Absolute stability analysis of Lure’s systems and some extensions

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Outline

1. Introduction of Lure systems
2. Stability establishment
3. Explore more properties using quadratic programme
4. Conclusion
Lure’s systems

Definition – Lure system

A linear time invariant (LTI) plant is connected with a nonlinearity:

- The plant $G(s) := (A, B, C, 0)$ is a minimal realization.
- The nonlinearity $\phi(\cdot)$ is memoryless, locally Lipschitz in $y$ and satisfies a sector (bound) condition.
Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ with $\phi(0) = 0$. We denote $\phi \in [k_1, k_2]$ if $\phi(\cdot)$ satisfies

$$[\phi(y) - k_1 y][\phi(y) - k_2 y] \leq 0 \quad \forall y \in \mathbb{R} \quad (1)$$

**Figure:** Sector bound condition (SISO case)
Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ with $\phi(0) = 0$. We denote $\phi \in [k_1, k_2]$ if $\phi(\cdot)$ satisfies

$$[\phi(y) - k_1 y][\phi(y) - k_2 y] \leq 0 \quad \forall y \in \mathbb{R} \quad (1)$$

**Figure:** Sector bound condition (SISO case)
Common nonlinearities

**Saturation**

Saturation nonlinearity satisfies:

\[ \phi(y)(\phi(y) - y) \leq 0 \]

which corresponds to the general case when \( k_1 = 0 \) and \( k_2 = 1 \). Denote \( \phi \in [0, 1] \).
Common nonlinearities

**Deadzone**

![Deadzone Diagram](image)

**Figure:** Deadzone

Deadzone nonlinearity satisfies:

\[
\phi(y)(\phi(y) - y) \leq 0
\]

which corresponds to the general case when \(k_1 = 0\) and \(k_2 = 1\). Denote \(\phi \in [0, 1]\).
Sector bound condition – Multivariable Case

- Diagonal case:

\[
[\phi(y) - K_1 y]^T [\phi(y) - K_2 y] \leq 0
\]

where \( K_1 \) and \( K_2 \) are diagonal matrices with \( K_1 - K_2 < 0 \) and

\[
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad \phi(y) = \begin{bmatrix} \phi_1(y_1) \\ \phi_2(y_2) \\ \vdots \\ \phi_m(y_m) \end{bmatrix}
\]

- Coupled case:

\[
[\phi(y) - K_1 y]^T [\phi(y) - K_2 y] \leq 0
\]

where \( K_1 \) and \( K_2 \) are positive definite matrices with \( K_1 - K_2 < 0 \), and \( \phi_i(y) \) can depend on \( y_j \).
A Lure’s system

\[ r = 0 \]

\[ u \]

\[ \dot{x} = Ax + Bu \]

\[ y = Cx \]

\[ \phi(\cdot) \]

**Figure:** A Lure System

**Definition – Absolute stability [Khalil(2000)]**

Suppose \( \phi \) satisfies a sector condition. The system is absolutely stable if the equilibrium point at the origin is globally uniformly asymptotically stable for any nonlinearity in the given sector.
Outline

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Some Classical Stability Criteria

There are a number of classical criteria for the absolute stability analysis of the Lure system:

- Circle criterion;
- Popov criterion;
- Zames-Falb multiplier theory [Zames and Falb(1968)];
- Small gain;
- Passivity theory;
  
  · · ·

These criteria are related to each other. Two key problems to investigate:

- Less conservative?
- Easier to use?
the S-procedure [Yakubovich(1971)]

Suppose there are two statements as:

\[ S1: \sigma_1 \geq 0, \sigma_2 \geq 0, \ldots, \sigma_n \geq 0 \Rightarrow \sigma_0 \leq 0 \]

\[ S2: \sigma_0 + \sum_{i=1}^{N} \tau_i \sigma_i \leq 0 \]

with \( \tau_i \geq 0, i = 1, \ldots, N \). It is obvious that \( S2 \Rightarrow S1 \). But it is not always true that \( S1 \Rightarrow S2 \). It is called lossless if \( S1 \Leftrightarrow S2 \).

