“Wave-train solutions of a spatially inhomogeneous amplitude equation arising in the subcritical instability of narrow-gap spherical Couette flow.”

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Introduction

We consider the axisymmetric flow of incompressible fluid confined between two concentric spherical shells of radius $R_1$ & $R_2$ ($R_1 < R_2$) rotating about a common axis with angular velocities $\Omega_1$ & $\Omega_2$ respectively.
Introduction

The basic flow is primarily zonal flow with a weaker meridional circulation between the equator and the poles.

We consider the narrow gap limit, $\epsilon \equiv (R_2 - R_1)/R_1 \ll 1$, first considered by Walton 1978, for which the geometry at the equator resembles the classic Taylor–Couette case of concentric cylinders.

With $\Omega_1 > \Omega_2$ the centrifugal forces that drive the instability are greatest in the vicinity of the equator.

Instability here takes the form of roughly square Taylor vortices of angular length scale $\mathcal{O}(\epsilon)$. 
Governing Equations

We seek a solution within the WKB framework proportional to

\[ \exp \left[ i \left( \varepsilon^{-1} \int k \, d\theta - \omega t^\dagger \right) \right], \]

where \(-\theta\) is the latitude and the dimensionless wave number \(k\) and frequency \(\omega\) are \(O(1)\).

Locally in the vicinity of a given latitude, this provides an eigenvalue problem for the local frequency \(\omega = \omega(\theta, k, T)\). Where \(T\) is the Taylor number – a measure of the angular velocity gradient driving the instability.

Minimising \(T\) over real \(\theta\) and \(k\) subject to constraint that the frequency is real fixes the critical values

\[ \theta = \theta_c = 0, \quad k = k_c, \quad T = T_c. \]
Governing Equations

A Taylor series of the local frequency in the neighbourhood of the critical values determines

$$\omega - \omega_c = \theta \omega,\theta + \frac{1}{2} \theta^2 \omega,\theta\theta + \theta(k - k_c) \omega,\theta k + \frac{1}{2} k^2 \omega,kk + (T - T_c) \omega,T,$$

where the notation $\omega,\star$ denotes partial differentiation of $\omega$ with respect to $\star$.

We now write

$$a^\dagger(\theta, t^\dagger) \exp \left[ i (\epsilon^{-1} k_c \theta - \omega_c t^\dagger) \right],$$

so that the amplitude $a^\dagger$ satisfies

$$\frac{\partial a^\dagger}{\partial t^\dagger} + \epsilon \omega,k\theta \frac{\partial a^\dagger}{\partial \theta} = -i \left[ (T - T_c) \omega,T + \omega,\theta \theta + \frac{1}{2} \omega,\theta\theta \theta^2 - \gamma |a^\dagger|^2 \right] a^\dagger - \frac{1}{2} \epsilon^2 i \omega,kk \frac{\partial^2 a^\dagger}{\partial \theta^2}. $$

The effects of nonlinearity are taken into account by invoking the Stuart–Landau term proportional to $-|a^\dagger|^2 a^\dagger$ first identified by Davey 1962.
The non-zero value of $\partial \theta / \partial \omega$ can be linked to the meridional circulation and the boundary curvature of the problem.

This leads to a mechanism sometimes referred to as ‘phase mixing’ – a name born out of Astrophysics.

The consequence of this ‘phase mixing’ is that vortices generated at the local cylinder critical value of the Taylor number, $T_c$, are phase mixed and decay.

Therefore the global critical Taylor number $T_{\text{crit}}$ exceeds that for the classic Taylor–Couette problem – as shown by Soward and Jones 1983.
**Governing Equations**

Finite amplitude solutions are time dependent and controlled primarily by the ‘phase mixing’ balance

\[
\frac{\partial a^\dagger}{\partial t^\dagger} \approx -i \frac{\partial \omega}{\partial \theta} \theta a^\dagger
\]  

(2a)

which has solution

\[
a^\dagger(\theta, t^\dagger) = a^\dagger(\theta, 0) \exp \left( -i \frac{\partial \omega}{\partial \theta} \theta t^\dagger \right)
\]  

(2b)

Then, in a local frame with origin \( \theta = \bar{\theta}(t^\dagger) \) moving with the group velocity, constructing the amplitude

\[
a \equiv a^\dagger \exp \left( i \frac{\partial \omega}{\partial \theta} \bar{\theta} t^\dagger \right)
\]

removes the local oscillation identified by (2) above.
Governing Equations

The modulation of $a$ on a suitable scaling of length $x \propto \varepsilon^{-2/3}(\theta - \bar{\theta})$, time $t \propto \varepsilon^{2/3}t^\dagger$ and ignoring the small group velocity coefficient, $\varepsilon \partial_k \omega \theta$, yields the inhomogeneous Complex-Ginzburg-Landau equation

$$\frac{\partial a}{\partial t} = [\lambda(x) + ix]a + \frac{\partial^2 a}{\partial x^2} - |a|^2 a,$$

where the driving coefficient $\lambda(x) = \lambda(0) - \gamma_\varepsilon^2 x^2$.

