HOPF-GALOIS THEORY AND GALOIS MODULE STRUCTURE

UNIVERSITY OF EXETER, 23–26 June 2015

Abstracts of talks

Victor Abrashkin:

Arithmetic structure of Galois groups of p-extensions of local fields with nilpotent class $< p$

Abstract: Suppose K is a complete discrete valuation field with finite residue field of characteristic p and M is a natural number. If K has characteristic 0 we assume that K contains a primitive p^M-th root of unity. For such fields K we give an explicit construction of the maximal quotient of their absolute Galois group of period p^M and nilpotent class $< p$ together with the appropriate filtration by ramification subgroups in upper numbering. Our constructions essentially use the nilpotent Artin-Schreier theory, the field-of-norms functor and the equivalence of the categories of p-groups and Lie algebras of nilpotent class $< p$.

Alex Bartel:

Heuristics for Arakelov class groups of number fields

Abstract: I will report on an ongoing joint project with Hendrik Lenstra, whose aim is to develop a general conceptual heuristic for the behaviour of Galois modules, such as ideal class groups, or unit groups, or Arakelov class groups, in families. To do that, one needs to be able to compare sizes of automorphism groups of finitely generated $\mathbb{Z}[G]$-modules, even when those automorphism groups are infinite. We develop a general theory of commensurability of groups, of rings, and of modules, which allows us to do just that. I will explain how to use this theory to recover, in a conceptual way, the original heuristic of Cohen-Lenstra-Martinet on ideal class groups, and if time permits, I will conclude with some mysteries that remain, which have to do with our poor understanding of multiplicative Galois module structure.

Nigel Byott:

Scaffolds and (Generalised) Galois Modules

Abstract: Scaffolds provide new approach to Galois module structure questions in totally ramified extensions of local fields. I will explain the motivation behind scaffolds from two perspectives, one algebraic (inseparable field extensions and divided power Hopf algebras) and one arithmetic (Galois module structure for extensions of prime degree). I will then explain how scaffolds, when they exist, give rise to a great deal of explicit information in a uniform manner, and illustrate this by considering

the case of weakly (and totally) ramified extensions. This is joint work with Griff Elder and Lindsay Childs.

Lindsay Childs:

Commutative nilpotent algebras and Hopf Galois structures

Abstract: Let L/K be a Galois extension of fields with Galois group G an elementary abelian p-group of order p^n, $p \geq 3$. The number of Hopf Galois structures on L/K of type G is equal to the number of regular subgroups of type G in $\mathrm{Hol}(G) \subset \mathrm{Perm}(G)$. By a result of Caranti et al., these in turn arise from commutative nilpotent algebra structures A on G (viewed as an additive group). We focus on algebras A with $A^3 = 0$ and obtain an asymptotic lower bound on the number of Hopf Galois structures on L/K for large n. We show how algebras A with $A^3 = 0$ yield Hopf Galois structures on L/K efficiently. For a two-parameter class of such algebras we determine for all n how many Hopf Galois structures correspond to each isomorphism type.

Alessandro Cobbe:

Equivariant epsilon constant conjectures for weakly ramified extensions

Abstract: We study the local epsilon constant conjecture as formulated by M. Breuning. This conjecture fits into the general framework of the equivariant Tamagawa number conjecture (ETNC) and should be interpreted as a consequence of the expected compatibility of the ETNC with the functional equation of Artin-L-functions.
Let $K/\mathbb{Q}_p$ be unramified. Under some mild technical assumption we prove Breuning's conjecture for weakly ramified abelian extensions N/K with cyclic ramification group. As a consequence of Breuning's local-global principle we obtain the validity of the global epsilon constant conjecture as formulated by Bley and Burns and of Chinburg's $\Omega(2)$-conjecture for certain infinite families F/E of weakly and wildly ramified extensions of number fields.
This is a joint work with Werner Bley.

Teresa Crespo:

Induced Hopf Galois extensions

Abstract: In this talk we shall consider a finite Galois extension K/k with Galois group G such that G is the semi-direct product of a subgroup G' and a normal subgroup H. For F the subfield of K fixed by G' we shall prove that a Hopf Galois structure on F/k of type N_1 together with a Hopf Galois structure on K/F of type N_2 induces a Hopf Galois structure on K/k of type the direct product $N_1 \times N_2$. We shall call induced such Hopf Galois structures and split those Hopf Galois structures of type a direct product. We shall present a characterization of the split Hopf Galois structures of K/k which

are induced. We shall give examples of induced and split non-induced Hopf Galois structures, determine the number of induced Hopf Galois structures for some Galois extensions and obtain some cases in which all split structures are induced.
This is a joint work with Anna Rio and Montserrat Vela.

