

Review of the Huang-Hsiung rotating two-dimensional shallow-water equations

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1 Introduction

Although DILLINGHAM [3] gave the first derivation of the shallow water equations (SWEs) relative to a moving frame of reference (see [1] for a review of DILLINGHAM's and other derivations), there was an independent derivation by HUANG & HSIUNG [4, 6] (hereafter HH). In this report we review the HH derivation and identify their key assumptions. The reason for this report is threefold. The HH derivation has similarities with, but also differences from, the derivation of DILLINGHAM. Secondly, they extend their derivation to the three-dimensional (3D) case so it gives some insight into their approach (their 3D derivation is reviewed in [2]). The third reason is to compare with the new surface SWEs proposed in [1].

2 HH SWEs in two-dimensions

Suppose the flowfield is two-dimensional (one horizontal dimension and one vertical dimension), with coordinates (x, y) , and the fluid lies in a tank of length L with fluid depth described by the graph $y = h(x, t)$. HH choose a representative horizontal velocity u which is independent of y , and they propose the following variant of the SWEs relative to the rotating frame

$$\begin{aligned} u_t + uu_x + a(x, t)^{HH} h_x &= b(x, t)^{HH} \\ h_t + (hu)_x &= 0, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} a(x, t)^{HH} &= g \cos \theta + \ddot{q}_2 + \dot{\Omega}(x + d_1) + 2\Omega u - \Omega^2(h + d_2), \\ b(x, t)^{HH} &= -g \sin \theta - \ddot{q}_1 + \Omega^2(x + d_1) + \dot{\Omega}d_2. \end{aligned} \tag{2.2}$$

We have altered their notation in order to compare with [1]. In [1] the offset is denoted by $\mathbf{d} = (d_1, d_2)$, and in the HH equations the offset is fixed at $d_1 = 0$ and $d_2 = -z_g$. In HH notation the acceleration $\ddot{\mathbf{q}}$ is denoted by $\ddot{q}_1 = \dot{u}_2$ and $\ddot{q}_2 = \dot{u}_3$; they are, respectively, the ship sway and heave accelerations relative to the

body frame. Also $\theta = e_1$ and $\Omega = u_4$ are respectively the roll angle and angular velocity. The angular velocity is relative to the body frame which in the case of two-dimensions coincides with the spatial angular velocity since

$$\mathbf{Q}^T \dot{\mathbf{Q}} = \dot{\mathbf{Q}} \mathbf{Q}^T = \dot{\theta} \mathbf{J}, \quad \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and any proper rotation matrix in \mathbb{R}^2 is of the form

$$\mathbf{Q}(t) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

2.1 Derivation of HH rotating SWEs

The continuity equation and Euler's equations of motion in two dimensions are

$$\begin{aligned} u_x + v_y &= 0, \\ u_t + uu_x + vv_y + \frac{1}{\rho} \frac{\partial p}{\partial x} &= -g \sin \theta + 2\Omega v + \dot{\Omega} (y + d_2) + \Omega^2 (x + d_1) - \ddot{q}_1, \\ v_t + uv_x + vv_y + \frac{1}{\rho} \frac{\partial p}{\partial y} &= -g \cos \theta - 2\Omega u - \dot{\Omega} (x + d_1) + \Omega^2 (y + d_2) - \ddot{q}_2. \end{aligned} \tag{2.3}$$

The boundary conditions are

$$u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L, \quad v = 0 \quad \text{at} \quad y = 0, \tag{2.4}$$

and

$$p = 0 \quad \text{and} \quad h_t + uh_x = v, \quad \text{at} \quad y = h(x, t). \tag{2.5}$$

Based on the shallow water assumption, HH assume that u is a function of horizontal space coordinate and time and does not depend on the vertical space coordinate

$$u = u(x, t). \tag{HH-1}$$

Then integrating the continuity equation from $y = 0$ to $y = h(x, t)$ leads to

$$\int_0^h (u_x + v_y) dy = hu_x + v|_0^h = hu_x + h_t + uh_x = h_t + (hu)_x = 0, \tag{2.6}$$

using the bottom and kinematic free surface boundary conditions. Note that u in equation (2.6) is equivalent to the depth-averaged horizontal velocity,

$$\bar{u} = \frac{1}{h} \int_0^h u(x, y, t) dy, \tag{2.7}$$

since

$$\begin{aligned} (h\bar{u})_x &= \frac{\partial}{\partial x} \int_0^h u(x, y, t) dy = h_x u|_0^h + \int_0^h u_x dy = h_x u|_0^h - \int_0^h v_y dy \\ h_x u|_0^h - h_t - uh_x|_0^h &= -h_t \quad \Rightarrow \quad h_t + (h\bar{u})_x = 0. \end{aligned}$$

The second assumption is to neglect the vertical acceleration

$$\frac{Dv}{Dt} \approx 0. \quad (\text{HH-2})$$

The vertical momentum equation then reduces to

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = -g \cos \theta - 2\Omega u - \dot{\Omega}(x + d_1) + \Omega^2(y + d_2) - \ddot{q}_2,$$

