1:1 resonance in the shallow-water model for Cooker's sloshing experiment

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Abstract

In this report an explicit proof is given showing how the $1:1$ resonance arises in the shallow water model for dynamic coupling in Cooker's sloshing experiment.

1 Introduction

In a linear model for Cooker's sloshing experiment [5], the motion of the vessel is horizontal only, and the system can be modelled by a vessel moving on a horizontal plane constrained by a spring force, as shown schematically in Figure 1. The rectangular vessel has length L

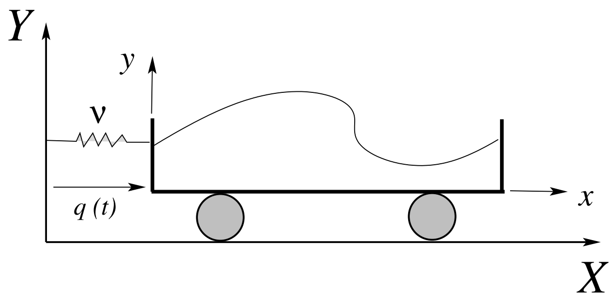

Figure 1: Schematic of a vehicle constrained by a spring force partially filled with fluid.

and unit width. It is partially filled with fluid of mean depth h_0. The tank is suspended by two rigid cables of equal length ℓ, and the cables make an angle θ with the vertical. In the linear approximation the horizontal displacement of the vessel is $q(t) = \ell\theta(t)$. The spring constant in the linear model is

$$\nu = \frac{g}{\ell}(m_v + m_f)\,, \tag{1.1}$$

where m_v is the mass of the dry vessel and $m_f = \rho h_0 L$ is the mass per unit width of the fluid, and ρ is the fluid density.

With the shallow water assumption, $\delta := h_0/L \ll 1$, the linear equations for the coupled model are [5]

$$\begin{aligned} h_t + h_0 u_x &= 0 \\ u_t + g h_x &= -\ddot{q} \\ m_v \ddot{q} + \nu q &= \rho g h_0 (h(L,t) - h(0,t))\,, \end{aligned} \tag{1.2}$$

with boundary conditions $u(0,t) = u(L,t) = 0$. In these equations, $h(x,t)$ is the free surface position of the fluid and $u(x,t)$ is the horizontal fluid velocity.

The purpose of this report is to show that the characteristic function for the natural frequencies can be expressed as a product of two functions

$$\Delta^{\rm SW}(s) = {\rm P}^{\rm SW}(s){\rm D}^{\rm SW}(s)\,, \tag{1.3}$$

where s is the dimensionless natural frequency

$$s = \frac{\omega}{\sqrt{gh_0}}\frac{L}{2}\,, \tag{1.4}$$

and ω is the dimensional natural frequency. It will be shown that the two functions in the product are

$$\begin{aligned} {\rm P}^{\rm SW}(s) &= \sin(s) \\ {\rm D}^{\rm SW}(s) &= \frac{G}{s} - Rs - \tan(s)\,. \end{aligned} \tag{1.5}$$

The function ${\rm D}^{\rm SW}(s)$ is the characteristic function derived in [5]. The other factor is new, and brings in the symmetric fluid modes.

The dimensionless parameters G and R were first introduced in [5] and they are defined by

$$R = \frac{m_v}{m_f} \quad \text{and} \quad G = \frac{\nu L^2}{4gh_0 m_f}\,. \tag{1.6}$$

With two factors in the characteristic function there are three principal classes of solutions and they are summarized below.

1. ${\rm D}^{\rm SW}(s) = 0$ but $\frac{d}{ds}{\rm D}^{\rm SW}(s) \neq 0$ and ${\rm P}^{\rm SW}(s) \neq 0$: anti-symmetric fluid mode coupled to vessel motion.

2. ${\rm P}^{\rm SW}(s) = 0$ but $\frac{d}{ds}{\rm P}^{\rm SW}(s) \neq 0$ and ${\rm D}^{\rm SW}(s) \neq 0$: symmetric fluid mode decoupled from vessel motion.

3. ${\rm D}^{\rm SW}(s) = 0$ and ${\rm P}^{\rm SW}(s) = 0$ but $\frac{d}{ds}{\rm D}^{\rm SW}(s) \neq 0$ and $\frac{d}{ds}{\rm P}^{\rm SW}(s) \neq 0$: internal $1:1$ resonance with a symmetric and anti-symmetric fluid mode coupled to the vessel motion.