The parameter $\gamma_\varepsilon$ is a measure of how similar the angular velocities of the two spheres are. It is zero unless they are really close.

We further restrict our attention to solutions within the symmetry class

$$a^*(-x,t) = a(x,t).$$
Pulse-Train Solutions

Work by Harris, Bassom and Soward 2003 (HBS) for the case where $x$ is measured from the equator ($\bar{\theta} = 0$), suggested that solutions are possible of the form

$$a(x, t) = \sum_{\forall n} \bar{b}_n (x - x_n) \exp (ix_n t) + \sum_{\forall n} \bar{b}_n^* (-x - x_n) \exp (-ix_n t),$$

(5a)

with

$$x_n = \left( 2n + \frac{1}{2} \right) L,$$

(5b)

where the frequency $L/2$ is a constant determined by the solution as a function of $\lambda(0)$.

They constitute a wave train of pulses, $\bar{b}_n (x - x_n)$, each localised in the vicinity of $x_n$ and separated by the non-dimensional distance $L$. 
Pulse-Train Solutions

Inspired by this, Bassom and Soward 2004 (BS) ignored the spatial variation of the driving coefficient ($\gamma_\varepsilon = 0$) and constructed infinite pulse–train structures valid on the intermediate length scale long compared to $\mathcal{O}(\varepsilon^{2/3})$ but short compared to $\mathcal{O}(\varepsilon^{1/3})$.

They successfully sought solutions of the governing equation in the form of pulse-trains with $\lambda = \text{constant}$. These possessed a spatial periodicity, reflected by the fact that all the functions $\bar{b}_n(x)$ were identical:

$$
\bar{b}_n(x) = \bar{b}(x) \equiv e^{i\pi/4} \bar{a}(x)
$$

($n = 0, 1, 2, \ldots$),

where $\bar{a}(x)$ possesses the complex reflectional symmetry

$$
\{C \bar{a}\}(x) = \bar{a}^*(-x) = \bar{a}(x).
$$
Pulse-Train Solutions

These BS pulse-train solutions were of higher amplitude and not linked in any obvious way to the HBS solutions for non-zero $\gamma_\varepsilon$.

Since the former is an asymptotic solution of the small $\gamma_\varepsilon$ problem we would expect to see consistent results.

We therefore considered again HBS’ problem for $\gamma_\varepsilon = 1/4$ with the objective of finding solutions that more closely resemble the BS $\gamma_\varepsilon = 0$ solutions.

We were fortunate to discover a new branch of solutions of type (5), apparently disconnected from the HBS–branch, that showed remarkable agreement with BS’s pulse-train solutions.
Pulse-Train Solutions

Pulse amplitudes $|\bar{b}_n(x - x_n)|$ and $|\bar{b}_n^*(-x - x_n)|$ for the case $\lambda = 10$, $T = 1.263$

Figure 1a: $\gamma_\varepsilon = \frac{1}{4}$ for $n = -1, 0, 1$

Figure 1b: $\gamma_\varepsilon = 0$ for $n = 0$
Pulse-Train Solutions

Contours of constant $\Re\{a\}$ and $\Im\{a\}$ for $\lambda = 10$, $T = 1.263$

Figure 2a: $\gamma = \frac{1}{4}$

Figure 2b: $\gamma = 0$
To obtain these pulse-train solutions BS examined a Fourier series in space with

\[ a(x, t) = \exp(\text{i}tx) \sum_{\forall n} A_n(t) \exp(\text{i}nTx). \]  

Substitution in our governing equation (3a) shows that the functions \( A_n(t) \) satisfy

\[ \frac{dA_n}{dt} - \left[ \lambda - (t + nT)^2 \right] A_n = - \sum_{\forall \alpha, \beta} A_{n+\alpha} A_{n+\beta} A^*_{n+\alpha+\beta} \]  

with initial conditions

\[ A_n(t) \rightarrow 0 \quad \text{as} \quad t + nT \downarrow -\infty. \]

