A SIMPLE PROOF OF THE RIEMANN HYPOTHESIS ©

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This paper is dedicated to Bernard Riemann, Leonhard Euler,
Daniel Bernoulli, Shrinivasa Ramanujan, and Marc Prevost,
and
All the great mathematicians who have
worked on this problem.

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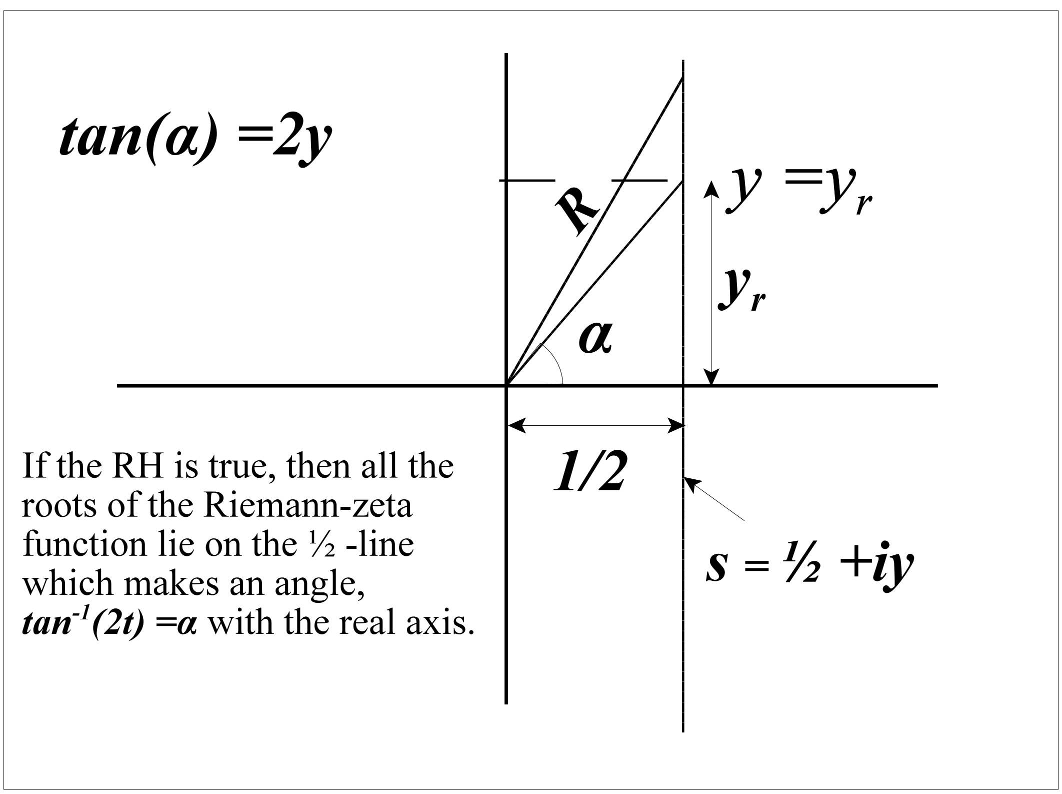
ABSTRACT: The history of the Riemann Hypothesis, (RH), is well known. In 1857, the German mathematician Bernard Riemann presented a paper to the Berlin Academy of Mathematics. In that paper, he proposed that a certain function, now called the Riemann-zeta function, $\zeta(s)$, takes the values, $\zeta(s)=0$, on the complex plane, when $s=½+it$, where t is real. This hypothesis has great significance for the world of mathematics and physics and the solution would lead to innumerable completions of theorems that rely upon its truth. Over a billion zeros of the function have been calculated by computers and shown to all lie on this line. In this paper, I use a general form of the Riemann-zeta function that becomes the Riemann-zeta function only when the Riemann Hypothesis is true. I prove that the complex roots of the Riemann's zeta function lie on the line, $s = ½ +it$. I further prove that the trivial roots of the Riemann-zeta function lie at points $s = -2n$, where n is an integer. The road to a solution is surprisingly not too complicated and requires the use of many well known special functions. In this short paper, I present a complete solution to the Riemann hypothesis. The solution is valid for all trivial and non-trial roots. The solution unifies the trivial and the non-trial roots as represented on the half-line.

§1 INTRODUCTION.

