Explicit integral Galois module structure of weakly ramified extensions of local fields

Thank members of audience, including:
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**Setup:** finite Galois extension of local fields

\[
\begin{align*}
\mathcal{E} & = \mathcal{O}_L / \mathcal{P}_L \\
K & = \mathcal{O}_K / \mathcal{P}_K
\end{align*}
\]

Residue fields assumed to be finite

Concerned with:
(i) \( \mathcal{P}_L \) as an \( \mathcal{O}_K [G] \)-module
(ii) \( \mathcal{O}_L \) as an \( \mathcal{U}_L \)-module \( \mathcal{U}_L = \{ x \in \mathcal{O}_K : x \mathcal{O}_L \subseteq \mathcal{O}_L \} \)

**Ramification groups** For \( i \geq 1 \)

\[
G_i = \{ g \in G : (g - 1)(\mathcal{O}_L) \subseteq \mathcal{P}_L^{i+1} \}
\]

Hence

\[
\begin{align*}
\mathcal{U}_K & \text{ unramified } \iff G_0 = 1 \\
\mathcal{U}_K & \text{ ramified } \iff G_1 = 1 \\
\mathcal{U}_K & \text{ weakly ramified } \iff G_2 = 1
\end{align*}
\]

Tame case
\[
\begin{align*}
\mathcal{U}_K & \text{ free over } \mathcal{O}_K [G] \text{ (Neither*)} \\
\text{Kawamoto (1986) constructed explicit generators}
\end{align*}
\]
Unramified case

\[ G(\frac{L}{E}) \quad G(\frac{L}{K}) \]

By Normal Basis Theorem, there exists \( \mathcal{O}_L = \mathbb{K}[G] \cdot \beta \).

Using Nakayama's Lemma, for any lift \( \beta \in \mathcal{O}_L \) of \( \beta \), \( \mathcal{O}_L = \mathbb{K}[G] \cdot \beta \). (Can make into an iff statement.)

Totally and tamely ramified case

\[ G(\frac{L}{K}) \quad \text{Let } \epsilon = [L:K]. \]

Uniformizes \( \pi_L \in \pi_K \) s.t. \( \pi_L^\epsilon = \pi_K \cdot \mathcal{O}_L = \mathcal{O}_K[G] \).

Let \( \alpha \in \mathcal{O}_L \). Then \( \alpha = u_0 + u_1 \pi_L + \ldots + u_{\epsilon - 1} \pi_L^{\epsilon - 1} \), \( u_i \in \mathbb{K} \).

\[ \mathcal{O}_L = \mathbb{K}[\alpha] , \alpha \Rightarrow u_i \in \mathbb{K} \quad \forall i \]

(In particular, \( \alpha = 1 + \pi_L + \pi_L^2 + \ldots + \pi_L^{\epsilon - 1} \) is a generator.)

Proof: Use that \( \pi_L \) is a Kummer generator, determinant calculations.

Idea: "Glue" the two cases together.

Weakly ramified case

Ullom:

(i) If \( \mathfrak{m} \in \mathcal{O}_L \) s.t. \( \beta_L^n \) free over \( \mathcal{O}_L[\mathfrak{m}] \), then \( \mathcal{O}_L \) weakly ramified.

(ii) If \( L/K \) totally & weakly ramified, then \( \beta_L^n \) free over \( \mathcal{O}_L[\mathfrak{m}] \).

Köck: \( \beta_L^n \) free over \( \mathcal{O}_L[\mathfrak{m}] \) if \( L/K \) weakly ramified & \( n \equiv 1 \mod |G| \).
Proof uses cohomological triviality argument 
(erez's work on square root of inverse different uses similar ideas)

Does not construct explicit generators.

Theorem 1. \( \mathbb{A}/\mathbb{C} \) weakly ramified. Let \( \mathbb{A} \) st. \( n \equiv 1 \mod \left| \mathbb{C} \right| \). Then one can explicitly construct \( \mathbb{E} \) s.t. \( \mathbb{P}_n = \mathbb{O}_L \mathbb{C}[\mathbb{E}] \).

Theorem 2. \( \mathbb{A}/\mathbb{C} \) weakly ramified. Take any uniformizer of \( \mathbb{A} \). Then \( \mathbb{U}/\mathbb{C} = \mathbb{O}_L \mathbb{C}[\mathbb{P}_n] \mathbb{C} \) and if \( \mathbb{P}_n = \mathbb{O}_L \mathbb{C}[\mathbb{E}] \) then \( \mathbb{O}_L = \mathbb{U}/\mathbb{C} \mathbb{C} \).

Idea of proof of Theorem 1: Explicitly construct generators in following cases:

(i) unramified
(ii) totally & finitely ramified
(iii) totally & weakly ramified extension

Then use "splitting lemma".

"Glue" generators together.

Take trace.

This is a generalisation of Kawamoto's approach.

Totally & weakly ramified extension

Thm. \( \mathbb{A}/\mathbb{C} \) totally & weakly ramified extension \( (p=char \mathbb{R} \neq) \):

(i) \( G \) elementary abelian \( p \)-group (standard)
(ii) \( \mathbb{P}_n \) free over \( \mathbb{O}_L \mathbb{C}[\mathbb{G}] \implies n \equiv 1 \mod \left| \mathbb{C} \right| \) (known by koch)
(iii) Suppose \( n \equiv 1 \mod \left| \mathbb{C} \right| \).