Ilaria del Corso:

On wild extensions of p-adic fields (joint work with Roberto Dvornicich)

Abstract: We consider the problem of classifying the isomorphism classes of the wildly ramified extensions of a p-adic field.
This essentially reduces to the study of the extensions of degree p^k. For the case $k = 1$ and also for $k > 1$ but restricting to the case of extensions without intermediate fields, we establish a correspondence between the isomorphism classes of these extensions and some Kummer extensions of a suitable field F containing K. We then describe such classes in terms of the representations of $\mathrm{Gal}(F/K)$.
For $k = 2$ and for each possible Galois group G, we count the number of isomorphism classes of the extensions whose normal closure has a Galois group isomorphic to G. As a byproduct, we get the total number of isomorphism classes.

Griff Elder:

From Galois scaffolds to Hopf orders in group rings.

Abstract: Given a Galois extension L/K of local fields with Galois group G, it is possible to determine the Galois module structure of the ring of ring of integers (or some other fractional ideal) if there exists a scaffold of sufficient tolerance for the action of the the group algebra $K[G]$ on L. Using results of Bondarko, it is then furthermore possible to explicitly describe the associated orders that are Hopf orders in $K[G]$. The presentation of these Hopf orders follows a certain pattern, leading to a conjecture concerning the explicit description of Hopf orders in group algebras K[G]. In this talk, I will discuss progress in the construction of Galois scaffolds (joint work with Nigel Byott), as well as evidence for the conjecture (joint work with Rob Underwood and Cornelius Greither).

Cornelius Greither:

Explicit constructions of normal bases for the square root of the codifferent

Abstract: In the first part we generalise work of Pickett, who constructed explicit self-dual normal bases for the inverse different in weakly ramified extensions M/K of local fields. Pickett's research was presumably motivated by the fact that the inverse different, as a module, is indeed self-dual under the trace form, and perhaps also by a counterexample (found by Vinatier) to

the existence of a self-dual NIB in the corresponding global situation. There are two main differences between our approach and Pickett's: we do not need the base field K to be unramified over the p-adic numbers, and we get by without using Dwork series. We only rely on basic facts about ramification, a little Kummer theory, and explicit cyclotomic descent. However we have to restrict ourselves to *cyclic* extensions M/K, just as in Pickett's work.
In a second part, we sketch an inductive construction of *noncyclic* weakly ramified extensions, which does lead to an explicit construction of normal bases for the inverse different. There are two things one has to put up with: self-duality is lost, and we need a technical tool, Artin-Hasse series, which is rather similar to but more flexible than the Dwork series in Pickett's paper. It is hoped that our approach will also have useful consequences in global Galois theory, similarly as Pickett's original paper, which led to the beautiful joint paper of Pickett and Vinatier in Compositio Math. 2013.

Henri Johnston:

Explicit integral Galois module structure of weakly ramified extensions of local fields

Abstract: Let L/K be a finite Galois extension of complete local fields with finite residue fields and let $G = \mathrm{Gal}(L/K)$. Let G_1 and G_2 be the first and second ramification groups. Thus L/K is tamely ramified when G_1 is trivial and we say that L/K is weakly ramified when G_2 is trivial. Let $\mathcal{O}_L$ be the valuation ring of L and let $\mathfrak{P}_L$ be its maximal ideal. We show that if L/K is weakly ramified and $n \equiv 1 \bmod |G_1|$ then $\mathfrak{P}_L^n$ is free over the group ring $\mathcal{O}_K[G]$, and we construct an explicit generating element. Under the additional assumption that L/K is wildly ramified, we then show that every free generator of $\mathfrak{P}_L$ over $\mathcal{O}_K[G]$ is also a free generator of $\mathcal{O}_L$ over its associated order in the group algebra $K[G]$.

Alan Koch:

Hopf orders in Hopf algebras with trivial Verschiebung

Abstract: Let R be a discrete valuation ring of characteristic p with field of fractions K. Let H be a finite abelian K-Hopf algebra of p-power rank such that $\mathrm{Spec}(H)$ is killed by the Verschiebung map. Using a theory of Dieudonné modules, we construct Hopf orders in H. Examples will be given, including the following cases: H is monogenic with primitive nilpotent generator; H is a product of simple local-local Hopf algebras; and $H = KG^*$, G an elementary abelian group.