We are concerned about the case when \( \sigma_k \) take the quadratic forms as

\[ \sigma_k(f) = \langle \Phi_k f, f \rangle, \quad k = 0, 1, \ldots, N. \]  \hspace{1cm} (2)

with \( \Phi_k^* = \Phi_k \).
Using the S-procedure to establish stability

Suppose we have chosen a Lyapunov function candidate $V > 0$.
If we denote $\sigma_0 := \dot{V}(x) < 0$ and use $\sigma_i \geq 0$ with $i = 1, \ldots, N$ to describe the $N$ quadratic constraints derived from the nonlinearity in the Lure system. Then a sufficient condition for the system to be stable is the satisfaction of the inequality:

$$\sigma_0 + \sum_{i=1}^{N} \tau_i \sigma_i < 0$$

(3)

with $\tau_i \geq 0$ and $i = 1, \ldots, N$. 
Using the S-procedure to establish stability

We are interested in the question: how to reduce the conservatism of the stability criterion.

We have two options in this direction:

- Choose different Lyapunov function candidates;
- Use more constraints describing the nonlinearity.
Suppose $\phi \in [0, K]$. Then the sector condition is
\[
\begin{bmatrix} x \\ u \end{bmatrix}^T \Pi_c \begin{bmatrix} x \\ u \end{bmatrix} \geq 0
\]
with
\[
\Pi_c = \begin{bmatrix} 0 & -C^T K^T \\ -KC & -2I \end{bmatrix}
\]
Choose a Lyapunov function $V(x) = x^T P x$, with $P > 0$. The system is g.a.s., if under the sector condition, we have
\[
\dot{V}(x) = \begin{bmatrix} x \\ u \end{bmatrix}^T \Pi_v \begin{bmatrix} x \\ u \end{bmatrix} < 0
\]
with
\[
\Pi_v = \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix}
\]
Circle Criterion

Then the Circle criterion can be derived by the S-procedure:

\[
\Pi_v + \Pi_c = \begin{bmatrix}
A^TP + PA & PB - C^TK^T \\
B^TP - KC & -2I
\end{bmatrix} < 0
\] (4)

This is a linear matrix inequality (LMI).
The feasibility of an LMI can be tested easily using some exiting LMI solvers, such as LMILAB (a toolbox of MATLAB), SEDUMI, etc.
Choose the candidate Lyapunov function (Lure and Postnikov type) as

\[ V(x) = x^T P x + 2r \int_0^y \phi(\tau) d\tau \]  

with \( r > 0 \) and \( P > 0 \). The derivative of \( V(x) \) is

\[ \dot{V}(x) = \begin{bmatrix} x \\ u \end{bmatrix}^T (\Pi_v + r \Pi_p) \begin{bmatrix} x \\ u \end{bmatrix} < 0 \]  

with

\[ \Pi_p = - \begin{bmatrix} 0 & A^T C^T \\ C A & C B + B^T C^T \end{bmatrix} \]
Popov Criterion – with positive multiplier

Then the Popov criterion can be derived by the S-procedure:

\[ \Pi_v + \Pi_c + r\Pi_p = \begin{bmatrix} A^T P + PA & PB - C^T K^T - rA^T C^T \\ B^T P - KC - rCA & -2I - r(CB + B^T C^T) \end{bmatrix} < 0 \] (8)
Choose the candidate Lyapunov function as

\[ V(x) = x^T P x + 2r_+ \int_0^y \phi(\tau) d\tau + 2r_- \int_0^y [K_\tau - \phi(\tau)] d\tau \quad (9) \]

with \( r_+ > 0, \ r_- > 0 \) and \( P > 0 \). The derivative of \( V(x) \) is

\[ \dot{V}(x) = \begin{bmatrix} x \\ u \end{bmatrix}^T (\Pi_V + r_+ \Pi_{p+} + r_- \Pi_{p-}) \begin{bmatrix} x \\ u \end{bmatrix} < 0 \quad (10) \]

with

\[ \Pi_{p+} = - \begin{bmatrix} 0 & A^T C^T \\ CA & CB + B^T C^T \end{bmatrix} \quad (11) \]

\[ \Pi_{p-} = - \begin{bmatrix} (KC)^T CA + (CA)^T KC \\ (CB)^T KC \\ (CB)^T KC \end{bmatrix} \quad (12) \]
Popov Criterion – with indefinite multiplier (SISO)

Then the Popov criterion can be derived by the S-procedure:

$$\Pi_v + \Pi_c + r_+\Pi_{p+} + r_-\Pi_{p-} < 0 \quad \text{with } r_+, r_- \geq 0$$  \hspace{1cm} (13)

which can be (in some cases) shown equivalent to

$$\Pi_v + \Pi_c + r\Pi_{p+} < 0$$  \hspace{1cm} (14)

with \( r \in \mathbb{R} \).

