# Chaos induced by delay in a linearly controlled system

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## Outline

A small nonlinearity that might be negligible in most cases can cause non-trivial dynamics, even chaos, if it is combined with delay effects. We demonstrate this using a simple mathematical model of an inverted pendulum that is balanced with linear feedback control.

## Equations of motion

$$\ddot{\theta}(t) = \sin\theta(t) - F\cos\theta(t)$$
$$\ddot{\delta}(t) = \frac{2}{3} L F$$

where $F$ is a **delayed** linear proportional-plus-derivative feedback control force

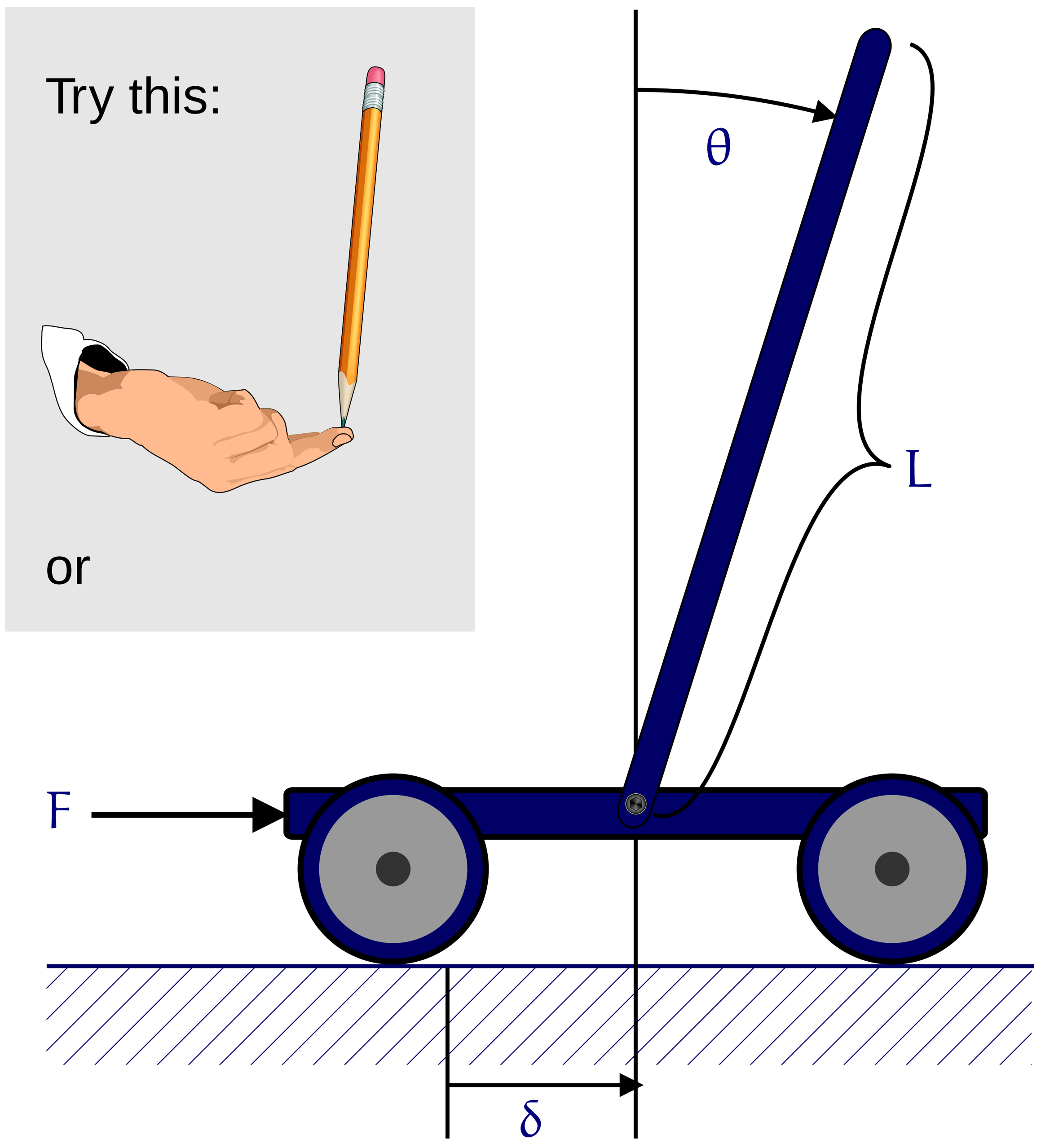
$$F = a\theta(t-\tau) + b\dot{\theta}(t-\tau).$$

Note that the nonlinearity is small; that is, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ for small $\theta$.