The Riemann hypothesis implies that the roots of the Riemann-zeta function, *ζ(z)=0,* all lie on a straight line, $z = ½ + iy$, in the complex plane. It is known that there exists trivial roots which lie on the negative real axis at negative even values of ***z***. Let, ***R,*** be defined as a *root vector*. ***R*** is the vector joining a root of the function, ***y,*** restricted to the vertical ½-line on the complex plane to the origin of the same complex plane. The geometric significance of the hypothesis is that on the two right quadrants of the complex plane, all the *root vectors,* ***R,*** must form *root angles* with the real axis such that:

$$\tan(\alpha) = 2y \quad \textbf{(1)}$$



The Riemann-hypothesis imposes on the Riemann-zeta function certain restrictions on the variable ***y,*** and implies that the non-trivial *root line* is the complex line $z = ½ + iy$. Then, if the RH is true, the function $\zeta(½+ iy)$ vanishes when, $z = ½ + it,$ for some values of ***y,*** that are roots of the function. As the angular variable, α, goes from $-\pi/2 \rightarrow +\pi/2$, the variable, ***y,*** takes all possible values, and the root vector should intersect all the complex roots on the straight line $z = ½ + iy$. Advantageously, one can envision that there are particular *root angles* at which these *"root vectors"* will intercept the line $z = ½ + iy$, when ***y is a root***. It is then convenient that a particular form of the Riemann-zeta function should be used to manifest these roots as particular cases when, $\zeta(z)=0$. Once one has established that the *roots vectors* can be represented as a geometric form (1), one can make the general form of the Riemann-zeta function identically zero when a *root vector* meets a root. There is a large number of zeroes that have been calculated so far, and all of these zeros are known to be symmetrically placed about the real line on the line $z = ½ + iy$. In consideration of this "angular proposition" as a clue to the solution of the hypothesis, one needs to find an appropriate equation that holds up to scrutiny.

§2.0 Solution of the Riemann Hypothesis for the trivial roots.

Define the Riemann-zeta function as:

$$\zeta(s) = \frac{2^{(s-1)} s}{(2^s - 1)(s-1)} + \frac{2}{2^s - 1} \int_0^\infty \frac{\left(\frac{1}{4} + t^2\right)^{\left(-\frac{s}{2}\right)} \sin(s \operatorname{acrtan}(2t))}{e^{(2\pi t)} - 1} dt \qquad \textbf{(2)}$$

This formula is found in reference [5], *Table of integrals, Series, and Products: I.S. Gradshteyn, M. Ryzhik, Allan Jeffrey.*

Theorem 1: *For left-half of the complex plane, the roots of* ***ζ (z)=0****, all lie on negative even values of z.*

This proof is already well known and will only be repeated here for completion.

Proof: The function (2) can be transformed to a generalized trigonometric equation by the substitution:

$$\tan(\theta) \quad = 2t \qquad \textbf{(3)}$$

Let ***s* = ½ + *it*,** then,

$$\zeta(z) = \frac{2^{(z-1)}\, z}{(2^z - 1)(z-1)} + \frac{2}{2^z - 1}\int_0^{\infty} \frac{\left(\frac{1}{2}\sec(\theta)\right)^{(-z)} \sin(z\,\theta)\left[\frac{1}{2}\sec(\theta)^2\right]}{\mathbf{e}^{(\pi\tan(\theta))} - 1}\, d\theta \qquad \textbf{(4)}$$

This equation is valued for the range of integration of the variable ***θ***. Now, I use the extended Chebyshev map (Euler form) to reduce the equation to a simpler equation in, ***θ***, for all values of ***z***. Noting that:

$$\sin(z\theta) = \sqrt{1 - \left(\frac{1}{2}\left[\cos(\theta) + i\sqrt{1-(\cos(\theta))^2}\right]^z + \frac{1}{2}\left[\cos(\theta) - i\sqrt{1-(\cos(\theta))^2}\right]^z\right)^2}$$

the function becomes:

$$\zeta(z) = \frac{2^{z-1}}{2^z - 1}\frac{z}{z-1} + \frac{2^z}{2^z - 1}\int_0^{\frac{\pi}{2}} \frac{\cos^{2z-2}\theta\sqrt{\cos^{-2z}\theta - \left(\frac{1}{2}\left[[1+i\tan\theta]^z + [1-i\tan\theta]^z\right]\right)^2}}{e^{\pi\tan\theta} - 1}\, d\theta \qquad \textbf{(6)}$$

Equation (6) is valid for all values of ***θ***. Now, modify the equation by inserting a factor, cos^z***θ***, outside the radical and then, reinserting, cos^{-2z}***θ*,** inside the radical. The next step is to change the variables again by using *X = tan θ,* then, advantageously, (6) becomes*:*

$$\zeta(z)=\frac{2^{z-1}}{2^z-1}\frac{z}{z-1}+\frac{2^z}{2^z-1}\int_0^\infty\frac{\left(\frac{1}{\sqrt{1+X^2}}\right)^{2s}\sqrt{\left(\frac{1}{\sqrt{1+X^2}}\right)^{-2z}-\left(\frac{1}{2}\left[\left[1+iX\right]^z+\left[1-iX\right]^z\right]\right)^2}}{e^{\pi X}-1}dX \qquad \textbf{(7)}$$