- For SISO case, the equivalence can be shown by loop transformation [Yakubovich(1967)];
- Diagonal MIMO case is shown in [Yakubovich(1967), Park(1997)];
- Coupled MIMO case is shown in [Heath and Li(2009)].
Choice of Lyapunov Function candidates

To summarize, we have used the same sector bound condition and chosen three different Lyapunov functions for three cases:

- **Circle**
  \[ V(x) = x^T P x \]

- **Popov (positive multiplier)**
  \[ V(x) = x^T P x + 2 r \int_0^y \phi(\tau) d\tau \]

- **Popov (indefinite multiplier)**
  \[ V(x) = x^T P x + 2 r_+ \int_0^y \phi(\tau) d\tau + 2 r_- \int_0^y [K_\tau - \phi(\tau)]^T d\tau \]
The Criteria in Frequency Domain

Using the Kalman- Yakubovich-Popov (KYP) Lemma, these criteria in the LMI forms can be transformed into frequency domain:

- **Circle**
  \[ \Re\{KG(j\omega) + I}\geq 0 \quad \forall \omega \in \mathbb{R} \]

- **Popov (positive multiplier)**
  \[ \Re\{(K + rj\omega I)G(j\omega) + I}\geq 0 \quad \forall \omega \in \mathbb{R} \]
  with \( r \geq 0 \).

- **Popov (indefinite multiplier)**
  \[ \Re\{(K + rj\omega I)G(j\omega) + I}\geq 0 \quad \forall \omega \in \mathbb{R} \]
  with \( r \in \mathbb{R} \).
Example 1

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

Figure: Absolute stability example

\[
A = \begin{bmatrix}
-2 & -2 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \quad B = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\]
Example 1

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

Figure: Absolute stability example

The range of \( k > 0 \) that guarantees the stability of the system:

- **Circle**: \( k \leq 8.12 \);
- **Popov – positive multiplier**: \( k \leq 8.12 \);
- **Popov – indefinite multiplier**: \( k \leq 8.9 \).
We are interested in the question: how to reduce the conservatism of the stability criterion.

We have two options in this direction:

- Choose different Lyapunov function candidates;
- Use more constraints describing the nonlinearity.
The constraints of the nonlinearity

- $D$ – The nonlinearity.
The constraints of the nonlinearity

- $D$ – The nonlinearity.
- $C_1$ – The constraint $\sigma_1 \geq 0$. 
The constraints of the nonlinearity

- $D$ – The nonlinearity.
- $C_1, C_2$ – The constraints $\sigma_i \geq 0$ with $i = 1, 2$ respectively.
The constraints of the nonlinearity

- $D$ – The nonlinearity.
- $C_1$, $C_2$, $C_3$ – The constraints $\sigma_i \geq 0$ with $i = 1, 2, 3$ respectively.
The constraints of the nonlinearity

- $D$ – The nonlinearity.
- $C_1$, $C_2$, $C_3$, $C_4$ – The constraints $\sigma_i \geq 0$ with $i = 1, 2, 3, 4$ respectively.
The constraints of the nonlinearity

- $D$ – The nonlinearity.
- $C_1, C_2, C_3, C_4, C_5$ – The constraints $\sigma_i \geq 0$ with $i = 1, 2, 3, 4, 5$ respectively.
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The nonlinearity $\phi(\cdot)$ can be expressed by a quadratic program (QP) [Primbs(2001)]:

$$u(t) = \phi(y(t)) = \arg\min_{\tilde{u}} \frac{1}{2} \tilde{u}^T H \tilde{u} + \tilde{u}^T y(t)$$

(subject to $L\tilde{u} + My \leq b$)
Common nonlinearities

**Saturation**

Saturation nonlinearity can be described by

\[
\begin{align*}
    u &= \arg \min_{\tilde{u}} \frac{1}{2}(\tilde{u} - y)^2 \\
    \text{s.t.} |\tilde{u}| &\leq 1
\end{align*}
\]

(16)

which corresponds to the QP with \( H = 1 \), \( F = -1 \), \( L = 1 \), \( N = 0 \) and \( b = 1 \).
Common nonlinearities