**Parameters**

$a$, $b$ feedback control gains
$\tau$ control loop latency
$L$ length of the pendulum

## Sketch of modelled setup




## Bifurcation analysis

The equations of motion form a system of **delay differential equations**, constituting a dynamical system with an infinite-dimensional phase space (like many partial differential equations). Bifurcation analysis enables us to draw a complete chart describing the dynamics of the system and its dependence on all parameters ($a$, $b$, and the delay $\tau$).

To this end, we use local center manifold theory for delay differential equations, bifurcation theory for finite-dimensional dynamical systems, numerical continuation methods, and numerical methods to compute invariant manifolds in maps.

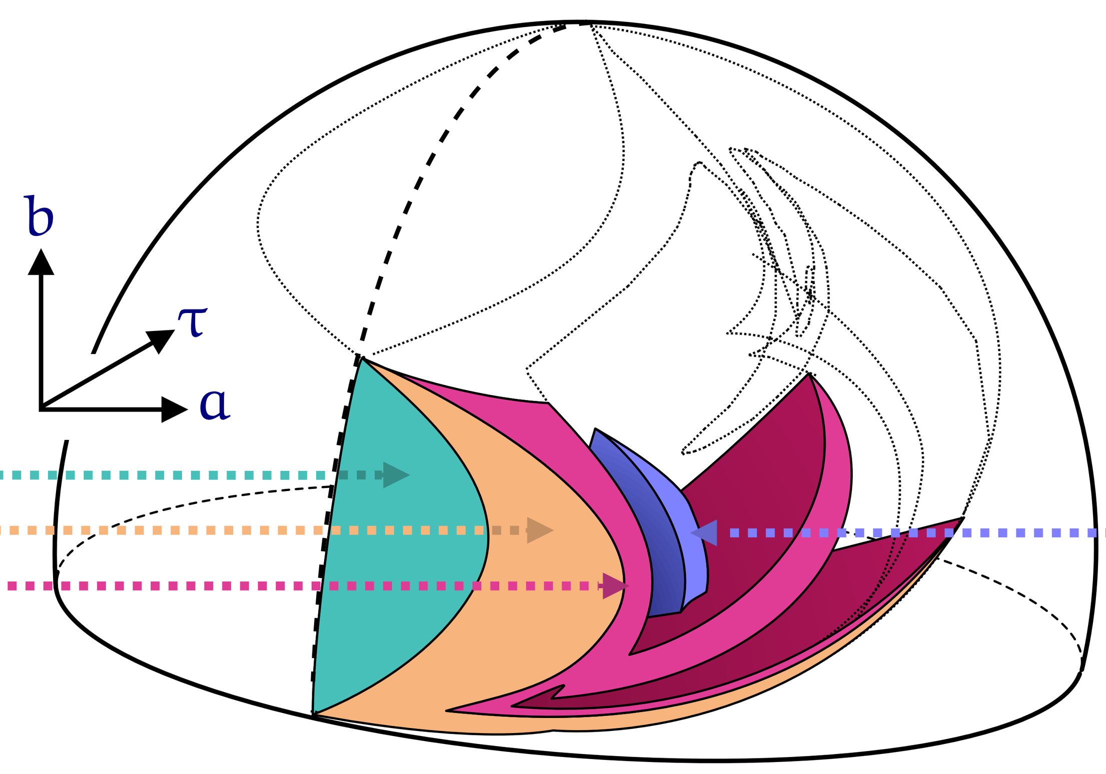
**Fine-print:** The equations assume that friction is negligible and the mass of the pendulum is much smaller than the mass of the cart. Furthermore, time is measured in multiples of $\sqrt{3L}/\sqrt{2g}$. Thus, decreasing the length of the pendulum effectively increases the delay. See references below for background, details and more general treatment.