Integrating this equation from any point y to the surface h gives an expression for the pressure field at any point y

$$\frac{1}{\rho} p(x, y, t) = \left(g \cos \theta + 2\Omega u + \dot{\Omega}(x + d_1) + \ddot{q}_2 \right) (h - y) - \frac{1}{2} \Omega^2 \left((h + d_2)^2 - (y + d_2)^2 \right),$$

applying the dynamic free surface boundary condition. $\frac{1}{\rho} \frac{\partial p}{\partial x}$ is needed for the x -momentum equation

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \left(g \cos \theta + 2\Omega u + \dot{\Omega}(x + d_1) - \Omega^2(h + d_2) + \ddot{q}_2 \right) h_x + \left(2\Omega u_x + \dot{\Omega} \right) (h - y). \quad (2.8)$$

Substituting equation (2.8) into the x -momentum equation gives

$$\begin{aligned} u_t + uu_x + vu_y + \left(g \cos \theta + 2\Omega u + \dot{\Omega}(x + d_1) - \Omega^2(h + d_2) + \ddot{q}_2 \right) h_x \\ + \left(2\Omega u_x + \dot{\Omega} \right) (h - y) = -g \sin \theta + 2\Omega v + \dot{\Omega}(y + d_2) + \Omega^2(x + d_1) - \ddot{q}_1. \end{aligned} \quad (2.9)$$

Note that $vu_y = 0$ as $u = u(x, t)$. The third assumption is to neglect the horizontal Coriolis force

$$2\Omega v \approx 0. \quad (\text{HH-3})$$

Then integrating equation (2.9) over the entire depth leads to

$$\begin{aligned} u_t + uu_x + \left(g \cos \theta + 2\Omega u + \dot{\Omega}(x + d_1) - \Omega^2(h + d_2) + \ddot{q}_2 \right) h_x \\ + \boxed{\Omega h u_x} = -g \sin \theta + \dot{\Omega} d_2 + \Omega^2(x + d_1) - \ddot{q}_1. \end{aligned} \quad (2.10)$$

The fourth assumption is to neglect the boxed term

$$\Omega h u_x \approx 0. \quad (\text{HH-4})$$

Then the x -momentum equation simplifies to

$$u_t + uu_x + a(x, t)^{HH} h_x = b(x, t)^{HH}, \quad (2.11)$$

with $a(x, t)^{HH}$ and $b(x, t)^{HH}$ given in (2.2).

3 Comparison with the surface SWEs in [1]

The new surface equations are derived in [1]. The surface momentum equation neglecting surface tension is

$$U_t + UU_x + a(x, t)h_x = b(x, t), \quad (3.12)$$

where

$$\begin{aligned} a(x, t) &= g \cos \theta + \dot{\Omega}(x + d_1) - \Omega^2(h + d_2) - \ddot{q}_1 \sin \theta + \ddot{q}_2 \cos \theta \\ b(x, t) &= 2\Omega h_t - g \sin \theta + \dot{\Omega}(h + d_2) + \Omega^2(x + d_1) - \ddot{q}_1 \cos \theta - \ddot{q}_2 \sin \theta. \end{aligned} \quad (3.13)$$

In equation (3.12) the translational accelerations \ddot{q}_1 and \ddot{q}_2 are relative to the spatial frame and if we replace them with the body translational accelerations

$$\ddot{\mathbf{q}} = \mathbf{Q}^T \ddot{\mathbf{q}} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}, \quad (3.14)$$

$$\begin{aligned} \ddot{q}_1 &= \ddot{q}_1 \cos \theta + \ddot{q}_2 \sin \theta \\ \ddot{q}_2 &= -\ddot{q}_1 \sin \theta + \ddot{q}_2 \cos \theta, \end{aligned} \quad (3.15)$$

then $a(x, t)$ and $b(x, t)$ simplifies to

$$\begin{aligned} a(x, t) &= g \cos \theta + \dot{\Omega}(x + d_1) - \Omega^2(h + d_2) + \ddot{q}_2 \\ b(x, t) &= 2\Omega h_t - g \sin \theta + \dot{\Omega}(h + d_2) + \Omega^2(x + d_1) - \ddot{q}_1. \end{aligned} \quad (3.16)$$

Assume that the horizontal surface velocity U in the surface momentum equation is equivalent to the horizontal velocity in HH momentum equation, then comparison of the coefficients shows that

$$\begin{aligned} a(x, t)^{HH} &= a(x, t) + 2\Omega u \\ b(x, t)^{HH} &= b(x, t) - 2\Omega h_t - \dot{\Omega}h. \end{aligned}$$

Hence we expect the two formulations of the SWEs to give similar results when

$$|2\Omega U| \ll 1 \quad \text{and} \quad |2\Omega h_t + \dot{\Omega}h| \ll 1.$$

Numerical results comparing the 2D surface SWEs with the 2D HH SWEs are presented in [1]. A review of the HH equations in 3D is given in [2].

References

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