**Deadzone**

Saturation nonlinearity can be described by

\[
    u = \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^2 \\
    \text{s.t.} |\tilde{u} - y| \leq 1
\]

which corresponds to the QP with \( H = 1, \ F = 0, \ L = 1, \ N = -1 \) and \( b = 1 \).
Using quadratic programme to express nonlinearities

The nonlinearity $\phi(\cdot)$ can be expressed by a quadratic program (QP) [Primbs(2001)]:

$$u(t) = \phi(y(t)) = \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^T H \tilde{u} + \tilde{u}^T y(t)$$

subject to $L \tilde{u} + My \preceq b$
Using quadratic programme to express nonlinearities

We can derive its KKT (Karush-Kuhn-Tucker) conditions:

\[
Hu + Fy + L^T \lambda = 0 \tag{19a}
\]

\[
Lu + My + s = b \tag{19b}
\]

\[
\lambda^T s = 0 \tag{19c}
\]

\[
\lambda \succeq 0 \tag{19d}
\]

\[
s \succeq 0 \tag{19e}
\]

where \(\lambda\) is the Lagrangian multiplier and \(s\) is the slack variable. The KKT conditions are the necessary and sufficient conditions for the solution of the QP to be optimal.
Using quadratic programme to express nonlinearities

From these KKT conditions and there first derivatives, we can derive three quadratic conditions

[Li et al.(2007)Li, Heath, and Lennox]
[Li et al.(2008)Li, Heath, and Lennox]:

\[
\begin{align*}
(u + L^\dagger My)^T (Hu + Fy) - ((I - LL^\dagger)My)^T \lambda & \leq 0 \\
(\dot{u} + L^\dagger M\dot{y})^T (Hu + Fy) - ((I - LL^\dagger)M\dot{y})^T \lambda & = 0 \\
(\ddot{u} + L^\dagger M\ddot{y})^T (H\dot{u} + F\dot{y}) - ((I - LL^\dagger)M\dot{y})^T \lambda & = 0
\end{align*}
\]

where \(L^\dagger := (LT L)^{-1} L^T\). The columns of \(L\) are linearly independent.
From these KKT conditions and there first derivatives, we can derive three quadratic conditions

[Li et al.(2007)Li, Heath, and Lennox]
[Li et al.(2008)Li, Heath, and Lennox]:

\[(u + L^\dagger My)^T(Hu + Fy) - ((I - LL^\dagger)My)^T\lambda \leq 0 \quad (20a)\]
\[(\dot{u} + L^\dagger M\dot{y})^T(Hu + Fy) - ((I - LL^\dagger)M\dot{y})^T\lambda = 0 \quad (20b)\]
\[(\dot{u} + L^\dagger M\dot{y})^T(H\dot{u} + F\dot{y}) - ((I - LL^\dagger)M\dot{y})^T\lambda = 0 \quad (20c)\]

where \(L^\dagger := (L^TL)^{-1}L^T\). The columns of \(L\) are linearly independent.
Assuming $M = LN$, three quadratic conditions can be further simplified:

$$
(u + Ny)^T (Hu + Fy) \leq 0 \quad (21a)
$$
$$
(\dot{u} + N\dot{y})^T (Hu + Fy) = 0 \quad (21b)
$$
$$
(\dot{u} + N\dot{y})^T (H\dot{u} + F\dot{y}) = 0 \quad (21c)
$$

where $y = Cx$ and $\dot{y} = C\dot{x} = CAx + CBu$. 

Using quadratic programme to express nonlinearities
Common nonlinearities

**Saturation**

![Saturation diagram](image)

Figure: Saturation

Its quadratic constraints are:

\[ u(u - y) \leq 0 \]  \hspace{1cm} (22)

\[ \dot{u}(u - y) = 0 \]  \hspace{1cm} (23)

\[ \ddot{u}(\ddot{u} - \dot{y}) = 0 \]  \hspace{1cm} (24)
Common nonlinearities

Deadzone

Figure: Saturation

Its quadratic constraints are:

\begin{align*}
  u(u - y) &\leq 0 \\
  u(\dot{u} - \dot{y}) &= 0 \\
  \dot{u}(\dot{u} - \dot{y}) &= 0
\end{align*}
Assuming $M = LN$, three quadratic conditions can be further simplified:

\[(u + Ny)^T(Hu + Fy) \leq 0\]  \hspace{1cm} (28a)

\[(\dot{u} + N\dot{y})^T(Hu + Fy) = 0\]  \hspace{1cm} (28b)

\[(\dot{u} + N\dot{y})^T(H\dot{u} + F\dot{y}) = 0\]  \hspace{1cm} (28c)

where $y = Cx$ and $\dot{y} = C\dot{x} = CAx + CBu$. 
Using quadratic programme to express nonlinearities

