Review of the Armenio-LaRocca rotating two-dimensional shallow-water equations

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1 Introduction

Dillingham [8] gave the first derivation of the shallow water equations (SWEs) relative to a moving frame of reference in two dimensions (2D). Independent derivations were later given in 1994 by Huang & Hsiung [6, 7] (the HH SWEs) and in 1996 by Armenio & La Rocca [5] (the ALR SWES). The 2D derivation of [6, 7] is reviewed in the technical report [3].

In this report we review the ALR derivation and identify their key assumptions. The reason for this report is fourfold. The ALR derivation has similarities with, but also differences from, the derivations of Dillingham and Huang & Hsiung. Secondly, their form of the SWEs has the novel property that it has a variational principle in the Eulerian setting. Thirdly, the ALR derivation is more precise than either [8] and [7]. The fourth reason is to compare with the new surface SWEs proposed in [1].

The notation in this report follows [1].

2 ALR SWEs in two-dimensions

A key part of the ALR derivation is the use of the mean velocity

$$\overline{u} = \frac{1}{h}\int_0^h u(x,y,t)\,\mathrm{d}y\,. \tag{2.1}$$

By using the mean velocity the mass equation is exact

$$h_t + (h\overline{u})_x = 0\,. \tag{2.2}$$

The reduction of the momentum equations proceeds as follows. The translation accelerations $\ddot{\mathbf{q}}$ are dropped since they are neglected by ALR. (They do not however affect the derivation of ALR and could be added.) The starting point of the ALR derivation is the momentum equations in the form

$$\begin{aligned}\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= -g\sin\theta + 2\Omega v + \dot{\Omega}(y+d_2) + \Omega^2(x+d_1)\,,\\ \frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} &= -g\cos\theta - 2\Omega u - \dot{\Omega}(x+d_1) + \Omega^2(y+d_2)\,,\end{aligned} \tag{2.3}$$

The first assumption is to neglect the vertical acceleration

$$\frac{Dv}{Dt} \approx 0\,. \qquad (\mathbf{ALR\text{-}1})$$

The vertical momentum equation then reduces to

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\cos\theta - 2\Omega u - \dot{\Omega}(x+d_1) + \Omega^2(y+d_2)\,,$$

Integrate this equation over the entire depth

$$\frac{1}{\rho}p(x,y,t)\Big|^h - \frac{1}{\rho}p(x,0,t) = -gh\cos\theta - 2\Omega\int_0^h u\,\mathrm{d}y - \dot{\Omega}(x+d_1)h + \tfrac{1}{2}\Omega^2(h^2+2hd_2)\,.$$

Applying the dynamic free surface boundary condition, and neglecting surface tension, gives an expression for the pressure field at $y=0$

$$\frac{1}{\rho}p(x,0,t) = gh\cos\theta + 2\Omega h\overline{u} + \dot{\Omega}(x+d_1)h - \tfrac{1}{2}\Omega^2(h^2+2hd_2)\,.$$

Now consider the $x-$momentum equation in (2.3). To simplify this equation two assumptions are invoked

$$2\Omega v \approx 0\,, \qquad (\mathbf{ALR\text{-}2})$$

and

$$\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{in the } x-\text{momentum equation is evaluated at } y=0\,. \qquad (\mathbf{ALR\text{-}3})$$

With these assumptions the $x-$momentum equation simplifies to

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x}\Big|_{y=0} = -g\sin\theta + \dot{\Omega}(y+d_2) + \Omega^2(x+d_1)\,,$$

with

$$\frac{1}{\rho}\frac{\partial p}{\partial x}\Big|_{y=0} = gh_x\cos\theta + 2\Omega(h\overline{u})_x + \dot{\Omega}h + \dot{\Omega}(x+d_1)h_x - \Omega^2(h+d_2)h_x\,.$$

However, the system is still not closed. One additional assumption is required

$$\frac{Du}{Dt} \approx \overline{u}_t + \overline{u}\,\overline{u}_x\,. \qquad (\mathbf{ALR\text{-}4})$$

With this hypothesis, the $x-$momentum equation becomes

$$\overline{u}_t + \overline{u}\,\overline{u}_x + a(x,t)^{ALR}h_x = b(x,t;y)^{ALR}\,, \qquad (2.4)$$

with coefficients

$$\begin{aligned} a(x,t)^{ALR} &= g\cos\theta + \dot{\Omega}(x+d_1) - \Omega^2(h+d_2) + 2\Omega\overline{u}\,,\\ b(x,t;y)^{ALR} &= -g\sin\theta - 2\Omega h\overline{u}_x + \dot{\Omega}(y-h+d_2) + \Omega^2(x+d_1)\,. \end{aligned} \qquad (2.5)$$

The first two assumptions (**ALR-1**)-(**ALR-2**) are analogues of the ones used in the derivation of the surface equations in [1]. It is difficult to quantify the error in assumption (**ALR-3**). The error in Assumption (**ALR-4**) can be clarified however, since

$$\frac{1}{h}\int_0^h \frac{Du}{Dt}\,\mathrm{d}y - \overline{u}_t - \overline{u}\,\overline{u}_x = \frac{1}{h}\frac{\partial}{\partial x}\left(\int_0^h u^2\,\mathrm{d}y - h\overline{u}^2\right)\,.$$

Hence the error is small if the the right-hand side of this expression is small. A sufficient condition for neglect is when the depth-averaged velocity squared is close to the square of the depth-averaged velocity.

With appropriate change of notation, equations (2.4) with (2.5) correspond to equation (18) in [5]. Note that the vehicle acceleration $\ddot{\mathbf{q}}$ is absent in [5] but it can be easily added. The pair of equations $h_t + (h\overline{u})_x = 0$ and (2.4) are called the ALR SWEs.

