REFUTATION OF THE RIEMANN HYPOTHESIS

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Abstract

We prove that the Riemann hypothesis for the Riemann ς function is false. We prove even that the real part of non trivial zeros of ς admits 0 and 1 for accumulation points. The proof is rather basic, using double extensions of analytic and harmonic functions.

Introduction

Let be $\frac{1}{2}+iu_k$ the non trivial zeros of the Riemann ς function of positive imaginary part. Our proof is based on the study of the function $F(t)=\sum_{k\geq 0} e^{-tu_k}$ où $(\Re t>0)$. For Pierre Cartier(IHES France) the study of this function (chapter one to seven) is known but not published. For Raoul Robert(CNRS France)other simpler proof is perhaps possible.

The result is that $F(t)$ can be extended to the domain $\{t \notin IR^{-}\}$and is meromorphic on this domain with simple pole at the point $\pm i\ln(n)$ with $\Lambda(n)\neq 0$ with residue $+\frac{1}{2\pi}\frac{\Lambda(n)}{n.^{\frac{1}{2}}}$.

Its real part on the positive imaginary half axis is $\frac{-e^{\frac{\Im t}{2}}}{2}+\frac{e^{-\frac{5}{2}\Im t}}{1-e.^{-2\Im t}}$.

The seven first paragraph are devoted to establish this result.

The end is devoted to study the function :

$$\Re(F_\lambda(t)+\frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}}\left[\frac{1}{t-i\ln n}+(\frac{1}{\bar{t}+i\ln n})\right] \text{ where } F_\lambda(t)=\sum_{k\geq 0} e.^{-t(u_k-i\lambda)}. \text{ definite for } \lambda>\frac{1}{2}$$

and $\Re t>0$.

We use method of harmonic extension in t and holomorphic extension in λ, to obtain a contradiction if we suppose $|\Re iu_k|<\frac{1}{2}-\delta,\ \delta>0$.

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Recall that the Riemann function ς is defined for $\Re(s) > 1$ by $\varsigma(s) = \sum_{n\geq 1} \frac{1}{n^s}$.

We shall use the Riemann function λ [1]

$\lambda(s) = s(s-1)\varsigma(s)\Gamma(\frac{s}{2})\pi.^{-\frac{s}{2}}$ which is .an integral function of order one[2] and especially the ξ.function of Riemann:

$\xi.(u) = \lambda(u + \frac{1}{2})$ which is even and real on the imaginary axis $\Re u = 0$.

Let be L a path in the complex plane composed of two half lines:

$\Re t \geq 0 \;\; \Im t = a \;\; (0 < a < \frac{1}{2})$.,$\Re t \geq 0 \;\; \Im t = -a$ and the half circle

$\Re t \leq 0 \;\; |t| = a$

We shall take the determination of neperian logarithm on the plane without negative reals. and we orient L in the direct sense .

1-computing $K_\lambda(\alpha) = \int_L \frac{\xi'(u+\lambda)du}{\xi(u+\lambda)(-u).^{\alpha+1}} (\Re\alpha > 0, |\Im\lambda| < u_0)$

We use the formula of products of the ξ function [2]:

$\xi(u) = C \prod_{k \in IN} (1 + \frac{u^2}{u_k{}^2})$ where C positive real.

Put $u_k = v_k + iw_k \quad v_k, w_k$ reals $|w_k| < \frac{1}{2}$.

We shall compute $I_N(\alpha) = \frac{1}{2i\pi}\int_L \frac{\xi'_N(u+\lambda)du}{\xi_N(u+\lambda)(-u).^{\alpha+1}}$ with

$\Re\alpha > 4 \quad N \in IN$ and . $\xi_N(u) = C\prod_{0.\leq k\leq N}(1+\frac{u^2}{{u_k}^2})$

then

$$\frac{\xi'_N(u)}{\xi_N(u)} = \sum_{k\leq k\leq N}\frac{2u}{{u_k}^2+u^2}.$$

We can apply the residues theorem for completing the integral by completing part of L into a path by a bit of circle of center 0 and with radius R almost high.

We obtain :

$$I_N(\alpha) = \sum_{0\leq k\leq N}\frac{1}{(iu_k+\lambda)^{1+\alpha}} + \sum_{0\leq k\leq N}\frac{1}{(-iu_k+\lambda)^{1+\alpha}}.$$

Now for computing $K\lambda(\alpha)$ we can replace L by a curve made by a circle of center 0 and of radius $<\frac{1}{2}$ completed by half lines supported by the positive reals and up and down.

When N grows to infinity

. $\frac{\xi'_N(u+\lambda).}{\xi_N(u+\lambda)}$ converge to $\frac{\xi'(u+\lambda).}{\xi(u+\lambda)}$ uniformly on the circle.

Then

$\left|\frac{2(u+\lambda)}{{u_k}^2+(u+\lambda)^2}\right| = \left|\frac{2(u+\lambda)}{{v_k}^2-{w_k}^2-2iu_k w_k u+(u+\lambda)^2}\right| \leq \left|\frac{2(u+\lambda)}{{v_k}^2-\frac{1}{4}.+(u+\lambda)^2}\right|$ thus we have a majored convergence when N grows to infinity.

Last by the Lebesgue theorem:

$$K_\alpha(\alpha) = \frac{1}{2i\pi}\int_L \frac{\xi'_.(u+\lambda)du}{\xi_.(u+\lambda)(-u).^{\alpha+1}} = \sum_{k\geq 0}\frac{1}{(iu_k+\lambda)^{1+\alpha}} + \sum_{k\geq 0}\frac{1}{(-iu_k+\lambda)^{1+\alpha}}$$

We can extend by holomorphy in α when $\Re\alpha > 0$ by using the property of the zeros of an integral function of order 1[2].

We can also extend by holomorphy in λ to the band $|\Im\lambda| < u_0$

2- Computing $.\frac{1}{2i\pi}\int_L \frac{\zeta'_{.}(\frac{1}{2}+u+\lambda).}{\zeta_{.}(\frac{1}{2}+u+\lambda)}.\frac{du}{(-u)^{\alpha+1}}$. $\Re\lambda > \frac{1}{2}$

We write:

$$\frac{\xi'(u)}{\xi(u)} = \frac{1}{u+\frac{1}{2}} + \frac{1}{u-\frac{1}{2}} + \frac{1}{2}\frac{\Gamma'(\frac{u+\frac{1}{2}}{2})}{\Gamma(\frac{u+\frac{1}{2}}{2})} + \frac{\zeta'(u+\frac{1}{2})}{\zeta(u+\frac{1}{2})} - \frac{1}{2}\ln\pi .$$

We have from[3], $\frac{\Gamma'(s)}{\Gamma(s)} = \gamma - \frac{1}{s} + \sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{s+n})$ where γ is the Euler constant.

On the positive reals, the function:

$u \to \sum_{n=1}^{n=M}(\frac{1}{n} - \frac{1}{\frac{1}{2}+u+\lambda+n}.)\frac{1}{(-u)^{1+\alpha}}$ is majored by an integrable function in u not depending upon M.

the terms $-\frac{1}{2}k\pi$ and- γ give an integral 0, the $\frac{1}{u+\lambda+\frac{1}{2}}$ cancels.