Bernhard Koeck:

Euler characteristics and epsilon constants of curves over finite fields - some wild stuff

Abstract: Given a smooth projective curve over a finite field equipped with an action of a finite group, we briefly introduce the corresponding

Artin L-function and a certain equivariant Euler characteristic. The main result is a precise relation between the epsilon constants appearing in the functional equations of Artin L-functions and that Euler characteristic. This generalises a theorem of Chinburg from the tamely to the weakly ramified case.

Lena Sundukova:

Galois module structure of ideals: some consequences of having a Galois scaffold

Abstract: Let L/K be a totally ramified Galois extension of local fields having degree p^n, where p is the residue characteristic. We suppose that L/K admits a Galois scaffold of "high enough" tolerance and that its ramification breaks satisfy the conclusion of the Hasse-Arf theorem. Then, without much loss of generality, we can assume that K has characteristic p, the scaffold has infinite tolerance, and there is only one ramification break. The existence of the Galois scaffold then gives an abundance of combinatorial data relating to the Galois module structure of ideals of the valuation ring of L. We will see how this can be unpacked to give explicit statements about the freeness, or otherwise, of these ideals over their associated orders.

Alex Torzewski:

Multiplicative Galois module structure and class groups

Abstract: Let K/F be a Galois extension of number fields with Galois group G, let be p be a prime such that G has a Sylow p-subgroup of order p. Then we explain how to obtain partial information on the Galois structure of the integral units modulo roots of unity from the structure of ideal class groups. As a corollary, we deduce that when G is cyclic of square-free order, the genus of the units modulo roots of unity is determined by the class groups of the intermediate fields. When G is dihedral of order $2p$, where p is an odd prime, we combine this approach with the technique of regulator constants to reach the same conclusion.

Dajano Tossici:

Kummer theory for models of finite diagonalizable groups

Abstract: Let R be a discrete valuation ring with residue field of characteristic $p > 0$. Let us consider a finite diagonalizable group scheme D over K, the fraction field of R. The group D can be seen as the kernel of an isogeny, essentially Kummer isogeny. As well known this isogeny is very important for the description of D-torsors. In a paper with Ariane Mézard and Matthieu Romagny we concentrate in the case D equal to $\mu_{p^n,K}$ and we classify all possible group schemes over R which are isomorphic to $\mu_{p^n,K}$ (we call models, groups with this property) and such that they are the kernel of an isogeny, between smooth group schemes over R, which extends the

isogeny over K. We call these groups Kummer group schemes, since they have naturally a sort of Kummer Theory to describe torsors under these groups schemes. In the same paper we conjectured that all models of $\mu_{p^n,K}$ arise in this way. The conjecture is true for $\mu_{p^2,K}$. In a joint work with Matthieu Romagny we are working on this conjecture, and in fact we are trying to consider the general case, with D any finite diagonalizable group. In the talk I will recall the results obtained with Mézard and Romagny and I will explain the strategy to prove the conjecture (which it is not yet proven).

Rob Underwood:

Finite Group Scheme Extensions in Characteristic p

Abstract: Let R be a discrete valuation ring with field of fractions K and residue class field k with $\mathrm{char}(k) = \mathrm{char}(K) = p$. Let C_p^2 denote the elementary abelian group of order p^2, and let H be an R-Hopf order in KC_p^2. In this talk we give a complete classification of Hopf orders in KC_p^2. This is joint work with G. Griffith Elder.

Stephane Vinatier:

Gauss sums, Jacobi sums and cyclotomic units related to torsion Galois modules

Abstract: Let G be a finite group and let N/E be a tamely ramified G-Galois extension of number fields whose inverse different $\mathcal{C}_{N/E}$ is a square. Let $\mathcal{A}_{N/E}$ denote the square root of $\mathcal{C}_{N/E}$. Then $\mathcal{A}_{N/E}$ is a locally free $\mathbb{Z}[G]$-module, which is in fact free provided N/E has odd order, as shown by Erez. Using M. Taylor's theorem, we can rephrase this result by saying that, when N/E has odd degree, the classes of $\mathcal{A}_{N/E}$ and $\mathcal{O}_N$ (the ring of integers of N) in $\mathrm{Cl}(\mathbb{Z}[G])$ are equal (and in fact both trivial). We show that the above equality of classes still holds when N/E has even order, assuming that N/E is locally abelian. This result is obtained through the study of the Fröhlich representatives of the classes of some torsion modules, which are independently introduced in the setting of cyclotomic number fields. Jacobi sums, together with the Hasse-Davenport formula, are involved in this study. Finally, when G is the binary tetrahedral group, we use our result in conjunction with Taylor's theorem to find a tame G-Galois extension whose square root of the inverse different has nontrivial class in $\mathrm{Cl}(\mathbb{Z}[G])$. This is joint work with Luca Caputo.