They isolated a zero-mean solution with half-period 2\( T \) satisfying \( A_{n+2}(t) = -A_n(t) \) with

\[ \hat{A}^0(t) = \hat{A}^1(t) = -\hat{A}^2(t) = -\hat{A}^3(t) \equiv \hat{A}(t) \quad (\text{say}) \]

when \( \hat{A}^\alpha(t) = A_\alpha(t - \alpha T) \).
BS Solution Class $[-A, -A, A, A]$

Since each Fourier coefficient $A_\alpha(t)$ is localised in the vicinity of $t = -\alpha T$, we label them $[-A, -A, A, A]$ in reverse order: $\alpha = 3, 2, 1, 0$. This provides a signature of this class of solution.

Figure 3: $\lambda = 10, T = 1.5$
Energy Contours

As a measure of the solutions, BS used the AUTO package (Doedel et al., 1997) to trace the locus of the square root of the time-averaged energy $\langle \mathcal{E}_0 \rangle$ for various values of $\lambda (\leq 5)$. Use of Parseval’s Theorem gives

$$\langle \mathcal{E} \rangle = \frac{1}{T} \int_{-\infty}^{\infty} |\hat{A}(t)|^2 \, dt = \frac{1}{L} \int_{-\infty}^{\infty} |\bar{a}(x)|^2 \, dx$$  \hspace{1cm} (8a)

The corresponding HBS style solution at specified $\lambda(0)$ values determine $L$ as part of the solution. For comparison purposes we calculate the appropriate $L$ and the value of the square root of

$$\langle \mathcal{E}_0 \rangle = \frac{1}{L} \int_{-\infty}^{\infty} |\bar{b}_0(x)|^2 \, dx$$  \hspace{1cm} (8b)

for the largest HBS pulse $\bar{b}_0(x)$ at the same values $\lambda(0) = \lambda$. 
Energy Contours

Contours of constant $\sqrt{\langle \varepsilon \rangle}$ vs. the spatial period $L$

Figure 4: $\lambda = 2.55$ (innermost curve), 2.6, 3, 4, and 5 (outermost)
Symmetries

Given a solution \( a(x, t) \) of (3) we can generate others by

\[
\{ \mathcal{R}_\phi a \}(x, t) = e^{i\phi} a(x, t), \quad \{ T a \}(x, t) = a(x, t + T),
\]

\[
\{ C a \}(x, t) = a^*(-x, t),
\]

(9a–c)

where the phase rotation \( \phi \) and the time-shift \( T \) are arbitrary constants, while the superscript star (\( * \)) denotes complex conjugate.

In addition to the transformations (9a-c), the spatial periodicity of the BS problem, \( \lambda = \text{constant} \), allows new solutions

\[
\{ L a \}(x, t) = e^{-iLt} a(x + L, t)
\]

(9d)

to be generated under translation on any length \( L \).
Symmetries

BS solutions were periodic with spatial period $2L$ and temporal half-period $2T$ with $L$ and $T$ related by

$$LT = \pi .$$

When $L$ and $T$ are related thus Fourier coefficients $A_n(t)$ are mapped under the operations $\{R_\phi a\}$, $\{Ca\}$, $\{Ta\}$ and $\{La\}$ to

$$\{R_\phi A\}_n(t) = e^{i\phi}A_n(t) , \quad \{CA\}_n(t) = A^*_n(t) , \quad (10a,b)$$

$$\{TA\}_{n+1}(t) = A_n(t + T) , \quad \{LA\}_n(t) = (-1)^nA_n(t) . \quad (10c,d)$$

Restriction to the symmetry class $\{Ca\}(x, t) = a(x, t)$ implies that all the Fourier coefficients $A_n(t)$ are real, $\text{Im}\{A_n(t)\} = 0$. 
General $2NT$ Periodic Solutions

We now generalise the findings of BS and consider solutions with spatial period $2NT$. We classify these periodic solutions by the general properties

\[
\{T^Na\}(x,t) = \begin{cases} 
  a(x,t) & \text{non-zero-mean, period } NT \text{ with } N \text{ odd;} \\
  -a(x,t) & \text{zero-mean, half-period } NT.
\end{cases} \tag{11a,b}
\]

Furthermore, when $N$ is even and $N/2$ is odd, solutions also exist with quarter-period space-shifted symmetry

\[
\{T^{N/2}La\}(x,t) = a(x,t). \tag{12}
\]

Solutions with this property are a subclass of (11b).