We have to compute

$$.\sum_{n=1}^{\infty}\frac{1}{2i\pi}\int_L(\frac{1}{n} - \frac{1}{\frac{1}{2}(\frac{1}{2}+u)+n})\frac{du}{(-u)^{\alpha+1}} .$$

by the residues theorem, appled like in the paragraph one , this quantity is

$$\sum_{n=1}^{\infty}\frac{1}{(\frac{1}{2}+2n+\lambda)^{1+\alpha}}$$

We have hence the formula :

$$.\frac{1}{2i\pi}\int_L \frac{\zeta'_{.}(u+\frac{1}{2}+\lambda)}{\zeta_{.}(\frac{1}{2}+u+\lambda)}+\frac{1}{u+\lambda-\frac{1}{2}})\frac{du}{(-u)^{\alpha+1}}$$

$$==\sum_{k\in IN}\frac{1}{(iu_k+\lambda)^{1+\alpha}}+\sum_{k\in IN}\frac{1}{(-iu_k+\lambda)^{1+\alpha}}+\sum_{n\in IN^*}\frac{1}{(2n+\frac{1}{2}+\lambda)^{1+\alpha}}.$$

We have for $\Re\lambda>\frac{1}{2}$

$$.\frac{1}{2i\pi}\int_L \frac{'_{.}1}{_{.}(-\frac{1}{2}+u+\lambda)}\frac{du}{(-u)^{\alpha+1}}=\frac{'_{.}1}{_{.}(\lambda-\frac{1}{2})^{1+\alpha}}.$$

The second member is analytic in λ on a domain D non containing the point $\frac{1}{2}$ and containing the half-line $\left[\frac{1}{2}+\delta,+\infty\right]$ and 0.

Hence:

$$.\frac{1}{2i\pi}\int_L (\frac{\zeta'_{.}(u+\frac{1}{2}+\lambda)}{\zeta_{.}(\frac{1}{2}+u+\lambda)})\frac{du}{(-u)^{\alpha+1}}=$$

$$=\frac{-1}{(\lambda-\frac{1}{2})^{1+\alpha}}=+\sum_{k\in IN}\frac{1}{(iu_k+\lambda)^{1+\alpha}}+\sum_{k\in IN}\frac{1}{(iu_k+\lambda)^{1+\alpha}}+\sum_{n\in IN^*}\frac{1}{(2n+\frac{1}{2}+\lambda)^{1+\alpha}}.$$

This function is analytic in λ for $\Re\alpha>0$ and λ appartenant à D.

3-Analytic explicit continuation of $\Gamma(\alpha+1)\int_L \frac{\zeta'_.(\frac{1}{2}+u)}{\zeta_.(\frac{1}{2}+u)}\frac{du}{(-u)^{\alpha+1}}$

We know [1] that

$\frac{\zeta'(s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$ si $\Re s>1$ with

$\Lambda(n)=\ln p$ *if* $n=p^m$ with p premier et m>1, 0 if not.

Let be λ in the complex plane with $|\Im\lambda|<\frac{u_0}{2}$ if $-1\le\Re\lambda\le\frac{1}{2}$ and all $\Im\lambda$ if $\Re\lambda>\frac{1}{2}$.

We have:

$$\frac{\zeta'_.(u+\frac{1}{2}+\lambda)}{\zeta_.(\frac{1}{2}+u+\lambda)}=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}+u+\lambda}} \quad \text{if } \Re u\ge 2..$$

Hence :

$$\sum_{n=1}^{\infty}\int_2^{+\infty}\left|\frac{\Lambda(n)}{n^{\frac{1}{2}+u+\lambda}}\right|\frac{1}{\left|(-ia-u)^{\alpha+1}\right|}du \quad \le \text{C, C>0 finite, if} \quad a\in IR \quad \forall\alpha\in C.$$

We deduce that the integral

$\int_L \frac{\zeta'_.(\frac{1}{2}+u+\lambda)}{\zeta_.(\frac{1}{2}+u+\lambda)}.\frac{du}{(-u)^{\alpha+1}}$ exists , that it is continue in (λ,α) by the Lebesgue theorem), that it

is holomorphic in λ for $\Re\lambda>\frac{1}{2}$ and integral in α by the Cauchy-Morera theorem.

We consider only analycity in one variable, the other being fixed. We don't use analicity in

(λ,α) . $\Re\lambda>\frac{1}{2}$ $\alpha\notin Z$.

For $\Re\lambda$ in D $\Re\lambda\le\frac{1}{2}$ we have this property for $\Re\alpha>0$ .cf paragraph3, where we have an

explicite expression of the function.

Let be C being a continuous path of C^1 class by pieces beginning in 0, not containing the point $\frac{1}{2}$ and being the half live $\Re\lambda > \lambda_0$ for $\frac{1}{2} < \lambda_0 < 1$ $\Im\lambda = 0$ .cf (2), where the function is explicite.

Let be $w \to C(w)$, w being a parameter of C with $w \in IR^+$ with $C(w_{0)} = \lambda_0$, $\lambda_0 > 1$ and $\Re C(w) > 0$ for $w > 0$ and $0 < \Im C(w) < u_0$ for $w \in \left]0, w_0\right]$.

We shall compute:

$$\frac{1}{n!}\int_C \lambda^n I_{\lambda,\varphi(\alpha)} d\lambda \quad 3 \le \Re\alpha \le 4 .$$

We can see first that this integral exists:

For $\lambda > \lambda_0$ we have :

$$I_{\lambda,\alpha} = \Gamma(\alpha+1)\int_L \sum_{m\ge 1} \frac{\Lambda(m)}{m^{u+\frac{1}{2}+\lambda}}.)\frac{du}{(-u)^{\alpha+1}}$$

$$=\Gamma(\alpha+1)\sum_{m\ge 1}\frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda}}\int_L \frac{1}{m^u}\frac{du}{(-u)^{\alpha+1}}$$

This formula is easily verified for a curve L belonging to a neighbourhood of 0 if $\Re u < \varepsilon$.

It results that $I_{\lambda,\alpha}$ has the order of a decreasing exponential in λ ,hence the integral can be compute for all n.

Let be now α with $\Re\alpha < 1$.

Computing $J(\alpha) = \frac{1}{n!}\int_C \lambda^n I_{\lambda,\varphi(\alpha)} d\lambda$

we have for $3 \le \Re\alpha \le 4$:

$$J(\alpha) = \frac{1}{n!}\int_C \lambda^n I_{\lambda,\varphi(\alpha)} d\lambda = \int_C (-\frac{\partial}{\partial\lambda} I_{\lambda,\alpha-1}) d\lambda =$$

$I_{0,\alpha-1}$.

Now $J(\alpha)$ is holomorphic in α when the real part of α is stricly between integers. $\alpha \notin Z$

For $\Re\alpha$ integer, we consider the two definitions of $\varphi(\alpha)$. By integration by parts , we see that the two definitions give the same $J(\alpha)$, using the relation

$\frac{\partial}{\partial\lambda} I_{\lambda,\alpha} = -I_{\lambda,\alpha+1}$. ($\alpha \notin Z$) and the functional property of Γ for $\Re\lambda \leq \frac{1}{2}$

Then by the Cauchy Morera theorem $J(\alpha)$ is analytic continuation of

$$I_{0,\alpha-1} = \Gamma(\alpha)\int_L \frac{\zeta'_.(\frac{1}{2}+u)}{\zeta_.(\frac{1}{2}+u)} \frac{du}{(-u)^\alpha} \text{ , } 2 \leq \Re\alpha \leq 3 \text{., to } \alpha \notin Z, \Re\alpha \leq 5$$

We have given the analytic continuation requisite.

