Hopf-Galois Theory and Galois Module Structure
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*Induced Hopf Galois structures*

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Let $K/k$ be a separable field extension of degree $n$, $\tilde{K}$ its Galois closure, $G = \text{Gal}(\tilde{K}/k)$, $G' = \text{Gal}(\tilde{K}/K)$. A Hopf Galois structure on $K/k$ may be given, equivalently, by

- a finite cocommutative $k$-Hopf algebra $\mathcal{H}$ and a Hopf action of $\mathcal{H}$ on $K$, i.e a $k$-linear map $\mu : \mathcal{H} \to \text{End}_k(K)$ inducing a bijection $K \otimes_k \mathcal{H} \to \text{End}_k(K)$. (Chase-Sweedler)

- a regular subgroup $N$ of $S_n$ normalized by $\lambda(G')$, where $\lambda : G \to S_n$ is the morphism given by the action of $G$ on the left cosets $G/G'$. (Greither-Pareigis)

If $N \subset \lambda(G')$, equivalently $N$ is a normal complement of $G'$ in $G$, $K/k$ is called almost classically Galois.

- a group monomorphism $\varphi : G \to \text{Hol}(N)$ such that $\varphi(G')$ is the stabilizer of $1_N$, where $\text{Hol}(N) = N \rtimes \text{Aut } N \hookrightarrow \text{Sym}(N)$ is defined by sending $n \in N$ to left translation by $n$ and $\sigma \in \text{Aut } N$ to itself. (Childs, Byott)

$N \subset S_n$ regular, normalized by $\lambda(G) \leftrightarrow \mathcal{H} = \tilde{K}[N]^G$

\{ Hopf subalgebras of $\mathcal{H}$\} $\leftrightarrow$ \{ $G$-stable subgroups of $N$\}

For $N'$ a $G$-stable subgroup of $N$, $K^{N'} := K^\mathcal{H'}$ for $\mathcal{H}'$ the Hopf subalgebra of $\mathcal{H}$ corresponding to $N'$. 
Theorem 1.

\[
\begin{array}{c|c}
K & K/k \text{ finite Galois} \\ 
G' & G = \text{Gal}(K/k) \\ 
F & G' = \text{Gal}(K/F') \\ 
k & G = G' \rtimes H \\
\end{array}
\]

Assume that

- \(N_1\) gives \(F/k\) a Hopf Galois structure and
- \(N_2\) gives \(K/F\) a Hopf Galois structure.

Then \(N_1 \times N_2\) gives \(K/k\) a Hopf Galois structure.
Proof.

$N_1$ gives $F/k$ a Hopf Galois structure $\iff \exists \varphi_1 : G \rightarrow \text{Hol}(N_1)$, with kernel $\text{Gal}(K/\tilde{F})$, such that $\varphi_1(G') = \text{Stab}(1_{N_1})$.

$N_2$ gives $K/F$ a Hopf Galois structure $\iff \exists \varphi_2 : G' \hookrightarrow \text{Hol}(N_2)$ such that $\varphi_2(1_{G'}) = \text{Stab}(1_{N_2})$.

If $g, g' \in G$, $g = xy, g' = x'y'$ with $x, x' \in H, y, y' \in G'$, since $H \triangleleft G$, we have $gg' = (xx'')yy'$, for some $x'' \in H$. Hence, the map

$$\varphi : G \rightarrow \text{Hol}(N_1) \times \text{Hol}(N_2)$$

$$g = xy \mapsto (\varphi_1(g), \varphi_2(y))$$

is a group monomorphism. We define now

$$\iota : \text{Hol}(N_1) \times \text{Hol}(N_2) \hookrightarrow \text{Hol}(N_1 \times N_2)$$

$$((n_1, \sigma_1), (n_2, \sigma_2)) \mapsto ((n_1, n_2), \sigma)$$

where $\sigma(n_1, n_2) := (\sigma_1(n_1), \sigma_2(n_2))$, and consider

$$\varphi : G \xrightarrow{\varphi} \text{Hol}(N_1) \times \text{Hol}(N_2) \xleftarrow{\iota} \text{Hol}(N_1 \times N_2).$$

We check $\varphi(1_G) = \text{Stab}(1_{N_1 \times N_2})$: for $g = xy \in G, x \in H, y \in G'$, $\varphi(g)(1_{N_1 \times N_2}) = 1_{N_1 \times N_2} \iff \varphi_1(g)(1_{N_1}) = 1_{N_1}$ and $\varphi_2(y)(1_{N_2}) = 1_{N_2} \iff g \in G'$ and $y = 1_{G'} \iff g = 1_G$. 
A Hopf Galois structure on a Galois extension $K/k$ with Galois group $G$ will be called **induced** if it is obtained as in Theorem 1 for some field $F$ with $k \subsetneq F \subsetneq K$ and given Hopf Galois structures on $F/k$ and $K/F$;

**split** if the corresponding regular subgroup of $Sym(G)$ is the direct product of two nontrivial subgroups.

**Corollary.** A Galois extension $K/k$ with Galois group $G = H \rtimes G'$ has at least one split Hopf Galois structure of type $H \times G'$.