The first class of modes was studied in [5, 2]. The second class of solutions decouple from the vessel motion and so were considered heretofore to be of less interest. The third class is new and couples classes 1 and 2.

In this report, first the product structure of the characteristic function (1.3) will be derived. Then the properties of the three classes of solutions identified.

2 Calculating the natural frequencies

In this section the characteristic equation (1.3) is derived. First, transform the coupling equation to a more convenient form using the $u-$equation in (1.2),

$$\frac{d}{dt}\left[\int_0^L \rho h_0 u \, \mathrm{d}x + (m_v + m_f)\dot{q}\right] + \nu q = 0\,. \tag{2.1}$$

Now look for solutions that are periodic in time of frequency ω

$$h(x,t) = \widehat{h}(x)\cos(\omega t)\,, \quad u(x,t) = \widehat{u}(x)\sin(\omega t)\,, \quad q(t) = \widehat{q}\cos(\omega t)\,. \tag{2.2}$$

Substitution into the governing equations gives

$$-\omega\widehat{h} + h_0\widehat{u}_x = 0 \quad \text{and} \quad \omega\widehat{u} + g\widehat{h}_x = \omega^2\widehat{q}\,, \tag{2.3}$$

which when combined give

$$\widehat{u}_{xx} + \alpha^2\widehat{u} = \omega\alpha^2\,\widehat{q}\,, \quad \widehat{u}(0) = \widehat{u}(L) = 0\,, \quad \alpha = \frac{\omega}{\sqrt{gh_0}}\,. \tag{2.4}$$

The solution of (2.4) satisfying only the left boundary condition $\widehat{u}(0) = 0$ is

$$\widehat{u}(x) = A\,\sin(\alpha x) + \omega(1 - \cos(\alpha x))\widehat{q}\,, \tag{2.5}$$

where A is an arbitrary constant. Application of the second boundary condition $\widehat{u}(L) = 0$ gives

$$A\,\sin(2s) + 2s\sqrt{gh_0}(1 - \cos(2s))\frac{\widehat{q}}{L} = 0\,. \tag{2.6}$$

There is a temptation to assume here that $\sin(s) \neq 0$ and then divide (2.6) by $\sin s$. However, $\sin(s)$ can be zero, so it should be retained.

Substitute the expression for $\widehat{u}$ into the vessel equation (2.1),

$$\frac{A}{2s}(1 - \cos(2s)) + 2\sqrt{gh_0}\left[s - \frac{1}{2}\sin(2s)\right]\frac{\widehat{q}}{L} = 2\sqrt{gh_0}\left[(1+R)s - \frac{G}{s}\right]\frac{\widehat{q}}{L}\,. \tag{2.7}$$

Equations (2.6) and (2.7) are two homogeneous equations for two unknowns. Combining them into one matrix equation

$$\begin{bmatrix} \sin(2s) & 2s(1-\cos(2s)) \\ \frac{1}{2s}(1-\cos(2s)) & 2\left[\frac{G}{s} - Rs\right] - \sin(2s) \end{bmatrix} \begin{pmatrix} \frac{A}{\sqrt{gh_0}} \\ \frac{\widehat{q}}{L} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For non-trivial solutions the determinant of the coefficient matrix must vanish, resulting in the characteristic equation

$$\Delta^{\mathrm{SW}}(s) = \det \begin{bmatrix} \sin(2s) & 2s(1-\cos(2s)) \\ \frac{1}{2s}(1-\cos(2s)) & 2\left[\frac{G}{s} - Rs\right] - \sin(2s) \end{bmatrix},$$

or

$$\Delta^{\mathrm{SW}}(s) = 4\sin(s)\left[\left(\frac{G}{s} - Rs\right)\cos(s) - \sin(s)\right].$$

However, if $\cos(s) = 0$ then $\Delta^{\mathrm{SW}}(s) = -4 \neq 0$. Hence $\cos(s)$ is never zero, and it can be divided out. Also dividing by 4 gives the characteristic equation (1.3).