4-Asymptotic estimation of the analytic extension of $I_{0,\alpha}$ to $\Re\alpha. < 0$. $\alpha \notin Z$

Let be with $\Re\alpha. < 0$, $\alpha = -n + 1 + \varphi(\alpha)$, $n \in IN$.

We have $\int_C \frac{\lambda^{n-1}}{(n-1)!} I_{\lambda,\varphi(\alpha)+1} d\lambda =$

$$\int_{C_1} \frac{\lambda^{n-1}}{(n-1)!} I_{\lambda,\varphi(\alpha)+1} d\lambda + \int_{\lambda_0}^{+\infty} \frac{\lambda^{n-1}}{(n-1)!} I_{\lambda,\varphi(\alpha)+1} d\lambda$$

Where C_1 is a part of C between 0 and λ_0 .

We have now

$$\int_{C_1} \frac{\lambda^{n-1}}{(n-1)!} I_{\lambda,\varphi(\alpha)+1} d\lambda =$$

$$.\int_{C_1} \frac{\lambda^{n-1}}{(n-1)!} \left\{ \sum_{k=0}^{\infty} \frac{1}{(iu_k+\lambda)^{1+\varphi(\alpha)}} + \sum_{k=0}^{\infty} \frac{1}{(-iu_k+\lambda)^{1+\varphi(\alpha)}} + \sum_{m=1}^{\infty} \frac{1}{(2m+\frac{1}{2}+\lambda)^{1+\varphi(\alpha)}} - \frac{1}{(\lambda-\frac{1}{2})^{1+\varphi(\alpha)}} \right\}$$
$$\Gamma(1+\varphi(\alpha) d\lambda .$$

Then

$$\int_{\lambda_0}^{+\infty} \frac{\lambda^{n-1}}{(n-1)!} I_{\lambda,\varphi(\alpha)+1} d\lambda = \int_{\lambda_0}^{+\infty} \frac{\lambda^{n-1}}{(n-1)!} A d\lambda \text{ with}$$

$$A = . \int_L \sum_{m \geq 1} \frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda+u}} \Gamma(1+\varphi(\alpha) \frac{du}{(-u)^{1+\varphi(\alpha)}}$$

We have:

$$A = \sum_{m \geq 1} \frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda}} \ln(m)^{\varphi(\alpha)} \int_L e^{-u \ln m} \frac{du}{(-u)^{1+\varphi(\alpha)}} \Lambda(m)$$

The function

$\int_L e^{-u \ln m} \frac{du}{(-u)^{1+\beta}}$ is integral in β .If β is negative real we have:

$$\int_L e^{-u \ln m} \frac{du}{(-u)^{1+\beta}} = 2 \sin \pi (1+\beta) \int_0^{+\infty} e^{-u \ln m} \frac{du}{(-u)^{1+\beta}} =$$

$$-2 \sin \pi\beta \ln^{\beta} m \int_0^{+\infty} e^{-u} \frac{du}{(-u)^{1+\beta}} =$$

$$2 \sin \pi (1+\beta) \ln^{\beta}(m) \Gamma(-\beta) =$$

$$+2 \sin \pi (1+\beta) \, \Gamma(1+\beta) \ln^{\beta}(m) =$$

$$2 \frac{\pi \ln^{\beta}(m)}{\Gamma(1+\beta)} .$$

Hence

$$\int_L e^{-u \ln m} \frac{du}{(-u)^{1+\varphi(\alpha)}} = 2\pi \frac{\ln(m)^{\varphi(\alpha)}}{\Gamma(1+\varphi(\alpha))}$$

Finally

$$A = 2\pi \sum_{m \geq 1} \frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda}} \ln(m)^{\varphi(\alpha)}$$

Then we have to compute

$$2\pi\int_{\lambda_0}^{+\infty}\frac{\lambda^{n-1}}{(n-1)!}\sum_{m\geq 1}\frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda}}\ln(m)^{\varphi(\alpha)}d\lambda=$$

$$2\pi\sum_{m\geq 1}\Lambda(m)\int_{\lambda_0}^{+\infty}\frac{\lambda^{n-1}}{(n-1)!}\frac{\ln(m)^{\varphi(\alpha)}}{m^{\frac{1}{2}+\lambda}}d\lambda$$

By successive integrations by parts we obtain:

$$\int_{\lambda_0}^{+\infty}\frac{\lambda^{n-1}}{(n-1)!}\frac{d\lambda}{m^{\frac{1}{2}+\lambda}}=\frac{1}{m^{\frac{1}{2}+\lambda_0}}\sum_{l=0}^{n-1}\frac{\ln m^{-l-1}\lambda_0^{\,n-l-1}}{(n-l-1)!}$$

$$\frac{(\ln m)^{-n}1}{m^{\frac{1}{2}+\lambda_0}}\sum_{l=0}^{n-1}\frac{\ln m^{n-l-1}\lambda_0^{\,n-l-1}}{(n-l-1)!}$$

Let be $\varepsilon>0$ the term $\frac{(\lambda_0\ln m)^{-n-l-1}}{(n-l-1)!}$ is maximum for $\lambda_0\ln m=n-l-1$ with the value

$$\frac{(n-1-l)^{-n-l-1}}{(n-l-1)!}.$$

By the Stirling formula this term is majored by $\mathrm{K}\frac{e^{n-l-1}}{\sqrt{n-l-1}}\leq Ke^n$.

Take now m_ε such that $(\ln m)^{-n+1}nKe^n<\varepsilon^n$ with $\alpha=-n+1+\varphi(\alpha)$ for $m\geq m_\varepsilon$.

Then $\left|\sum_{m\geq m_\varepsilon}\frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda_0}}\int_{\lambda_0}^{+\infty}\frac{\lambda^{n-1}}{(n-1)!}\ln(m)^{\varphi(\alpha)}d\lambda\right|\leq\varepsilon^n\sum_{m\geq m_\varepsilon 1}\frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda_0}}$

Compute now the sum since m_ε. By the Taylor formula we have:

$$\sum_{l=0}^{n-1}\frac{(\ln m)^{n-l-1}}{(n-l-1)!}=m^{\lambda_0}-\frac{(\ln m)^n\lambda_0}{n!}m^{\theta\lambda_0},0<\theta<1$$

Hence:

$$\sum_{m\geq 1}\frac{\Lambda(m)}{m^{\frac{1}{2}+\lambda_0}}\int_{\lambda_0}^{+\infty}\frac{\lambda^{n-1}}{(n-1)!}\ln(m)^{\varphi(\alpha)}d\lambda=\sum_{m<m_\varepsilon}\frac{\Lambda(m)\ln(m^{-n+\varphi(\alpha)}}{m^{\frac{1}{2}0}}+R\ \text{ with}\left|R<3\varepsilon^n\right|\ (*)$$

We recall that $\alpha=-n+1+\varphi(\alpha)$

We have of course

$$\sum_{m<m_\varepsilon}\frac{\Lambda(m)\ln(m)^{-n+\varphi(\alpha)}}{m^{\frac{1}{2}}}=\sum_{m<m_\varepsilon}\frac{\Lambda(m)\ln(m)^{\alpha-1)}}{m^{\frac{1}{2}}}$$

(*) is true on the domains that we need like $-\Re\,\alpha \geq P\left|\Im\,\alpha\right|, P>0, \Re\,\alpha<0$

Finally the part corresponding to the integral on the path C_1 is easily majored by $K'_p\,\varepsilon^n$.