Denote \( \nu := [x^T, u^T, \dot{u}^T]^T \), then the three quadratic conditions can be expressed as

\[
\begin{align*}
\nu^T \Pi_1 \nu &\geq 0 \quad & (29a) \\
\nu^T \Pi_2 \nu &= 0 \quad & (29b) \\
\nu^T \Pi_3 \nu &= 0 \quad & (29c)
\end{align*}
\]

with

\[
\begin{align*}
\Pi_1 &= - \text{He} \left( \begin{bmatrix} NC & I & 0 \end{bmatrix} \begin{bmatrix} FC & H & 0 \end{bmatrix} \right) \quad & (30a) \\
\Pi_2 &= \text{He} \left( \begin{bmatrix} NCA & NCB & I \end{bmatrix} \begin{bmatrix} FC & H & 0 \end{bmatrix} \right) \quad & (30b) \\
\Pi_3 &= \text{He} \left( \begin{bmatrix} NCA & NCB & I \end{bmatrix} \begin{bmatrix} FCA & FCB & H \end{bmatrix} \right) \quad & (30c)
\end{align*}
\]

Here \( \text{He}(M) = M^* + M \).
Using quadratic programme to express nonlinearities

Choose a candidate Lyapunov function

\[ V(x, u) = [x^T, u^T]P[x^T, u^T]^T \]  

with \( P > 0 \). The first derivative of \( V \) is

\[ \dot{V}(x, u) = \begin{bmatrix} x \\ u \\ \dot{u} \end{bmatrix}^T \begin{bmatrix} A^TP_{11} + P_{11}A & A^TP_{12} + P_{11}B & P_{12} \\ P_{12}^TA + B^TP_{11} & P_{12}^TB + B^TP_{12} & P_{22} \\ P_{12}^TP_{12} & P_{22} & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ \dot{u} \end{bmatrix} \]

that is

\[ \dot{V}(x, u) = v^T \Pi_V v \]
Using quadratic programme to express nonlinearities

The system is stable if there exists a positive definite matrix $P > 0$ such that the following LMI is satisfied:

$$\Pi_v + \sum_{i=1}^{3} r_i \Pi_i < 0$$

(34)

with scalar variables $r_1 \geq 0, r_2, r_3 \in \mathbb{R}$. 

Example 1 – continued

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

\[
A = \begin{bmatrix} -2 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]

**Figure:** Absolute stability example
Example 1 – continued

The range of $k > 0$ that guarantees the stability of the system:
- **Circle**: $k \leq 8.12$;
- **Popov (indefinite multiplier)**: $k \leq 8.9$;
- **QP method**: $k < +\infty$.
- **Zames-Falb multiplier**: $k < +\infty$.

*Figure*: Absolute stability example
Example 2

- Two cases for the nonlinearity \( \phi \): saturation or deadzone;
- Two cases for the plant

\[
G_1(s) = \frac{1}{s^4 + s^3 + 8s^2 + 2s + 1}
\]
\[
G_2(s) = \frac{1}{s^4 + s^3 + 8s^2 + 3s + 1}
\]

- Two Lyapunov function candidates:

\[
V_1(x, u) = [x^T, u^T]P[x^T, u^T]
\]
\[
V_2(x, u) = [x^T, u^T]P[x^T, u^T] + r \int_0^y \phi(\tau) \, d\tau \quad \text{with } r \geq 0
\]

Note: \( V_2 \) is inspired by the Lyapunov function corresponding to the Popov criterion.
Example 2

Table: A comparison of the existing approaches and the new approach: continuous case. In each case the maximum value of $K$ for which the system is guaranteed stable is shown.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>Plant</th>
<th>Circle</th>
<th>Popov</th>
<th>QP ($V_1$)</th>
<th>QP ($V_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sat</td>
<td>$G_1$</td>
<td>1.69</td>
<td>4.03</td>
<td>3.98</td>
<td>9.63</td>
</tr>
<tr>
<td>dz</td>
<td>$G_2$</td>
<td>2.94</td>
<td>5.41</td>
<td>6.93</td>
<td>11.56</td>
</tr>
<tr>
<td>dz</td>
<td>$G_1$</td>
<td>1.69</td>
<td>1.69</td>
<td>10.16</td>
<td>10.16</td>
</tr>
</tbody>
</table>
Extension to stability analysis of model predictive control (MPC)

Discrete version

The stability analysis based on quadratic programme has a discrete time version.