**Proof.** Let $F = K^{G'}$ and let $\tilde{F}$ be the normal closure of $F$ in $K$. Then $K/F$ is Galois with group $G'$ and $F/k$ is almost classically Galois of type $H$ since $H$ is a normal complement of $\text{Gal}(\tilde{F}/F)$ in $\text{Gal}(\tilde{F}/k)$. These two Hopf Galois structures induce a Hopf Galois structure on $K/k$ of type $H \times G'$. 
A Galois extension with Galois group $G$ has an induced Hopf Galois structure of type $N$ in each of the following cases.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_3 = C_3 \rtimes C_2$</td>
<td>$C_6 = C_3 \times C_2$</td>
</tr>
<tr>
<td>$D_{2n} = C_n \rtimes C_2$</td>
<td>$C_n \times C_2$</td>
</tr>
<tr>
<td>$S_n = A_n \rtimes C_2$</td>
<td>$A_n \times C_2$</td>
</tr>
<tr>
<td>$A_4 = V_4 \rtimes C_3$</td>
<td>$V_4 \times C_3$</td>
</tr>
<tr>
<td>Frobenius group</td>
<td>$H \rtimes G'$</td>
</tr>
<tr>
<td>$G = H \rtimes G'$</td>
<td>$H \times G'$</td>
</tr>
<tr>
<td>Hol$(M) = M \rtimes \text{Aut}(M)$</td>
<td>$M \times \text{Aut}(M)$</td>
</tr>
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</table>

A Frobenius group $G$ is a transitive permutation group of some finite set $X$, such that every $g \in G \setminus \{1\}$ fixes at most one point of $X$ and some $g \in G \setminus \{1\}$ fixes a point of $X$. We have $G = H \rtimes G'$, where $H$ is the Frobenius kernel, i.e. the subgroup of $G$ whose nontrivial elements fix no point of $X$, and $G'$ is a Frobenius complement, i.e. the stabilizer of one point of $X$.

A semi-direct product $G = H \rtimes G'$ is a Frobenius group iff $C_G(h) \subset H$ for all $h \in H \setminus \{1\}$, and $C_G(g') \subset G'$ for all $g' \in G' \setminus \{1\}$. 
**Split non-induced Hopf Galois structures**

1. The quaternion group

\[ H_8 = \langle i, j | i^4 = 1, i^2 = j^2, ij = ji^3 \rangle = \{1, i, i^2, i^3, j, ij, i^2j, i^3j \} \]

is not a semi-direct product of two subgroups. The action of \( H_8 \) on itself by left translation induces

\[
\begin{align*}
\lambda : H_8 & \to \text{Sym}(H_8) \\
i & \mapsto (1, i, i^3)(j, ij, i^2j, i^3j) \\
j & \mapsto (1, j, i^2j)(i, i^3j, i^3, ij)
\end{align*}
\]

Then, \( \lambda(H_8) \) normalizes

\[ N = \langle (1, i^2)(i, i^3)(j, i^2j)(ij, i^3j), (1, i^3)(i, i^2)(i, ij)(i^2j, i^3j), (1, i^3j)(i, j)(i^2, ij)(i^3, i^2j) \rangle \]

which is a regular subgroup of \( \text{Sym}(H_8) \) isomorphic to \( C_2 \times C_2 \times C_2 \). Hence a Galois extension with Galois group \( H_8 \) has a split Hopf Galois structure of type \( C_2 \times C_2 \times C_2 \).
2. In the case $G = H \times G'$, i.e. $F/k$ Galois, the Galois structures of $K/F$ and $F/k$ induce the Galois structure on $K/k$:

$$G \to G/G' = H \overset{\rho}{\to} \text{Hol}(H) \text{ and } G' \overset{\rho}{\to} \text{Hol}(G') \text{ give } G \overset{\rho}{\to} \text{Hol}(G').$$

Let us consider a Galois extension $K/k$ with Galois group $G \cong C_p \times C_p$ (with $p$ prime). There are $p^2$ different Hopf Galois structures for $K/k$ (Byott, 1996).