5-Computing

$$\int_D \frac{t^{-\alpha-1}\Gamma(\alpha+1)}{\sin\frac{\pi}{2}\alpha}\sum_{k\geq 01}(\frac{1}{(iu_k)^{\alpha+1}}+\frac{1}{(-iu_k)^{\alpha+1}})d\alpha \text{ Where } t>0\,, D=\left\{\frac{1}{2}+iv, v\in IR\right\}$$

Remark that the integral $\int_D \frac{t^{-\alpha-1}\Gamma(\alpha+1)}{\sin\frac{\pi}{2}\alpha}(\frac{1}{(iu_k)^{\alpha+1}}+\frac{1}{(-iu_k)^{\alpha+1}})d\alpha$ is convergent by using the complex Stirling formula [3] for $\Gamma(\alpha+1)$ and the fact that the terms iu_k have an argument near $\frac{\pi}{2}$.

Let be $u>1$ we first compute $\int_D \frac{\Gamma(\alpha+1)}{u^{-\alpha-1}}t^{-\alpha-1}d\alpha$.

By the Phragmen-Lindelof method [4], $\frac{\Gamma(\alpha+1)}{u^{-\alpha-1}}t^{-\alpha-1}d\alpha$ will be smaller and smaller on pieces of circle with center 0 and radius $R\notin IN$ limited by the line D and containing -IR.

We can apply the residue theorem and so obtain

$$\frac{1}{2i\pi}\int_D \frac{\Gamma(\alpha+1)}{u^{-\alpha-1}}t^{-\alpha-1}d\alpha=\sum_{n\geq 1}\frac{(-1)^{n-1}t^{n-1}u^{n-1}}{(n-1)!}=e^{-tu}$$

Now for the iu_k we remark that

$\int_D \frac{t^{-\alpha-1}\Gamma(\alpha+1)}{\sin\frac{\pi}{2}\alpha}(\frac{1}{v^{\alpha+1}}+\frac{1}{(-v)^{\alpha+1}})d\alpha$ is analytic in v on a domain containing iu with u real and iu_k. Then $\frac{1}{2i\pi}\int_D \frac{t^{-\alpha-1}\Gamma(\alpha+1)}{\sin\frac{\pi}{2}\alpha}(\frac{1}{(iu_k)^{\alpha+1}}+\frac{1}{(-iu_k)^{\alpha+1}})d\alpha$=-2 e^{-tu_k} .

Finally using the theorem of convergence of the integral of series absolutly majored we obtain

$\frac{1}{2i\pi}\int_D \frac{t^{-\alpha-1}\Gamma(\alpha+1)}{sin\frac{\pi}{2}\alpha}\sum_{k\geq 01}(\frac{1}{(iu_k)^{\alpha+1}}+\frac{1}{(-iu_k)^{\alpha+1}})d\alpha=-2\sum_{k\geq 0}e^{-tu_k}$ .Remark that the function $\sum_{k\geq 0}e^{-tu_k}(-2)$ is real for t real strictly positive.

Chapitre 6

Meromorphic extension of de $\sum_{k\geq 0}e^{-tu_k}\ \Re t>0$ to $\Im t\notin IR^-$

We consider the function $f(t)=\frac{1}{2i\pi}\int_D \frac{I_{0,\alpha}t^{-\alpha+1}}{\sin\frac{\pi}{2}\alpha(-\alpha)(-\alpha+1)}d\alpha$ for t positive real

This functionnis well definite by finite sumes normelly convergent by integrals by the expression of $I_{0,\alpha}$ of point 2.

We can upperbound $I_{0,\alpha}$ by $Ke^{\pi|\mathrm{Im}\alpha|}Ke^{-\frac{\pi}{2}|\mathrm{Im}\alpha|}$ $K>0$ en by using the Stirling complex formula..

As we have divided $(\sin\frac{\pi}{2}\alpha)(-\alpha)(-\alpha+1)$ the integral is well defined.

let ε >0 be

We consider $H_\alpha^{\ \varepsilon}(t)=\frac{I_{0,\alpha}t^{-\alpha+1}}{\sin\frac{\pi}{2}\alpha(-\alpha)(-\alpha+1)}-\frac{t^{-\alpha+1}}{(\sin\frac{\pi}{2}\alpha)(-\alpha)(-\alpha+1)}\sum_{m\leq m_\varepsilon}\frac{\Lambda(m)}{m^{\frac{1}{2}}}(\ln m)^\alpha$.

(m_ε defined at point 4).

By the complex Stirling formula and the Pragmen Lindelof method for a sector, we have that: $\frac{1}{2i\pi}\int_D H_\alpha^{\ \varepsilon}(t)d\alpha$ is equal to $\frac{1}{2i\pi}\int_\Delta H_\alpha^{\ \varepsilon}(t)d\alpha$ where Δ is the union of two conjugated half-lines $\frac{1}{2}+e^{i\vartheta}v\ \ v\geq 0,\theta=\pi-\delta>0$ and $\frac{1}{2}+e^{-i\vartheta}v\ \ ,v\geq 0$. the value onr Δ is bounded by $(\varepsilon|t|)^{-\Re\alpha}$. So: $\frac{1}{2i\pi}\int_{\Delta}H_{\alpha}^{\varepsilon}(t)d\alpha$ is an analytic function. $t\notin IR^{-},|t|<A_{\varepsilon}$ where $A_{\varepsilon}\to+\infty$ when $\varepsilon\to 0$. We can consider $\frac{1}{2i\pi}\int_{\Delta}\frac{\Gamma(\alpha+1)t^{-\alpha+1}}{(\sin\frac{\pi}{2}\alpha)(\frac{-1}{2})^{\alpha+1}(-\alpha)(-\alpha+1)}d\alpha$ instead of $\frac{1}{2i\pi}\int_{D}\frac{\Gamma(\alpha+1)t^{-\alpha+1}}{(\sin\frac{\pi}{2}\alpha)(\frac{-1}{2})^{\alpha+1}(-\alpha)(-\alpha+1)}d\alpha$ by using again the Phagmen Lindehof method on a sector [4]. Thus that appears that the integral we compute as for second derivate in t :

$$\frac{1}{2i\pi}\int_{\Delta}\frac{\Gamma(\alpha+1)t^{-\alpha-1}}{(\sin\frac{\pi}{2}\alpha)(\frac{-1}{2})^{\alpha+1}}d\alpha+\frac{1}{2i\pi}\int_{D}\frac{\Gamma(\alpha+1)t^{-\alpha-1}}{\sin\frac{\pi}{2}\alpha}\sum_{n\geq 1}\frac{1}{(2n+\frac{1}{2})^{1+\alpha}}d\alpha+$$

$\frac{1}{2i\pi}\int_{D}\frac{\Gamma(\alpha+1)t^{-\alpha-1}}{\sin\frac{\pi}{2}\alpha}\sum_{k\geq 0}(\frac{1}{(iu_k)^{1+\alpha}}+\frac{1}{(-iu_k)^{1+\alpha}})d\alpha$ which is defined for $\Re t>0$. We note that the second derivate in t of $H_{\alpha}^{\varepsilon}(t)$ on Δ is bounded by the Cauchy estimates by $[\varepsilon(|t|+\frac{1}{4})]^{-\Re\alpha}\times 16$ considering the discs of centre t an of radius $\frac{1}{4}$.Then we have to compute :