3 Solution Class 1: $\mathrm{D}^{\mathrm{SW}}(s) = 0$

The first class of modes are associated with roots of $\mathrm{D}^{\mathrm{SW}}(s)$ with the conditions

$$\mathrm{D}^{\mathrm{SW}}(s) = 0 \quad \text{but} \quad \frac{d}{ds}\mathrm{D}^{\mathrm{SW}}(s) \neq 0 \quad \text{and} \quad \mathrm{P}^{\mathrm{SW}}(s) \neq 0\,. \tag{3.1}$$

An explicit solution for $\mathrm{D}^{\mathrm{SW}}(s) = 0$ has not been found, but the qualitative position of the roots can be established by plotting $G/s - Rs$ and $\tan(s)$ (see Figure 2 in [5]). The second condition in (3.1) is satisfied since

$$\frac{d}{ds}\mathrm{D}^{\mathrm{SW}}(s) = -\frac{G}{s^2} - R - 1 - \tan^2(s)\,,$$

which is clearly not zero for any $s > 0$. The third condition in (3.1) is satisfied as long as

$$G \neq m^2\pi^2 R \quad \text{for any } m \in \mathbb{N}\,. \tag{3.2}$$

The mode shapes are determined as follows. In this case A and $\widehat{q}$ in (2.5) are related by

$$\frac{A}{\sqrt{gh_0}} = -2s\tan(s)\frac{\widehat{q}}{L}\,.$$

and so

$$\widehat{u}(x) = 2s\sqrt{gh_0}\Big[1 - \cos(\alpha x) - \tan(s)\,\sin(\alpha x)\Big]\,\frac{\widehat{q}}{L}\,, \tag{3.3}$$

with free surface

$$\widehat{h}(x) = 2sh_0\Big[\sin(\alpha x) - \tan(s)\,\cos(\alpha x)\Big]\,\frac{\widehat{q}}{L}\,, \tag{3.4}$$

with the value of $\widehat{q}$ arbitrary (determined by the initial data), and the value of s (and $\alpha = 2s/L$) one of the roots of $\mathrm{D}^{\mathrm{SW}}(s) = 0$.

To see that the free surfaces is anti-symmetric about the vessel centerline, rewrite $\widehat{h}(x)$ in terms of $x - \frac{1}{2}L$,

$$\sin(\alpha x) - \tan(s)\,\cos(\alpha x) = \frac{1}{\cos(s)}\sin\left(\alpha\big(x - \tfrac{1}{2}L\big)\right)\,.$$

Hence

$$\widehat{h}(x) = \frac{2sh_0}{\cos(s)}\frac{\widehat{q}}{L}\,\sin\left(\alpha\big(x - \tfrac{1}{2}L\big)\right)\,,$$

and so $\widehat{h}(L) = -\widehat{h}(0)$.

4 Solution Class 2: $\mathrm{P}^{\mathrm{SW}}(s) = 0$

The conditions for Class 2 are

$$\mathrm{P}^{\mathrm{SW}}(s) = 0 \quad \text{but} \quad \frac{d}{ds}\mathrm{P}^{\mathrm{SW}}(s) \neq 0 \quad \text{and} \quad \mathrm{D}^{\mathrm{SW}}(s) \neq 0\,. \tag{4.1}$$

Setting $\mathrm{P}^{\mathrm{SW}}(s) = 0$ gives $\sin(s) = 0$ and so $s = m\pi$ for some $m \in \mathbb{N}$. The second condition in (4.1) is satisfied since $\frac{d}{ds}\mathrm{P}^{\mathrm{SW}}(s) = \cos(s)$ which is non-zero when $s = m\pi$. The third condition in (4.1) is satisfied as as long as (3.2) is satisfied.

With $\mathrm{D}^{\mathrm{SW}}(s) \neq 0$ it follows from (2.7) that $\widehat{q} = 0$ so the vessel is stationary. The $m-$dependent velocity solution is

$$\widehat{u}_m(x) = A_m \sin\left(2m\pi\frac{x}{L}\right),$$

with A_m arbitrary (determined by the initial conditions). The $m-$dependent free-surface mode shape is

$$\widehat{h}_m(x) = \frac{h_0}{c_0} A_m \cos\left(2m\pi\frac{x}{L}\right).$$

The symmetry of this mode follows from the fact that $\widehat{h}_m(L) = \widehat{h}_m(0)$.

