phi_0)]^{-1}$. Thus, to the south of ϕ_2, the mean zonal velocity is given by the equation

$$3\bar{u}\cos\phi/a\Omega = 3C - A[\cos\phi_0\sin^3\phi + \sin\phi_0\cos^3\phi], \qquad (27)$$

where C is a constant of integration to be determined by the continuity of $\bar{u}$ at ϕ_2, and therefore depends upon the values of ϕ_0 and ϕ_2.

During the winter months, the disturbances (migratory cyclones and anticyclones) are produced at a lower latitude, say around 30–35°N; thus, we may take $\phi_0 = 32.5$°N. The latitude ϕ_2, where the cyclonic vortices produced at ϕ_0 will find their equilibrium position, depends upon the relative vorticity $\zeta = 2\omega$ of the vortices (relative to the surrounding fluid) and the gradient of the absolute vorticity in the environment, taken as equal to β. Therefore, for the same vortex strength 2ω, the distance traveled will be greater during summer (with a higher ϕ_0, where β is smaller) than during winter. Thus, if we take the relative vorticity of the vortex at ϕ_0 to be one-tenth of the vorticity of the earth's rotation, $\zeta = 2\omega = \Omega/5$, the distance L is roughly equal to 7 deg lat in winter, with $\phi_0 = 32.5$°N. Therefore $\phi_2 = 39.5$°N. The constant B is taken to be four-ninths, and the corresponding values of A and C in (27) are 4.2578 and 0.7272, respectively. The zonal velocities computed from (25) and (27), with these constants, are given by the curve in fig. 3, while the circles are the observed mean winds at 12 km along 80°W during winter, taken from Hess's (1948) computations.

During the summer months, the latitude along which the disturbances are produced shifts farther north, presumably north of 45°N. If we take ϕ_0 to be

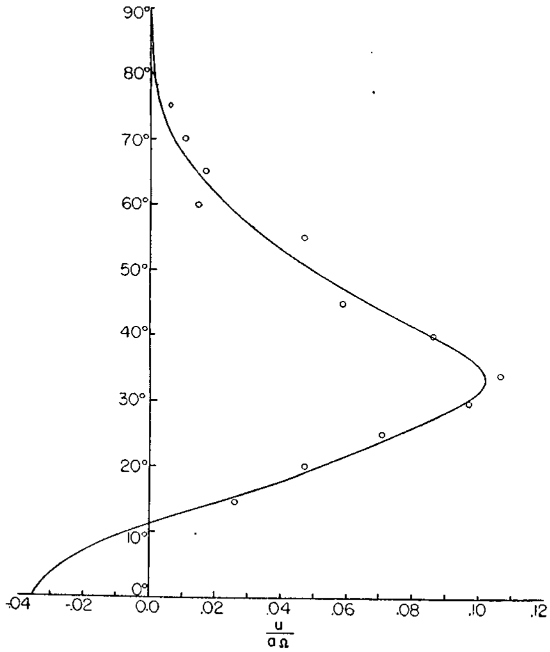

FIG. 3. Computed zonal winds with $\phi_0 = 32.5$, $\phi_2 = 39.5$ deg. Circles are observed mean zonal winds at 12 km along 80°W for winter (after Hess, 1948).

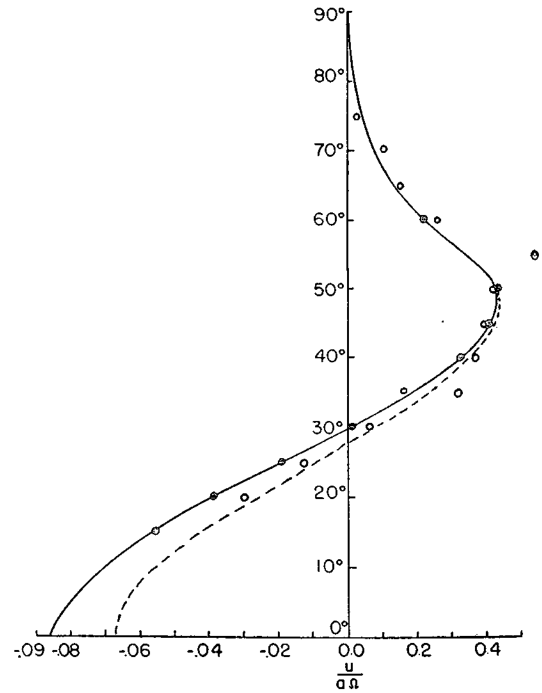

FIG. 4. Computed zonal winds for summer; full curve for $\phi_0 = 46$, $\phi_2 = 55$ deg; dashed curve for $\phi_0 = 45$, $\phi_2 = 55$ deg. Circles are Hess's data for summer; dots are Willett's data.

46°N and the vorticity of the disturbance relative to the environment to be one-tenth of 2Ω again, the mixing length L will be roughly equal to 9 deg lat, so that $\phi_2 = 55$°N. With the same B, the corresponding constants A and C in (27) are 1.5625 and 0.2889, respectively. The zonal winds, computed from (25) and (27), with these constants, are given in fig. 4 by the full curve, while the dashed curve is that for $\phi_0 = 45°$ ($A = 1.407$, $C = 0.2642$). The circles are Hess's data at 12 km, and the dots are the observed mean winds over North America at the average tropopause level given by Willett (1944).

Equations (23) to (27) are designed to describe the mean seasonal zonal winds, in which the maximum and minimum of the absolute vorticity have been canceled out by the meridional shifts of the zone of disturbances. When individual conditions are considered, the maximum and minimum of the absolute vorticity must be pronounced; therefore different expressions should be used.

Since the motions have been assumed to be purely horizontal (or along isentropic surfaces) and barotropic, the calculated results are necessarily applicable only to the given level. But since the vertical variation of wind is given approximately by the thermal wind relation, conditions at other levels can be inferred from conditions at the level under consideration.

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