$\frac{1}{2i\pi}\int_{D}\frac{1}{\sin\frac{\pi}{2}\alpha}t^{-\alpha-1}\sum_{m\leq m_{\varepsilon}}(\frac{\Lambda(m)}{m^{\frac{1}{2}}}(\ln m)^{\alpha})d\alpha$.for t réel >0 such as $\frac{t}{\ln m}>1$ we have :

$$\frac{1}{2i\pi}\int_{D}\frac{1}{\sin\frac{\pi}{2}\alpha}t^{-\alpha-1}(\ln m)^{\alpha}d\alpha=\frac{-1}{\frac{\pi}{2}}\sum_{l\geq 1}(-1)^{l}(\ln m)^{2l}t^{-2l-1}=\frac{2}{\pi}\frac{1}{t}\frac{(\ln m)^2}{t^2+(\ln m)^2}$$

(by the résidue theorem on half circle of center $\frac{1}{2}$ on the side of positive reals limitedby D)

For $\frac{t}{\ln m}<1$, we obtain $\frac{2}{\pi}\frac{1}{t}\frac{(\ln m)^2}{t^2+(\ln m)^2}$

We have the résidues at the points $\pm i\ln m,\Lambda(m)\neq 0$

We write $\frac{(\ln m)^2}{t^2+(\ln m)^2}=\frac{1}{2(1-\frac{it}{\ln m})}+\frac{1}{2(1+\frac{it}{\ln m})}$

So the résidue is $-\frac{\Lambda(m)}{m^{\frac{1}{2}}}\frac{1}{\pi}$.

We have to substract $-\frac{1}{2i\pi}\int_{\Delta}\frac{\Gamma(\alpha+1)t^{-\alpha-1}}{\sin\frac{\pi}{2}\alpha(\frac{-1}{2})^{\alpha+1}}d\alpha+\frac{1}{2i\pi}\int_{D}\frac{\Gamma(\alpha+1)t^{-\alpha-1}}{\sin\frac{\pi}{2}\alpha}\sum_{n\geq 1}\frac{1}{(2n+\frac{1}{2})^{1+\alpha}}d\alpha$, for make the function $(-2)\sum_{k\geq 0}e^{-tu_k}$ $,\Re t>0$ appears which admits a meomorphic extension with the simple poles at the points $\pm i\ln m, \Lambda(m)\neq 0$ on C - IR⁻ with the résidu $-\frac{\Lambda(m)}{m^{\frac{1}{2}}}\frac{1}{\pi}$ and no other poles.

To compute $(-2)\sum_{k\geq 0} e^{-tu_k}$ on the imaginary axes, we use the fact that the function is real on the positive real axes. Hence, the real part is the half sum of values in $iv, v>0$ and $-iv$. We consider first $\Re\frac{1}{2i\pi}\int_D \frac{\Gamma(\alpha+1)t^{-\alpha-1}}{\sin\frac{\pi}{2}\alpha}\sum_{n\geq 1}\frac{1}{(2n+\frac{1}{2})^{1+\alpha}}d\alpha = -2\sum_{n\geq 1} e^{-|\Im t|(2n+\frac{1}{2})} = \ -2\frac{e^{-\frac{5}{2}|\Im t|}}{1-e^{-2|\Im t|}}$

$(t=iv)$

Thus is obtain by simplification by $\sin\frac{\pi}{2}\alpha$ and the computation of $\int_D \frac{\Gamma(\alpha+1)}{u^{\alpha+1}}d\alpha$ (cf point5).

We have to compute: $\frac{1}{2i\pi}\int_\Delta H_\alpha^{\ \varepsilon}(iv)d\alpha + \frac{1}{2i\pi}\int_\Delta H_\alpha^{\ \varepsilon}(-iv)d\alpha$ with v>0. But $(iv)^{-\alpha-1}+(-iv)^{-\alpha-1} = 2\cos\frac{\pi}{2}(\alpha+1)v^{-\alpha-1} = -2\sin\frac{\pi}{2}\alpha v^{-\alpha-1}$.We finally have to compute

$\frac{1}{2i\pi}\int_\Delta \frac{\Gamma(\alpha+1)(iv)^{-\alpha-1}}{\sin\frac{\pi}{2}\alpha(\frac{-1}{2})^{\alpha+1}}d\alpha + \frac{1}{2i\pi}\int_\Delta \frac{\Gamma(\alpha+1)(-iv)^{-\alpha-1}}{(\frac{-1}{2})^{\alpha+1}\sin\frac{\pi}{2}\alpha}d\alpha$ with $v\neq 0$.

By the Phragmen-Lindelof method and the comples Stirling formula, we have to compute the 'intégral on E which is the edge of the half band $|\Im\alpha|\leq\delta \quad \Re\alpha\leq 0$.

Then we obtain by the residue theorem by bounding sucessively by the segments $t_n + iw \ |w|\leq\delta \quad t_n = \frac{n+(n+1)}{2}$ that the intégral has for value $-2\sum_{n\geq 0}\frac{(-1)^n}{n!}(-\frac{1}{2})^n v^n = -2\ e^{\frac{v}{2}}$.

Finally $\sum_{k\geq 0} e^{-tu_k}\ \Re t>0$ has a meromorphic extension à $C-R^-$, as simple poles $\pm(i\ln m)\ \ \Lambda(m)\neq 0$ with the résidues $\frac{1}{2\pi}\frac{\Lambda(m)}{m^{\frac{1}{2}}}$. Its real parts on the imaginary axis if $t\neq\mp i\ln m$: $e^{\left|\frac{\Im t}{2}\right|}+\frac{e^{\frac{-5}{2}|\Im t|}}{1-e^{-2|\Im t|}}$.

Then the study of the function $\sum_{k\geq 0} e^{-tu_k}\ \Re t>0$ is completed . The other points are devoted:to refute the Riemann Hypothesis

7- Computing an explicit analytic continuation $t \to \sum_{k\geq 0} e^{-t(u_k - i\lambda)} \Re t > 0 \quad \lambda \in IR, \Im t > 0$

On the positive imaginary axis the real part of $F(t) = \sum_{k\geq 0} e^{-tu_k} \Re t > 0$, $t \neq i \ln n(\Lambda(n)) \neq 0$ is

$$-\frac{1}{2}e^{\frac{\Im t}{2}} + \frac{e^{-\frac{5}{2}\Im t}}{1-e^{-2\Im t}}.$$

The function $F(t)$ is so continuous in *t,* $\Re t \geq 0, t \neq i \ln n, \Lambda(n) \neq 0, \Im t > 0$.

We can then apply the Schwarz reflection principle to the function

$$M(-it) = iF(t) + \frac{i}{2}e^{\frac{1}{2}(-it)} - i\frac{e^{\frac{5}{2}it}}{1-e^{2it}}.$$

We write: $iF(t) = iF(i(-it))$

The analytic continuation at the point $\overline{-it}$, $\Im t > 0$ is

$$\Re(iF(t) + \frac{i}{2}e^{\frac{1}{2}(-it)} - i\frac{e^{\frac{5}{2}it}}{1-e^{2it}}) - -i\Im(\ (iF(t) + \frac{i}{2}e^{\frac{1}{2}(-it)} - i\frac{e^{\frac{5}{2}it}}{1-e^{2it}})$$

It is sufficient now to m ultiply by $e^{-i\lambda t}$ to obtain the meromorphic continuation wished : it will be holomorphic in λ when we fix t and meromorphic in t when we fix λ.