5 Solution Class 3: $\mathrm{D}^{\mathrm{SW}}(s) = \mathrm{P}^{\mathrm{SW}}(s) = 0$

The conditions for this class of solutions are

$$\mathrm{P}^{\mathrm{SW}}(s) = 0 \quad \text{and} \quad \mathrm{D}^{\mathrm{SW}}(s) = 0 \quad \text{but} \quad \frac{d}{ds}\mathrm{P}^{\mathrm{SW}}(s) \neq 0 \quad \text{and} \quad \frac{d}{ds}\mathrm{D}^{\mathrm{SW}}(s) \neq 0. \tag{5.1}$$

The conditions (5.1) are equivalent to

$$\Delta^{\mathrm{SW}}(s) = \frac{d}{ds}\Delta^{\mathrm{SW}}(s) = 0,$$

which is the usual necessary condition for a $1:1$ resonance. It is also sufficient since the eigenfunction associated with $\mathrm{P}^{\mathrm{SW}}(s) = 0$ is linearly independent from the eigenfunction associated with $\mathrm{D}^{\mathrm{SW}}(s) = 0$.

The requirement $\mathrm{P}^{\mathrm{SW}}(s) = 0$ gives, as in the previous section, that $s = m\pi$ for some $m \in \mathbb{N}$. Substituting into the second factor

$$\mathrm{D}^{\mathrm{SW}}(m\pi) = \frac{G}{m\pi} - Rm\pi - \tan(m\pi) = \frac{G}{m\pi} - Rm\pi,$$

and so the condition for $1:1$ resonance is a condition on the parameters G and R.

$$G = m^2\pi^2 R, \quad \text{for some} \quad m \in \mathbb{N}.$$

This observation makes explicit the resonance noted in §3.4 of [5]. The third and fourth conditions in (5.1) are satisfied as noted in §3 and §4.

In terms of physical parameters the resonance condition is

$$1 + \frac{m_f}{m_v} = 4m^2\pi^2 \frac{\ell}{L}\frac{h_0}{L}. \tag{5.2}$$

Although experiments at resonance were not reported in [5], a check of the parameter values in the experiments of [5] shows that the resonances condition (5.2) is physically achievable for at least $m = 1$ and $m = 2$. Potentially, with different design parameters, higher order resonances can also be observed experimentally.

At resonance the vessel natural frequency equals one of the symmetric free modes of the fluid oscillation. The symmetric fluid modes exert no horizontal force on the vessel. However, at resonance, these symmetric modes can mix with the vessel motion. For each $m \in \mathbb{N}$, there is a continuum of such solutions in the linear problem, with eigenfunctions

$$\begin{aligned} u_m(x,t) &= 2\sqrt{gh_0}\sin(\tfrac{1}{2}m\kappa x)\left[\tfrac{A_m}{\sqrt{gh_0}}\cos(\tfrac{1}{2}m\kappa x) + m\kappa\sin(\tfrac{1}{2}m\kappa x)\widehat{q}_m\right]\sin(\omega_m t)\,, \\ h_m(x,t) &= \left[\tfrac{A_m h_0}{\sqrt{gh_0}}\cos(m\kappa x) + m\kappa h_0\widehat{q}_m\sin(m\kappa x)\right]\cos(\omega_m t)\,, \\ q_m(t) &= \widehat{q}_m\cos(\omega_m t)\,. \end{aligned} \tag{5.3}$$

In these expressions, A_m and $\widehat{q}_m$ are arbitrary real numbers, $\kappa := 2\pi/L$, and

$$\omega_m = m\kappa\sqrt{gh_0}\,. \tag{5.4}$$

6 Wave equation resonance

At the linear level, $h(x,t)$ satisfies a wave equation. Combining the first two equations of (1.2) gives

$$h_{tt} - gh_0h_{xx} = 0\,.$$

The normal modes of this equation on the finite domain (without coupling) are just constant multiples of the natural numbers. Hence, in the case of shallow water the $1:1$ resonance is much more severe than in finite depth. This severity is due to the fact that the fluid natural frequencies have an infinite order resonance: $\omega_m = m\omega_1$. Therefore an infinite number of symmetric fluid modes resonate with the vessel mode at the $1:1$ resonance. However, this degeneracy disappears when the depth is finite leaving a pure $1:1$ resonance. In [3, 4] this theory is extended to the finite depth model.

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