## Chart of dynamics

The chart below shows the upper half of a ball in the **3D parameter space** for $(a, b, \tau)$ around the critical value

$$a = 1, \qquad b = \sqrt{2}, \qquad \tau = \sqrt{2},$$

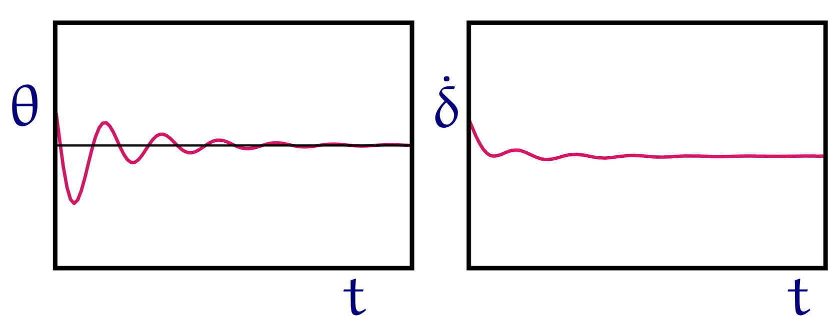
a codimension-3 bifurcation. At $\tau = \sqrt{2}$, the **critical delay**, it becomes impossible to stabilize the pendulum. The panels at the left and the right show the qualitative behavior of the system in the colored regions as time profiles of the angular displacement $\theta$ and the cart velocity $\dot{\delta}$.




White regions and the lower half of the ball are either unstable (mostly) or not yet fully explored. Dotted lines on the sphere are other bifurcations not directly related to balanced regimes.

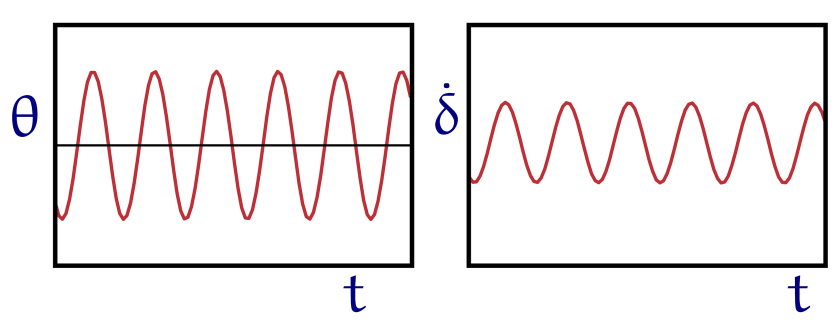
## Perfect stabilization

Upward position $\theta = 0$ is stable equilibrium; cart comes comes to a standstill.



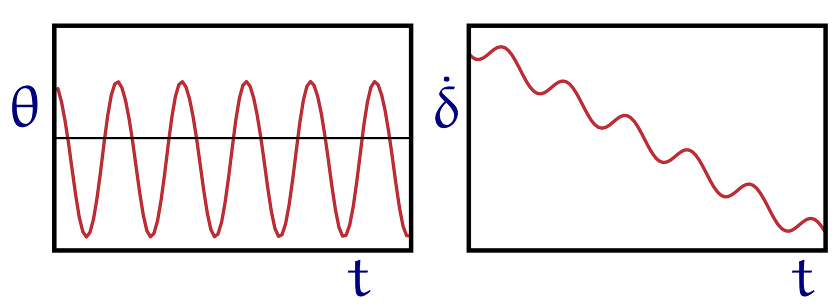

## Small oscillations

System has a symmetric stable periodic orbit close to $\theta = 0$; cart performs small oscillations.