Extension to MPC analysis

MPC resolves a constrained optimization problem at each sampling instant. The optimization can be expressed as a QP. Hence we can extend this method to MPC analysis.
Extension to stability analysis of model predictive control (MPC)

MPC controller – minimization of cost function

An optimization problem is resolved at each sampling instant $k$

$$U_k^* = \arg \min_{U_k} x_{k+N_p} S x_{k+N_p} + \sum_{i=0}^{N_p-1} \left[ x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \right]$$

s.t. $L_k u_k \preceq b_k$ with $b_k \succeq 0$

(35)

where $U_k = [u_k^T, u_{k+1}^T, \ldots, u_{k+N_p-1}^T]$ is the optimizer. Only the first element of $U_k^*$, i.e. $u_k^*$, is used as the control input.
Extension to robustness analysis of model predictive control (MPC)

MPC controller – quadratic programme

If only input constraints are considered, the MPC optimization problem can be written as a quadratic programme

\[
U_k^* = \arg \min_{U_k} U_k^T H U_k + F_k^T U_k \\
\text{s.t.} L U_k \leq b \text{ with } b \succeq 0
\]  

(36)

where \(H\) is a positive definite Hessian matrix, and \(F_k\) contains the estimated or measured state \(x_k\).
MPC framework

Figure: MPC framework
Example 3 – Robustness analysis of MPC

\[ \Delta_k \]

\[ q_k \rightarrow p_k \]

\[ u_k \rightarrow x_k \]

\[ U_k^* \rightarrow \bar{E} \]

\[ E \rightarrow U_k^* \rightarrow \text{Quadratic Program} \]

\[ \text{Plant} \]

Figure: MPC framework
Example 3 – Robustness analysis of MPC

An MPC example with uncertainties

Suppose a plant with unstructured uncertainties is

\[ x_{k+1} = Ax_k + B_u u_k + B_w w_k \]
\[ p_k = Cx_k + D_u u_k + D_w w_k \]
\[ w_k = \Delta_k p_k \]

where \( \Delta_k \) satisfies

\[ \|\Delta_k\|_2 \leq 1 \]

with

\[ A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} \quad B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_w = \begin{bmatrix} \theta & 0 \end{bmatrix} \]
\[ C = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad D_u = 0 \quad D_w = 0 \]
Example 3 – Robustness analysis of MPC

The plant is subject to input constraints $|u| \leq 1$.

The cost function is chosen as

$$U_k^* = \arg \min_{U_k} x_{k+N_p} S x_{k+N_p} + \sum_{i=0}^{N_p-1} \left[ x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \right]$$

s.t. $L_k u_k \leq b_k$ with $b_k \geq 0$

(37)

with

$$Q = \begin{bmatrix} 1 & -2/3 \\ -2/3 & 3/2 \end{bmatrix} \quad R = 1 \quad S = Q \quad N_p = 3$$
Example 3 – Robustness analysis of MPC

The Lyapunov function is chosen to be

\[ V = [x_k^T, U_k^T]^T P [x_k^T, U_k^T] \]

We use the three constraints derived from the QP and the quadratic constraints from the uncertainty \( \|\Delta_k\| \leq 1 \) to establish stability by the S-procedure. The range of \( \theta \) for the system to be stable is \( 0 \leq \theta \leq 0.19 \).
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Lure system stability analysis can be established using the S-procedure.

Two directions for reducing the stability conservatism:
- Choose suitable Lyapunov function candidate;
- Exploit more useful constraints for the nonlinearity.

Quadratic program can be used for the stability analysis of Lure system.

QP method can be further extended for the robust stability analysis of MPC systems.
Conclusion

- Lure system stability analysis can be established using the S-procedure.
- Two directions for reducing the stability conservatism:
  - Choose suitable Lyapunov function candidate;
  - Exploit more useful constraints for the nonlinearity.
- Quadratic program can be used for the stability analysis of Lure system.
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