We have also an harmonic continuation of $t \to \sum_{k\geq 0} e^{-\bar{t}(u_k + i\lambda)} \Re t > 0$, $\lambda \in C, \Im t > 0$.

In fact we use the analytic continuation of $\sum_{k\geq 0} e^{-t'(u_k + i\lambda)} \Re t > 0, \lambda \in C, \Im t' < 0.$

8-Continuation to $\delta < \lambda < \frac{1}{2} <$ of the function $\lambda \to \sum_{n\geq 2} \frac{\Lambda(n)}{n^{\frac{1}{2}+\pi}} \frac{1}{(t - i\ln n)} \quad \lambda > \frac{1}{2}$, t fixed,

$\Re t \neq 0, \Im t > 0$., if $\sup_k |\Im u_k| < \delta < \frac{1}{2}$, and of the function $\lambda \to \sum_{n\geq 2} \frac{\Lambda(n)}{n^{\frac{1}{2}+\pi}} \frac{1}{(\bar{t} + i\ln n)} \quad \lambda > \frac{1}{2}$

Let be $\varphi_\lambda(t)$ the sum of the two functions, which is harmonic.

For $\lambda \in C, \Re\lambda > \frac{1}{2}$ the series(t fixed $\# i \ln n$), $\sum_{n\geq 2} \frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}} \frac{1}{(t - i\ln n)}$ $\lambda > \frac{1}{2}$ is convergent.

We separe n=p prime from the $p^m, m \geq 2$. Let be $\varepsilon \in]0,1[$.

We have using integration by part for Stiejles measure

$$\sum_{p\geq 2} \frac{\ln(p)}{p^{\frac{1}{2}+\lambda-i\varepsilon}} \frac{1}{(t - i\ln p)} \quad (\lambda > \frac{1}{2})$$

$$= \int_2^{+\infty} \pi(x) \left[\frac{\partial}{\partial X} \frac{\ln X}{X^{\frac{1}{2}+\lambda-i\varepsilon}} \times \frac{1}{t - i\ln X} \right]_{X=x} dx$$

Here $\pi(x)$ is the number of prime integers $\leq x$.

Suppose now that the zeros of ξ satisfied

$\sup_k |\Im u_k| < \delta < \frac{1}{2}$; then classicaly (Marc Hindry, oral communication):

$$\pi(x) = \int_2^x \frac{dv}{\ln v} + O(x^{\frac{1}{2}+\delta})$$

Hence we have

$$(1) \int_2^{+\infty} \pi(x) \left[\frac{\partial}{\partial X} \frac{\ln X}{X^{\frac{1}{2}+\lambda-i\varepsilon}} \times \frac{1}{t - i\ln X} \right]_{X=x} dx =$$

$$\int_2^{+\infty} (\int_2^x \frac{dv}{\ln v}) \left[\frac{\partial}{\partial X} \frac{\ln X}{X^{\frac{1}{2}+\lambda-i\varepsilon}} \times \frac{1}{t - i\ln X} \right]_{X=x} dx + \int_2^{+\infty} O(x^{\frac{1}{2}+\delta}) \left[\frac{\partial}{\partial X} \frac{\ln X}{X^{\frac{1}{2}+\lambda-i\varepsilon}} \times \frac{1}{t - i\ln X} \right]_{X=x} dx$$

Then

$$\int_2^{+\infty} (\int_2^x \frac{dv}{\ln v}) \left[\frac{\partial}{\partial X} \frac{\ln X}{X^{\frac{1}{2}+\lambda-i\varepsilon}} \times \frac{1}{t - i\ln X} \right]_{X=x} dx =$$

$$\left[\int_2^x \frac{dv}{\ln v}) \frac{\ln x}{x^{\frac{1}{2}+\lambda-i\varepsilon} (t - i\ln x)} \right]_{x=2}^{x=+\infty} - \int_2^{+\infty} (\frac{1}{\ln x} \frac{\ln x}{x^{\frac{1}{2}+\lambda-i\varepsilon}} \frac{1}{(t - i\ln x)} dx$$

$$=\int_2^{+\infty}(.\frac{1}{x^{\frac{1}{2}+\lambda-i\varepsilon}}\frac{1}{(t-i\ln x)}dx$$

if $\Re\lambda > 4$.

The terms containing $O(x^{\frac{1}{2}+\delta})$ have an easy analytic continuation .

Now , if we consider the $p^m, m \geq 2$ using the integration by parts, they give an holomorphic function in λ for real t #0 $\Im t > 0$ $\Re\lambda > \delta$. We must now find a continuation of :

$\int_2^{+\infty}(.\frac{1}{x^{\frac{1}{2}+\lambda-i\varepsilon}}\frac{1}{(t-i\ln x)}dx$. We make the variable change $x = e^v$ and obtain:

$$\int_2^{+\infty} e^{v(\frac{1}{2}-\lambda+i\varepsilon)}\frac{dv}{(t-iv).}$$

We write now

$$\int_{\ln 2}^{+\infty} e^{v(\frac{1}{2}-\lambda+i\varepsilon)}\frac{dv}{(t-iv).} = \int_0^{+\infty} e^{v(\frac{1}{2}-\lambda+i\varepsilon)}\frac{dv}{(t-iv).}$$

The last term is an integral function in λ , holomorphic in t for $\Im t > \ln 2$ and converging to zero when $\Im t \to +\infty$ real t fixed.

We now make an integration by parts.We obtain:

$$\int_0^{+\infty} e^{v(\frac{1}{2}-\lambda+i\varepsilon)}\frac{dv}{(t-iv).} = \frac{i}{\frac{1}{2}-\lambda+i\varepsilon}\int_2^{+\infty}\left[\frac{e^{v(\frac{1}{2}-\lambda+i\varepsilon)}dv}{(t-iv)^2}+\frac{i}{t}\right] =(*)$$

We consider the quadrant (in t) limited by the real positive half axes and the positive imaginary half axis.

We close the pass by a quarter of circle with center O and radius R growing t the infinity.

We can apply the residue theorem and obtain:

(1) if $\Re$ t < 0 and $\Im t > 0$.

there is a pole in $v = -it$.

Hence:

$$(*)=\frac{-1}{\frac{1}{2}-\lambda+i\varepsilon}\int_2^{+\infty}\frac{e^{iw(\frac{1}{2}-\lambda+i\varepsilon)}dw}{(t+w)^2}+2\pi e^{-it(\frac{1}{2}-\lambda+i\varepsilon)}+\frac{1}{t(\frac{1}{2}-\lambda+i\varepsilon)}$$

2) If $\Re t>0,\Im t>0$ there is no pole.

$$(*)=\frac{-1}{\frac{1}{2}-\lambda+i\varepsilon}\int_2^{+\infty}\frac{e^{iw(\frac{1}{2}-\lambda+i\varepsilon)}dw}{(t+w)^2}+\frac{1}{t(\frac{1}{2}-\lambda+i\varepsilon)}.$$

if we fix ε, we can have an alone analytic continuation in λ with $|\Im\lambda|<\varepsilon$.

Finally, if $\varepsilon\to 0$ we have then continuation by continuity for $\delta<\lambda<\frac{1}{2}$, t fixed

$\Re t\neq 0,\Re t>\ln 2$ of $\lambda\to\sum_{n\geq 2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}}\frac{1}{(t-i\ln n)}\quad \lambda>\frac{1}{2}$.

It preserves the analycity in t $\Re t\neq 0$ as we can see by the Cauchy-Morera theorem [4].