**Case $p = 2$:** There is only one structure of type $C_2 \times C_2$, which is the classical one. The remaining 3 are of cyclic type. The extension $K/k$ has 3 different quadratic subextensions but all of them give rise to the same Hopf Galois structure, corresponding to $N = V_4 \subset S_4$.

**Case $p > 2$:** $\text{Hol}(C_{p^2})$ has no transitive subgroup isomorphic to $C_p \times C_p$. All $p^2$ Hopf Galois structures are split: $N \cong C_p \times C_p$. Only the classical structure is induced. The extension $K/k$ has $p + 1$ different subextensions of degree $p$ but all of them give rise to the classical structure.

We obtain then that a split Hopf Galois structure on a Galois extension $K/k$ may be induced by Hopf Galois structures on $K/F$ and $F/k$, for different intermediate fields $F$. 
Given a Galois extension $K/k$ of degree $n$ with Galois group $G$ and a regular subgroup $N = N_1 \times N_2$ of $S_n$ giving $K/k$ a split Hopf Galois structure, under which conditions is this Hopf Galois structure induced?

Theorem 1 gives that the following conditions are necessary.

1) $N_1$ and $N_2$ are $G$-stable,

2) If $F = K^{N_2}$ and $G' = \text{Gal}(K/F)$, then $G'$ has a normal complement in $G$.

**Theorem 2.** Let $K/k$ be a finite Galois field extension, $n = [K : k]$, $G = \text{Gal}(K/k)$. Let $K/k$ be given a split Hopf Galois structure by a regular subgroup $N$ of $S_n$ such that $N = N_1 \times N_2$ with $N_1$ and $N_2$ $G$-stable subgroups of $N$. Let $F = K^{N_2}$ be the subfield of $K$ fixed by $N_2$ and let us assume that $G' = \text{Gal}(K/F)$ has a normal complement in $G$.

Then $K/F$ is Hopf Galois with group $N_2$ and $F/k$ is Hopf Galois with group $N_1$. Moreover the Hopf Galois structure of $K/k$ given by $N$ is induced by the Hopf Galois structures given by $N_1$ and $N_2$. 
**Proof.** Since $K/k$ is Hopf Galois with group $N$, we have a monomorphism

$$
\varphi : G \rightarrow \text{Hol}(N) = N \rtimes \text{Aut} N
$$

$$
g \mapsto \varphi(g) = (n(g), \sigma(g))
$$

such that $\varphi(1_G)$ is the stabilizer of $1_N$.

Let us see $\varphi(G) \subset \iota(\text{Hol}(N_1) \times \text{Hol}(N_2))$, for

$$
\iota : \text{Hol}(N_1) \times \text{Hol}(N_2) \hookrightarrow \text{Hol}(N_1 \times N_2)
$$

$$
((n_1, \sigma_1), (n_2, \sigma_2)) \mapsto ((n_1, n_2), \sigma).
$$

For $i = 1, 2$, $N_i G$-stable and $N_i \lhd N \Rightarrow$

$$
\text{for } n_i \in N_i, g \in G, n(g)\sigma(g)(n_i)n(g)^{-1} \in N_i \Rightarrow \sigma(g)(n_i) \in N_i.
$$

We obtain then morphisms

$$
\varphi_1 : G \rightarrow \text{Hol}(N_1) \quad \varphi_2 : G' \rightarrow \text{Hol}(N_2)
$$

$$
g \mapsto (\pi_1(n(g)), \sigma(g)|_{N_1}) \quad g \mapsto (\pi_2(n(g)), \sigma(g)|_{N_2})
$$

Since $F = K^{N_2}$ and $G' = \text{Gal}(K/F')$, we have for $g \in G, g \in G' \Leftrightarrow \varphi(g)(1_N) \in N_2$.

Hence $\varphi_1(G') = \text{Stab}(1_{N_1})$.

Now for $y \in G', \varphi_2(y)(1_{N_2}) = 1_{N_2} \Rightarrow \varphi_2(y)(1_N) \in N_1$. But we had $\varphi(y)(1_N) \in N_2$, hence $\varphi(y)(1_N) = 1_N$, which implies $y = 1_G$, so $\varphi_2(1_{G'}) = \text{Stab}(1_{N_2})$. 

Counting Hopf Galois structures

1. The alternating group $A_4$

$K/k$ Galois with group $A_4$ has only two types of Hopf Galois structures: $A_4$ and $V_4 \times C_3$.

$$e(A_4, A_4) = 10 \text{ (Carnahan-Childs, 1999).}$$

Let us determine the number of induced Hopf Galois structures of type $V_4 \times C_3$.