With $\ddot{\mathbf{q}}$ neglected, and assuming $U \approx \overline{u}$, the ALR coefficients relative to a and b in [1] are

$$\begin{aligned} a(x,t)^{ALR} &= a(x,t) + 2\Omega\overline{u}\,, \\ b(x,t;y)^{ALR} &= b(x,t) + \dot{\Omega}(y-2h) - 2\Omega(h_t + h\overline{u}_x)\,. \end{aligned}$$

The comparison can be simplified by using the mass equation to eliminate the second term in $a(x,t)^{ALR}$ and the third term in $b(x,t;y)^{ALR}$. After this change the two systems are very close (assuming $U \approx \overline{u}$). The coefficient $b(x,t;y)^{ALR}$ stil depends on y. There are a number of choices for y: $y=0$, $y=\frac{1}{2}h$ (obtained by averaging), $y=h$ and $y=2h$. The most natural choice is $y=\frac{1}{2}h$ which is obtained by averaging. Another interesting choice is $y=h$. In this case the equations have an interesting variational principle (see §3 below). However, the conservation form is lost unless $y=2h$ which is not physically reasonable. Henceforth in discussing the ALR SWEs we will use the choice $y=h$.

It is remarkable that the ALR SWEs are very close to the surface equations, when U and $\overline{u}$ are identified, especially since the surface equations have only two assumptions, and the ALR SWEs have four assumptions. Although the two sets of SWEs are similar, there are still two principal advantages to using the surface SWEs: first it is very clear what the assumptions are in the derivation, and secondly, the derivation extends in a straightforward way to the case of three-dimensional rotating shallow-water flow, whereas deriving the SWEs in 3D with the average velocity is very difficult and not always unambiguous [2].

3 Variational principle: Eulerian form of SWEs

The ALR SWEs have a variational formulation. Take the ALR SWEs in (2.4) and (2.5) with $y=h$, and unit fluid density.

Introduce the Lagrangian functional

$$\mathscr{L}(h,u,\phi) = \int_{t_1}^{t_2}\int_0^L [KE - PE + \phi(h_t + (hu)_x)]\,\mathrm{d}x\,\mathrm{d}t\,,$$

with

$$\begin{aligned} KE &= \tfrac{1}{2}hu^2 - \Omega hu(h+d_2) + \tfrac{1}{2}\Omega^2\left(\tfrac{1}{3}h^3 + d_2h^2 + d_2^2h\right) \\ &\quad + \tfrac{1}{2}h\Omega^2(x+d_1)^2 - \tfrac{1}{2}\dot{\Omega}(x+d_1)(h+d_2)^2 \\ PE &= g\left(\tfrac{1}{2}h^2 + d_2h\right)\cos\theta + gh(x+d_1)\sin\theta\,. \end{aligned}$$

Note that mass conservation is introduced as a constraint, with Lagrange multiplier ϕ, which turns out to be a generalized velocity potential. The first variation of this Lagrangian functional with respect to h, u and ϕ recovers the ALR shallow-water equations.

To confirm, take variations

$$\frac{\partial \mathscr{L}}{\partial \phi} = 0 \quad \Rightarrow \quad h_t + (hu)_x = 0$$
$$\frac{\partial \mathscr{L}}{\partial u} = 0 \quad \Rightarrow \quad \phi_x = u - \Omega(h + d_2)\,,$$

and $\frac{\partial \mathscr{L}}{\partial h} = 0$ gives

$$\begin{aligned}\phi_t + u\phi_x &= \tfrac{1}{2}u^2 - \Omega u(h+d_2) - \Omega h u + \tfrac{1}{2}\Omega^2(h+d_2)^2 \\ &\quad + \tfrac{1}{2}\Omega^2(x+d_1)^2 - \dot{\Omega}(x+d_1)(h+d_2) \\ &\quad - g(h+d_2)\cos\theta - g(x+d_1)\sin\theta\,.\end{aligned}$$

Substituting in for ϕ_x and simplifying

$$\begin{aligned}\phi_t + \tfrac{1}{2}u^2 &= -\Omega h u + \tfrac{1}{2}\Omega^2(h+d_2)^2 \\ &\quad + \tfrac{1}{2}\Omega^2(x+d_1)^2 - \dot{\Omega}(x+d_1)(h+d_2) \\ &\quad - g(h+d_2)\cos\theta - g(x+d_1)\sin\theta\,.\end{aligned}$$

This is a form of Bernoulli's equation for the shallow-water equations in a rotating vessel. Differentiate with respect to x and substitute again for ϕ_x

$$\begin{aligned}u_t - \dot{\Omega}(h+d_2) - \Omega h_t + u u_x &= \Omega h_t + \Omega^2(h+d_2)h_x \\ &\quad + \Omega^2(x+d_1) - \dot{\Omega}(h+d_2) - \dot{\Omega}(x+d_1)h_x \\ &\quad - g\cos\theta\, h_x - g\sin\theta\,.\end{aligned}$$

or

$$u_t + u u_x + (g\cos\theta - \Omega^2(h+d_2) + \dot{\Omega}(x+d_1))h_x = 2\Omega h_t + \Omega^2(x+d_1) - g\sin\theta\,,$$

recovering the momentum equation in the ALR form of the SWEs.

In this derivation, the rotation motion is assumed to be prescribed. θ can also be considered an unknown, and then the variation with respect to θ, $\frac{\delta \mathscr{L}}{\delta \theta} = 0$, gives an equation for the vessel motion. For the complete equation, including the motion of the vessel, the kinetic and potential energy of the vessel need also to be included. The coupled problem is studied in [4].

References

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