We have now to consider $\sum_{n\geq 2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}}\frac{1}{(\bar{t}+i\ln n)}\quad \Re\lambda>\frac{1}{2}$.

We obtain for $\Re t\neq 0$ the value in $\bar{t}$ of analytic function the alone problem being to find a

continuation in λ of $\int_0^{+\infty}e^{v(\frac{1}{2}-\lambda+i\varepsilon)}\frac{dv}{(\bar{t}+iv).}$.

By the theorem of residue we have to find continuation by the same way than later.

The pole is for $v=\frac{-\bar{t}}{i}=i\bar{t}$, hence the residue appears for $\Re t>0,\Im t>0$ and gives:

$$\frac{-1}{\frac{1}{2}-\lambda+i\varepsilon}\int_{2}^{+\infty}\frac{e^{iw(\frac{1}{2}-\lambda+i\varepsilon)}}{(\bar{t}-w)^{2}}dw+\frac{1}{\bar{t}(\frac{1}{2}-\lambda+i\varepsilon)}$$. for $\Re t<0$ and

$$\frac{-1}{\frac{1}{2}-\lambda+i\varepsilon}\int_{2}^{+\infty}\frac{e^{iw(\frac{1}{2}-\lambda+i\varepsilon)}}{(\bar{t}-w)^{2}}dw+\frac{1}{\bar{t}(\frac{1}{2}-\lambda+i\varepsilon)}+2\pi e^{i\bar{t}(\frac{1}{2}-\lambda+i\varepsilon)}$$

9-Some properties of conformal transformations and the Poisson kernel.

We consider a rectangle in the complex plane with summits *A=0 B=1 C= iT+1 D=iT* $T>0$

Then [1] the half plane $\Re z\geq 0$ is conformly equivalent to a similar rectangle by the conform

transformation Z= $K\int_{0}^{z}v^{\frac{-1}{2}}\,(v-1)^{\frac{-1}{2}}(v-x)^{\frac{-1}{2}}dv$ with x such that

$$\int_{0}^{+\infty}v^{\frac{-1}{2}}\,(v+1)^{\frac{-1}{2}}(v+x)^{\frac{-1}{2}}dv=\quad T\int_{0}^{1}v^{\frac{-1}{2}}\,(1-v)^{\frac{-1}{2}}(x-v)^{\frac{-1}{2}}dv.$$

To put exactly the half plane upon *ABCD* we must take $K=x^{\frac{1}{2}}a_{x}$ with a_{x} such that

$$a_{x}\,x^{\frac{1}{2}}\int_{0}^{1}v^{\frac{-1}{2}}\,(1-v)^{\frac{-1}{2}}(1-\frac{v}{x})^{\frac{-1}{2}}dv=1.$$

When $T\rightarrow\infty\;\; x\rightarrow+\infty\; a_{x}\rightarrow a$ constant .

Then the conform transformation upon the rectangle converges uniformly on every compact to the conform transformation of the half plane A and B fixed, this half band is

$\{z|\Im z\geq 0, 0\leq\Re z\leq 1\}$We consider now a pentagon A'B'C'D'E' with:

A'=-1, B'=1,C'=1+iT,E'=-1+i, given by the property that the angle E' D'C' is $\theta\in\left]0,\pi\right[,\theta$ fixed and D' on the imaginary axis.

Let be D'=iv_T

Consider the function

$\Re\frac{e^{+it(\frac{1}{2}-\lambda)}}{(t-iv_T)^{1+\eta}.}$ $0<\eta<\frac{1}{2}$ which is harmonic in t and given by its Poisson kernel cf [1] .

This function is positive on the pentagon .

Consider the restriction of the function to the segment E' C', then it is bounded by $\Re\frac{e^{iT(\frac{1}{2}-\lambda)}}{2^{1+\eta}.}$'.

This function is positive on the rectangle A'B'C'E'and given by its Poisson Kernel.

The contribution in the Poisson kernel of the value of the function on the segment E'C'

converges to 0 when $T\to+\infty$

10- Computing the function

$$t\to\sum_{k\geq0} e^{-t(u_k-i\lambda)}+\sum_{k\geq0} e^{-\bar{t}(u_k+i\lambda)}+\frac{1}{2\pi}\sum_{n\geq2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}(t-i\ln n)}+\frac{1}{2\pi}\sum_{n\geq2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}(\bar{t}+i\ln n)}=G_0{}^{\lambda}(t)$$

$\Re\lambda>\frac{1}{2}$ $\Re t\neq0$ continued at $\delta<\lambda<\frac{1}{2}$ $\Im t\geq1$ $|\Re t\leq1|$ if $\underset{k}{Sup}|\Im u_k|<\delta$.

On the half band $\Im t\geq1$ $|\Re t|\leq1$. The function introduced is harmonic in t everywhere for

$\Re\lambda>\frac{1}{2}$ and defined.

We have that the function $\frac{1}{2\pi}\sum_{n\geq2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}}\frac{1}{(t-i\ln n)}+\frac{1}{2\pi}\sum_{n\geq2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda}}\frac{1}{(\bar{t}+i\ln n)}$ can be continued for

$\Re t\neq0$. and $\delta<\lambda<\frac{1}{2}$,introducing $\lambda-i\varepsilon$, by an alone way.

The function $t\to\sum_{k\geq0} e^{-t(u_k-i\lambda)}$ is meromorphic on the half band and $\sum_{k\geq0} e^{-\bar{t}(u_k+i\lambda)}$ is

harmonic for *t#i*ln*n*. $\Lambda(n)\neq0$

On the half band ,the function

$$t \to \sum_{k\geq 0} e^{-t(u_k - i\lambda-\varepsilon)} + \sum_{k\geq 0} e^{-\bar{t}(u_k + i\lambda+\varepsilon)} + \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda-i\varepsilon}(t-i\ln n)} + \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda-i\phi}(\bar{t}+i\ln n)} = G_\varepsilon{}^{\lambda}(t)$$

is majored if $\lambda \in [2,+\infty[\cup]\delta,\frac{1}{2}[$ by $e^{\eta|t|} > 0$ because:

-for $\Re t > 2$ the function is bounded on the half band .

-for $\delta < \lambda < \frac{1}{2}$ the band is $e^{\eta|t|}$ Cf paragraph 8

We have to prove now that $G_\varepsilon{}^{\lambda}(t)$ is given by the Poisson kernel on the half band.

Let be *v*>0 such that the polygonal line which limits the intersection of the sector of summit *iv* and angular at the summit $0 < \theta < \frac{\pi}{4}$ and $|\Re t| < 1$ we have that

$$\left|\frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda-i\varepsilon}(t-i\ln n)} + \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{n^{\frac{1}{2}+\lambda-i\varepsilon}(\bar{t}+i\ln n)}\right| \leq K\frac{v}{\ln v}$$ for $\Re\lambda > 2$ and for *v* going to infinity . Such *v* can obtain by driving to absurdity.