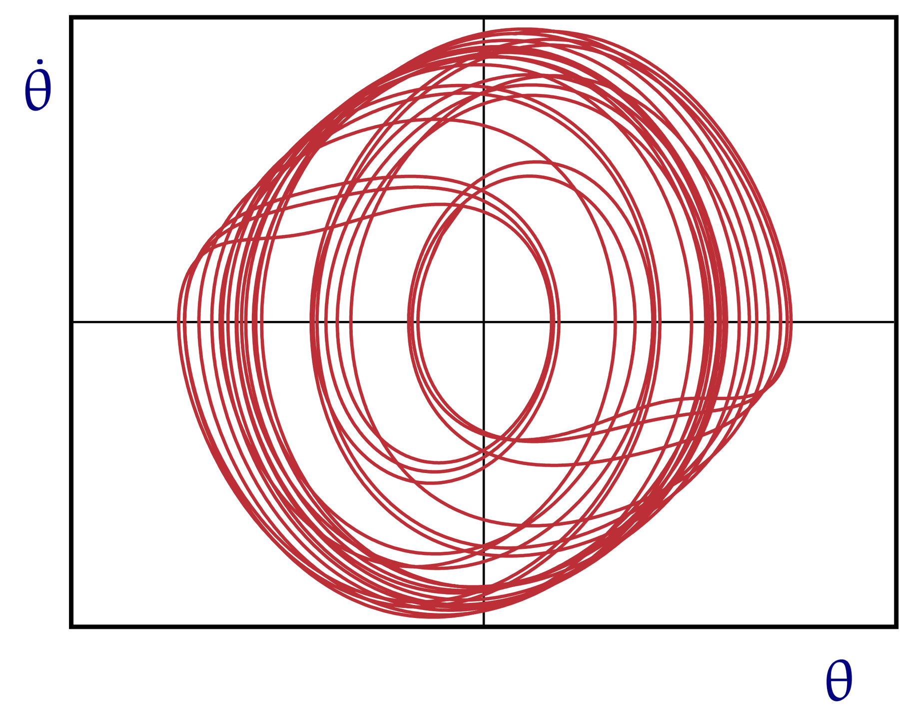

## Runaway oscillations

System has non-symmetric stable periodic orbit close to $\theta = 0$; cart accelerates away.

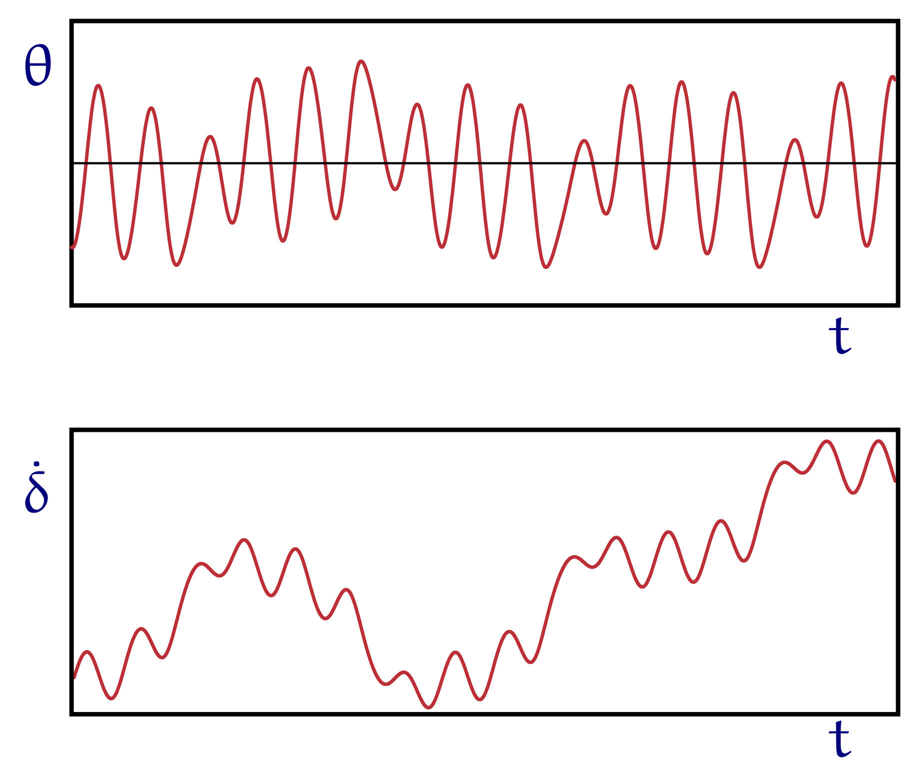



## Chaos

System has symmetric chaotic attractor, visualized by a long-time trajectory in $(\theta, \dot{\theta})$-projection below.




Pendulum performs small chaotic oscillations; cart performs “random walk”.





**References** (⇒ www.enm.bris.ac.uk/anm)
J.S., B.K.: Bifurcation analysis of an inverted pendulum with delayed feedback control near a triple-zero eigenvalue, *Nonlinearity*, 17(1), 2004.
J.S., B.K.: Complex balancing motions of an inverted pendulum subject to delayed feedback control, *submitted*, 2004
G. Stépán, L. Kollár: Balancing with reflex delay, *Mathematical and Computer Modelling*, 31, 2000.