We have a unique choice for the nontrivial normal subgroup $H$, the Klein subgroup $V_4 = \{id, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$. It has four different complements in $G$

$$G'_1 = \langle (2, 3, 4) \rangle, \ G'_2 = \langle (1, 3, 4) \rangle, \ G'_3 = \langle (1, 2, 4) \rangle, \ G'_4 = \langle (1, 2, 3) \rangle.$$

For a fixed $G'$, $F = K^{G'}/k$ is a quartic extension with Galois closure $K$ and has a unique Hopf Galois structure of type $V_4$ given by $\varphi_1 : A_4 \hookrightarrow \text{Hol}(V_4)$, such that $\varphi_1(G') = \text{Stab}(1_{V_4})$. The extension $K/F$ is Galois with group $G'$. This is the unique Hopf Galois structure for $K/F$. We obtain then a unique induced Hopf Galois structure for each $G'$, given by $\varphi : A_4 \hookrightarrow \text{Hol}(V_4 \times C_3)$ such that $\varphi(G') = \text{Stab}(\{1_{V_4}\} \times C_3)$. Therefore $K/k$ has four different induced Hopf Galois structures of type $V_4 \times C_3$. We obtain then

$$e(A_4, V_4 \times C_3) \geq 4.$$
2. Groups of order $4p$

$p$ odd prime, $G$ nonabelian group of order $4p$, $K/k$ Galois extension with group $G$. $G$ has a unique $p$-Sylow subgroup $H$ and $p$ 2-Sylow subgroups isomorphic either to $C_4$ or to $C_2 \times C_2$. Let $G'$ be a 2-Sylow subgroup of $G$ and $F = K^{G'}$. Since $F/k$ has degree $p$ and $G$ is solvable, $F/k$ is Hopf Galois (Childs 1989). Furthermore, $F/k$ is almost classically Galois and has a unique Hopf Galois structure given by the normal complement $H$ of $G'$ in $G$.

The number of Hopf Galois structures for Galois extensions with group isomorphic to $G'$ is

<table>
<thead>
<tr>
<th>$G' \simeq C_4$</th>
<th>$N_2 \simeq C_4$</th>
<th>$N_2 \simeq C_2 \times C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G' \simeq C_2 \times C_2$</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence the number of induced Hopf Galois structures of type $H \times N_2$ for $K/k$ is

<table>
<thead>
<tr>
<th>2-Sylow subgroup $\simeq C_4$</th>
<th>Structures $C_4 \times C_p$</th>
<th>Structures $C_2 \times C_2 \times C_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p</td>
<td>p</td>
<td>p</td>
</tr>
<tr>
<td>$3p</td>
<td></td>
<td>p</td>
</tr>
</tbody>
</table>

These are exactly the numbers of split Hopf Galois structures for $K/k$ of type $C_4 \times C_p$ or $C_2 \times C_2 \times C_p$ (Kohl, 2007).
3. Groups of order $pq$

$G$ group of order $pq$, $p$ and $q$ primes, $p > q$, $K/k$ Galois extension with group $G$.

- If $q \nmid p - 1$, $pq$ is a Burnside number and $K/k$ has a unique Hopf Galois structure, the classical Galois one (Byott, 1996).

- If $q \mid p - 1$, $G$ is either cyclic or metacyclic $C_p \rtimes C_q$.
  - If $G \cong C_{pq}$, there are $2q - 1$ different Hopf Galois structures for $K/k$, the classical one with $N \cong C_{pq}$ (split) and $2q - 2$ structures with $N \cong C_p \rtimes C_q$ (nonsplit).
  - If $G \cong C_p \rtimes C_q$, it has a unique $p$-Sylow subgroup and $p$ $q$-Sylow subgroups. Let $G'$ be a $q$-Sylow subgroup of $G$ and $F = K^{G'}$. Since $F/k$ has prime degree $p$ and $G$ is solvable, $F/k$ is Hopf Galois (Childs, 1989). Furthermore, in this case $F/k$ is almost classically Galois and has a unique Hopf Galois structure. The Galois structure of $K/F$ is also the unique Hopf Galois structure. Therefore, for each $G'$, we obtain exactly one induced Hopf Galois structure for $K/k$ and all together we obtain in this way $p$ induced Hopf Galois structures for $K/k$. This covers all split structures for $K/k$ (Byott, 2004).
In particular, if \( p \) is an odd prime and \( K/k \) is a dihedral extension of degree \( 2p \), its Hopf Galois structures are the two given by \( G \) and \( G^{opp} \) (dihedral type) and the \( p \) split structures of type \( C_2 \times C_p \) (cyclic type), induced by the structures of \( K/F \) and \( F/k \), for \( F = K^{G'} \) with \( G' \) ranging over the set of complements in \( G \) of the cyclic subgroup of order \( p \).