Recall that the number of zeros in iu_k with $|u_k| < v$ is given by

$: \frac{1}{2i\pi}v\ln v + \frac{1+\ln 2\pi}{2\pi} + O(\ln v)$.

hence on the polygonal line we have:

$$\left|\sum_{k\geq 0} e^{-t(u_k - i\lambda-\varepsilon)} + \sum_{k\geq 0} e^{-\bar{t}(u_k + i\lambda+\varepsilon)}\right| \leq A\frac{|\ln|t-iv||}{|t-iv|}e^{|t|\left|\frac{1}{2}-\lambda\right|}$$

Now the function $\Re\frac{e^{+it(\frac{1}{2}-\lambda)}}{(t-iv)^{1+\eta}}$ $0 < \eta < \frac{1}{2}$ will be harmonic on the pentagon limited by the edge of the half band and the edge of the sector with the explicite writting of the conform transformation of this domain on to the unit disc(cf[1]) . The contribution of $\Re\frac{e^{+it(\frac{1}{2}-\lambda)}}{(t-iv)^{1+\eta}}$ on

the edge of the sector converges to 0 when $v\to+\infty$ by the paragraph 9.

So $G_\varepsilon{}^\lambda(t)$ is bounded on the edge of the half band and given by its Poisson kernel.

It is harmonic in t when λ is fixed.

Let be $Q(T,t)$ the Poisson kernel on the half band transported from the unit disc .(T is on the edge Δ, dT is the Lebesgue mesure on the edge). We have :

$G_\varepsilon{}^\lambda(t)=\int_\Delta Q(T,t)G_\varepsilon{}^\lambda(T)dT$ if $\Re\lambda>2$.

This equality in extended when t is fixed analyticaly (in λ) at $\delta<\lambda<\frac{1}{2}$ when t is in the half band, by an alone way.

But when $\varepsilon\to 0$, the two members converge if $\Re t\neq 0$ so : $G_0{}^\lambda(t)=$

$\int_\Delta Q(T,t)G_0{}^\lambda(T)dT$, $\delta<\lambda<\frac{1}{2}$.

Finally $\int_\Delta Q(T,t)G_0{}^\lambda(T)dT$ is harmonic inside the half band, so $G_0{}^\lambda(t)$ is harmonic in t inside then half band . Its expression for $\Re t\neq 0$ is given by paragraphs7 and 8.

The function $G_0{}^\lambda(t)$, $\delta<\lambda<\frac{1}{2}$ is given by $2\pi e^{\bar{i}\bar{t}(\frac{1}{2}-\bar{\lambda})}+K_\lambda(t)$ for $\Re t<0$ where $K_\lambda(t)$ is harmonic in t and converges to 0 when $\Im t\to+\infty$, $-1<\Re t<0$, $\Re t$ fixed and given by

$2\pi e^{\bar{i}\bar{t}(\frac{1}{2}-\bar{\lambda})}+J_\lambda(t)$ if $0<\Re t<1$, $J_\lambda(t)\to 0$ when $\Im t\to+\infty$ $\Re t$ fixed

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11-End of the proof

Following points 8 and 10, $G_0^\lambda(t)$ is extended to t fixed $|\Re(t)| \le 1$ $\delta < \lambda_0 < \frac{1}{2}$ if

$\sup_k |\Im u_k| < \delta < \frac{1}{2}$ in tthe same and unique way.

By the part 8, we have that the function $\varphi_{\lambda_0}(t)$ can be written for $\Re t > 0$ $\Im t > \ln y_0$:

$$\frac{1}{(\frac{1}{2}-\lambda_0)}\int_0^\infty \frac{e^{iw(\frac{1}{2}-\lambda_0)}}{(t+w)^2}dw + \frac{1}{(\frac{1}{2}-\lambda_0)}\int_0^\infty \frac{e^{iw(\frac{1}{2}-\lambda_0)}}{(\bar{t}-w)^2}dw + 2\pi e^{i(\frac{1}{2}-\lambda_0)\bar{t}} +$$

$$\int_{y_0}^\infty 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{t-i\ln X})\right]_{X=x} dx +$$

$$\int_{y_0}^\infty 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{\bar{t}+i\ln X})\right]_{X=x} dx + \sum_{m\ge 2}\sum_{p\ premier}\frac{\ln p}{(t-i\ln p^m)}\frac{1}{p^{(\frac{1}{2}+\lambda_0)m}} +$$

$$\sum_{m\ge 2}\sum_{p\ premier}\frac{\ln p}{(\bar{t}+i\ln p^m)}\frac{1}{p^{(\frac{1}{2}+\lambda_0)m}} + K_{\lambda_0}(t)$$ with $K_{\lambda_0}(t)\to 0$ when $\Im t\to\infty$ $\Re t$ fixed(possibily zero)

We substract $2\pi e^{i(\frac{1}{2}-\lambda)\bar{t}}$ and we obtain : $\Re[\sum_{k\ge 0} e^{-t(u_k - i\lambda_0)}]$

$$+\Re\int_{y_0}^\infty 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{t-i\ln X})\right]_{X=x} dx +$$

$$\Re\int_{y_0}^\infty 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{\bar{t}+i\ln X})\right]_{X=x} dx$$

$=\Re M_{\lambda_0}(t)$ with $M_{\lambda_0}(t)$ bounded when $\Im t\to\infty$ $\Re t$ fixed in the half band $|\Re t \le 1|\Im t \ge 1$, Let $w = n$ be with $[n-2, n+2]\cap[p^m - 1, p^m + 1] = \phi$ $\forall m\in N$ $\forall p$ prime.

Let C_n be the half circle of center in and or radius 1located on the side of real negative parts.

Then $0(x^{\frac{1}{2}+\delta})$ is analytic on the disc D_n of center in and radius 1. Then :

$$\lim_{\substack{t\to i\ln n\\ \Re t>0}}\Re\{\int_{y_0}^\infty 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{t-i\ln X})\right]_{X=x} dx$$

$$+\int_{y_0}^\infty 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{\bar{t}+i\ln X})\right]_{X=x} dx\} =$$

$$\Re\{\int_{C_n} 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{i\ln n - i\ln X})\right]_{X=x} dx$$

$$+\int_{C_n} 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{-i\ln n+i\ln X}) \right]_{X=x} dx \ \}$$

$$+\Re\{ \int_{[y_0,n-1]\cup[n+1,+\infty]} 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{i\ln n-i\ln X}) \right]_{X=x} dx+$$

$$\int_{[y_0,n-1]\cup[n+1,+\infty]} 0(x^{\frac{1}{2}+\delta})\frac{\partial}{\partial X}\left[(\frac{\ln X}{X^{\frac{1}{2}+\lambda_o}})\times(\frac{1}{-i\ln n+i\ln X}) \right]_{X=x} dx \ \}$$

These expressions are null . But $\Re(\sum_{k\geq 0} e^{-t(u_k-i\lambda_0)})$ for t=iw $1t \neq i\ln m\Lambda(m)\neq 0$ has the following value $e^{(\frac{1}{2}-\lambda_0)w}+\frac{e^{-(\frac{5}{2}+\lambda_0)w}}{1-e^{-2w}}$ obtained by continuity.

We have to substract $e^{i(\frac{1}{2}-\lambda)\bar{t}}$ on the imaginary axis. But for t $\in C$ $\Re(t)>0, \Im(t)\neq i\ln m$ sufficiently near iw $w>0$, we have a value $e^{(\frac{1}{2}-\lambda_0)w}+\frac{e^{-(\frac{5}{2}+\lambda_0)w}}{1-e^{-2w}}+\varepsilon(t)$, $\varepsilon(t)$ near 0. So the function is not bounded on the half band and we get a contradiction..

14-Conclusion

We have proved the :

Theorem: The Riemann hypothesis is false . There exist an infinity of non trivial zeros s_k of $\zeta(s)$ such that they real part admits for accumulation points 0 and 1

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Palaiseau, le 4/03/2016