We note that the BS result belongs to class (12) – and hence (11b) – above with $N = 2$. 
$N = 2$ Symmetry-Broken Solution

For $\lambda$ just below 4 a zero-mean, half-period $2T$ symmetry-broken mode appears with the property that not all the $\bar{b}^n(x)$ are equal.

At this value of $\lambda$ it is unstable but just before $\lambda = 8$ this new symmetry-broken solution is preferred over the old BS solution.

The new mode has amplitude functions $\hat{A}_\alpha(t) = A_\alpha(t - \alpha T)$ satisfying

$$\hat{A}^0(t) = -\hat{A}^2(t), \quad \hat{A}^1(t) = -\hat{A}^3(t) \quad \text{with} \quad \hat{A}^0(t) \neq \hat{A}^1(t)$$

Where the original solution was said to have signature $[-A, -A, A, A]$ we label this symmetry-broken solution as $[-B, -A, B, A]$. 
$N = 2$ Symmetry-Broken Solution

The amplitude sequences $A_\alpha(t)$ ($\alpha = 3, 2, 1, 0$) for the case $\lambda = 10, T = 1.5$.

Figure 5a: $[-A, -A, A, A]$

Figure 5b: $[-B, -A, B, A]$
$N = 2$ Symmetry-Broken Solution

Pulses, $\tilde{b}^\beta(x)$, for the symmetry broken $N = 2$ case at $\lambda = 10$, $T = 1.5$

Figure 6: $|\tilde{b}(x - L/2)|$ and $|\tilde{b}^*)(-x - L/2)|$
New N=2 Solutions

Contours of constant $\sqrt{\langle \mathcal{E} \rangle}$ vs. $L$ for the $N = 2$ case at $\lambda = 10$

Figure 7: BS solution solid, new solution dashed and HBS energy is the dot.
General Solutions – Odd $N$

We have found solutions for odd values of $N$ equal to 3, 5 and 7. The $N = 3$ solutions class were mostly stable but only exist for $\lambda > 14.45$. Meanwhile the $N = 5$ and $N = 7$ solutions were generally unstable but exist for lower values of $\lambda$.

When $N$ is odd every $2NT$-periodic, zero-mean solution $a(x,t)$ has a corresponding non-zero-mean, $NT$-periodic solution $\{La\}(x,t)$.

For example the zero-mean, half-period $3T$ solution is characterised by six amplitudes $\hat{A^\alpha}(t) = A^\alpha(t - \alpha T)$ ($\alpha = 0, 1, \ldots, 5$), only three of which are independent.

We label them from left to right as $[-C, -B, -A, C, B, A]$ in reverse integer order ($\alpha = 5, 4 \ldots, 0$). The corresponding non-zero-mean period $3T$ solution $\{La\}(x,t)$ has signature $[C, -B, A, C, -B, A]$. 
$N = 3$ Solutions

$A_n(t)$ and $\{L A\}_n$ vs. time for $n = 7, \ldots, -4$ for the case $N = 3$, $\lambda = 22$, $T = 1.15$

Figure 8a: $A_n(t)$ zero-mean $[-C, -B, -A, C, B, A]$

Figure 8b: $\{L A\}_n$ non-zero-mean $[C, -B, A, C, -B, A]$
\( N = 3 \) Solutions

\[ b^0(x) \quad \text{and} \quad b^{\pm 2}(x \pm 2L/3) \quad \text{vs.} \quad x/L_{PS} \quad \text{for} \quad N = 3 \quad \text{with} \quad \lambda = 22, \ T = 1.15 \]

Figure 9: \( b^0(x) \) is drawn broken and \( b^{\pm 2}(x \pm 2L/3) \) are drawn continuous
General Solutions – Even $\mathcal{N}$

In addition to the $2T$-half-periodic symmetry-broken solution detailed earlier we found numerous $6T$-half-periodic solutions.

These exist for low values of $\lambda$ when solutions bifurcate off the BS solution branch with signature $[-\mathcal{C}, -\mathcal{B}, -\mathcal{A}, -\mathcal{C}, \mathcal{B}, -\mathcal{A}, \mathcal{C}, \mathcal{B}, \mathcal{A}, \mathcal{C}, -\mathcal{B}, \mathcal{A}]$.