Then \( \mathbb{S} \mathbb{C} \) free gen of \( \mathbb{P}_n \) over \( \mathbb{O}_L \mathbb{C}[\mathbb{G}] \).

\[ \Rightarrow \mathbb{V}_L(\mathbb{S}) = n \]
(iii) Already shown by others (sometimes with restrictions) by Vostokov, Vinaver (tByott), Byott, & Elder.

(In fact, works for perfect residue fields of the characteristic)

Proof Elementary
- Use Hilbert's formula to compute different of $L/K$.
- Obtain formula for $Tr_{L/K}(B_L)$.
- "Mod out" by $B_K$ as work over $\mathbb{F}[G]$.
- Use (minor variant) of result of Childs.
- Lift using Nbakoyama.

Example

$K(\tilde{\mathfrak{p}}^2)$

(lift time)

\[
\begin{array}{c}
\text{weakly ramified } \left( \left. \begin{array}{c} L \\ K \end{array} \right| \text{degree } p \\
\left. \begin{array}{c} K \\ \mathbb{Q}_p \end{array} \right| \text{unramified} \\
\text{Totally } \& \text{ weakly ramified extensions of arbitrary degree}
\end{array}
\right.
\]

$L/K$ $I = \mathcal{W} \times C$ by Schur-Zassenhaus

\[
\begin{array}{c}
\text{with } \mathcal{W} \text{ cyclic} \\
\text{and } C = \text{elementary abelian } p\text{-extension}
\end{array}
\]

L/E, F/K totally weakly ram $p$-extensions

$L/\mathbb{Q}_p, \mathbb{F}_p/K$ totally and tamely ramified.

Define $c$ by $|\mathcal{W}|=p^c$, let $c=1c$.

By Bézout $\exists a, b \in \mathbb{Z}$

$st. \quad ap^c+bc=1$. 
Proposition. \( \pi_E \) any uniformizer \( \alpha = \tau + \pi_E + \pi_E^2 + \ldots + \pi_E^{e-1} \).
Then \( \pi_P \pi_E^a \cdot \alpha \) free gen of \( P_L \) over \( \mathcal{O}_K[\mathbb{I}] \).

**Proof**

(i) \( v_L(\pi_P \pi_E^a) = 1 \) so \( \mathcal{P}_L = \mathcal{O}_E[W] \cdot (\pi_P \pi_E^a) \)

(ii) \( \pi_E^a \mathcal{O}_E = \mathcal{O}_K[\mathbb{I}] \cdot (\pi_E^a \alpha) \)

Do explicit calculation using semidirect product

and hence \( \pi_F \mathcal{P}_E = \mathbb{F}_L \cdot \pi_E, \alpha \in \mathcal{E} = \mathcal{L}^W \).

(Optional) \( W = \{ 2, 3 \} \quad C = \{ 6, 3 \} \).

Starts like:

\[
P_L = \mathcal{O}_E(W) \cdot (\pi_P \pi_E^a) \]

\[
= \bigoplus \pi \mathcal{O}(\pi_P \pi_E^a) \pi \mathcal{O}_E \]

\[
= \bigoplus \pi \mathcal{O}(\pi_P \pi_E^a) \pi \mathcal{O}_E \quad \text{since} \quad \pi_E \mathcal{E} = \mathcal{L}^W
\]

\[
= \bigoplus \pi \mathcal{O}(\pi_P \pi_E^a) \pi \mathcal{O}_E \quad \text{(use vii)} \quad \cdots
\]

\[
= \bigoplus \pi \mathcal{O}(\pi_P \pi_E^a) \pi \mathcal{O}_E
\]

Remark. If \( L/K \) abelian, totally \& wildly ramified, not of \( p \)-power degree then \( L/K \) cannot be weakly ramified.

*\( \mathbb{Q}_p(\sqrt[p]{2}) \)\* not weakly tamely ramified \( \quad \text{But} \quad \mathbb{Q}_3(\sqrt[p]{3^3 \sqrt{2}}) / \mathbb{Q}_3 \) Galois group \( S_3 \) totally \& weakly ramified.

*\( \mathbb{Q}_p\)\* weakly ramified

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(typeo in paper)
Doubly split extensions

$L/k$ finite Galois ext. of local fields.
$G = \text{Gal}(L/k)$, $I = G_0$, $W = G_1$.

We say $L/k$ is:

i) split wrt inertia if $G = I \times U$
   for some (cyclic) $U$ (so $L/U$ unramified).

ii) split wrt wild inertia if $G = W \times T$
    for some $T$ (so $L/T$ tamely ramified).

iii) doubly split if $T, C \leq I$ and $W \times C \leq U$.

So

$G = W \times T = W \times (C \times U) = (W \times C) \times U = I \times U$.

Rank: Automorphic in totally ramified case by Schur Zassenhaus.

Glue generators together for doubly split extension.

Lemma $L/k$ finite Galois ext. of local fields.

Let $k'/k$ be unique unramified extension of degree $[L:k]$. Let $L' = L[k']$. Then

(i) $L'/k$ Galois

(ii) $\text{Gal}(L'/k')$ inertia subgroup

(iii) $L'/k$ doubly split

(iv) if $L/k$ weak ram then $L'/k$ weak ram.

Proof Group Theory.

For general $L/k$, construct gen $E'$ for $L'/k$.
Then $E' = \text{Tr}_{L/L}(E)$ is gen for $L/k$. 