Further bifurcation yields $[-\mathcal{F}, -\mathcal{E}, -\mathcal{D}, -\mathcal{C}, \mathcal{B}, -\mathcal{A}, \mathcal{F}, \mathcal{E}, \mathcal{D}, \mathcal{C}, -\mathcal{B}, \mathcal{A}]$ – the general $6T$ solution signature. This is stable for higher values of $\lambda$. 
Solution Comparisons

Contours of constant $\sqrt{\langle \epsilon \rangle}$ vs. $L_{PS}$ at $\lambda = 22$

Figure 10: $N = 2$ BS case (solid), $N = 2$ SB case (broken) and $N = 3$ case (chain). The $N = 6$ case is shown long-dashed in the enlargement.
Solution Comparisons

Contours of constant $\sqrt{\langle \varepsilon \rangle}$ vs. $L_{PS}$ for $\lambda = 5$
References


General $2NT$ Periodic Solutions

To encompass both periodicity classes (11), we regard them as period $2NT$ solutions and write

$$a(x, t) = \sum_{\alpha=0}^{2N-1} a^\alpha(x, t + \alpha T),$$

(13a)

in which

$$a^\alpha(x, t) = \exp(itx) \sum_{\forall n} \hat{A}^\alpha(t + 2nNT) \exp(i2nNTx),$$

(13b)

where

$$\begin{align*}
A_\alpha(t - \alpha T) &= \hat{A}^\alpha(t) = \pm \hat{A}^{\alpha+N}(t), \\
a^{\alpha+N}(x, t + NT) &= a^\alpha(x, t) \\
&\quad \quad \quad \quad \quad \quad (0 \leq \alpha \leq N - 1).
\end{align*}$$

(13c)
General $2NT$ Periodic Solutions

Use of (13b) determines the definite integral

$$\frac{N}{L} \int_{-L/2N}^{L/2N} \exp(-i x t) a^{\alpha}(x, t) \, dx = \hat{A}^{\alpha}(t).$$

Rewriting $a^{\alpha}(x, t)$ in the form

$$a^{\alpha}(x, t) = \frac{1}{\sqrt{2N}} \sum_{m} \bar{a}^{\alpha}\left(x - \frac{mL}{N}\right) \exp\left(i \frac{mL}{N} t\right),$$

(14)

reversing the order of integration and summation and taking inverses gives

$$\bar{a}^{\alpha}(x) = \frac{1}{\sqrt{2T}} \int_{-\infty}^{\infty} \hat{A}^{\alpha}(t) \exp(itx) \, dt.$$  

(15)
General $2NT$ Periodic Solutions

Substitution of (14) into (13) and reorganising gives

$$a(x,t) = \sum_{\forall n} \sum_{\beta=0}^{2N-1} \bar{b}^\beta (x - x_n^\beta) \exp\left( i x_n^\beta t \right)$$  \hspace{1cm} (16a)

with

$$x_n^\beta = \left( 2n + \frac{\beta}{N} \right) L,$$  \hspace{1cm} (16b)

where

$$\bar{b}^\beta(x) = \frac{1}{\sqrt{2N}} \sum_{\alpha=0}^{2N-1} \exp\left( i \frac{\beta\alpha\pi}{N} \right) \bar{a}^\alpha(x)$$

$$= \frac{1 \pm (-1)^\beta}{\sqrt{2N}} \sum_{\alpha=0}^{N-1} \exp\left( i \frac{\beta\alpha\pi}{N} \right) \bar{a}^\alpha(x).$$  \hspace{1cm} (16c)
General $2NT$ Periodic Solutions

This ‘Fourier Transform’ representation (16a-c) reveals the character of $a(x,t)$ in terms of pulses $b^\beta(x - x_n^\beta)$, oscillating at the frequency $x_n^\beta$, with local origins $x = x_n^\beta$.

Significantly, (16c) indicates that under unit increases of $\beta$ each alternate $b^\beta(x - x_n^\beta)$ vanishes. This means that each pulse is separated by the distance

$$L_{PS} = x_n^{\beta+2} - x_n^\beta = 2L/N,$$  

so that $L_{PS}/L$ decreases with $N$.

However as $L_{PS}$ is controlled by the balance $(\lambda + ix)a + \partial^2 a/\partial x^2$ in (3) it is essentially independent of $N$ and so we would expect $L$ to decrease with increasing $N$ and vice-versa.