By expanding inside the radical and factoring the quadratic that results, equation (7) can be reduced to the equation:

$$\zeta(z)=\frac{2^{z-1}}{2^z-1}\frac{z}{z-1}+\frac{2^z}{2^z-1}\int_0^\infty\frac{\sqrt{\left(\frac{1}{1+X^2}\right)^z-\left(\frac{1}{2}\left[\frac{\left[X+iX\right]^z}{\left[1+X^2\right]^z}+\frac{\left[X-iX\right]^z}{\left[1+X^2\right]^z}\right]\right)^2}}{e^{\pi X}-1}dX \qquad \textbf{(8)}$$

This can again be reduced by factoring the conjugates;

$$\zeta(z)=\frac{2^{z-1}}{2^z-1}\frac{z}{z-1}+\frac{2^z}{2^z-1}\int_0^\infty\frac{\sqrt{\left(\frac{1}{1+X^2}\right)^z-\left(\frac{1}{2}\left[\frac{1}{\left[1-iX\right]^z}+\frac{1}{\left[1+iX\right]^z}\right]\right)^2}}{e^{\pi X}-1}dX \qquad \textbf{(9)}$$

The equation (9), can now be further reduced to the simpler form conjugates:

$$\zeta(z) = \frac{2^{z-1}}{2^z - 1}\frac{z}{z-1} + \frac{2^z}{2(2^z - 1)}\int_0^\infty \frac{i\left[\left[1+iX\right]^{-z} - \left[1-iX\right]^{-z}\right]}{e^{\pi X} - 1} dX \qquad \textbf{(10)}$$

Note that the conjugate terms are a feature of the Abel-Plana type relation (10), and this fact is a recurring property of the function.

Putting:

$$\frac{\left[\left[1+iX\right]^{-z} - \left[1-iX\right]^{-z}\right]}{e^{\pi X} - 1} = \frac{A}{e^{\pi X} + 1} + \frac{2A}{e^{2\pi X} - 1} \qquad \textbf{(11)}$$

Now, this can be reduced to two separate functions by the substitution:

$$A := \left[\left[1+iX\right]^{-z} - \left[1-iX\right]^{-z}\right]$$

so that:

$$\zeta(z) = \frac{2^{z-1}}{2^z - 1}\frac{z}{z-1} + \frac{2^z}{2(2^z - 1)}\int_0^\infty \frac{\left[\left[1+iX\right]^{-z} - \left[1-iX\right]^{-z}\right]}{i\left(e^{\pi X} + 1\right)} + \frac{2\left[\left[1+iX\right]^{-z} - \left[1-iX\right]^{-z}\right]}{i\left(e^{2\pi X} - 1\right)} dX \qquad \textbf{(13)}$$

This equation identically disappears when ***z*** *= -2n for the integral.*

$$\zeta(z) = \frac{2^{-2n-1}}{2^{-2n} - 1}\frac{-2n}{-2n-1} - \frac{2^{-2n-1}}{(2^{-2n} - 1)}\frac{-2n}{-2n-1} = 0 \qquad \textbf{(14)}$$

this proves that the function vanishes for negative even values of ***z.***

§3.0 Solution of the Riemann Hypothesis For the half-line.

I propose a solution for the Riemann Hypothesis that is only valid, iff, certain restrictions apply. These restriction are specially that the roots lie symmetrically on the half-line (a fact already proved) and that the function vanishes identically for roots on both positive and negative sides of the real line. However, this proof also shows that the real negative solutions are continuations of the complex roots at infinity.

***Theorem 2*: *All the roots of ζ (z) = 0, lie on ½ - line, z = ½ + iy, where y is a positive, negative, or complex number*.**

Consider equation (10) above,

$$\zeta(z) = \frac{2^{z-1}}{2^z - 1}\frac{z}{z-1} + \frac{2^z}{2(2^z-1)}\int_0^\infty \frac{i\left[\left[1+iX\right]^{-z} - \left[1-iX\right]^{-z}\right]}{e^{\pi X}-1}dX \quad (10)$$

Now, the terms $(1 \pm iX)^{-z}$ in (11) can be expanded as a convergent series, for $X > 1$, so that one can now represent the function for values of z as:

$$\left[1 \pm iX\right]^{-z} = \sum_{n=0}^{\infty}\left[\frac{(\pm iX)^n \Gamma(1-z)}{\Gamma(-z-n+1)n!}\right] \quad \textbf{(16)}$$

and so,

$$\zeta(z) = \frac{2^{z-1}}{2^z-1}\frac{z}{z-1} + \frac{z^{z-1}}{(2^z-1)}\left(\int_0^\infty\left(\frac{i\left[-(-i)^n\left[\sum_{n=1}^{\infty}\left[\frac{(X)^n\Gamma(1-z)}{\Gamma(-z-n+1)n!}\right]\right]+(i)^n\left[\sum_{n=1}^{\infty}\left[\frac{(X)^n\Gamma(1-z)}{\Gamma(-z-n+1)n!}\right]\right]\right]}{e^{\pi X}-1}\right)\right)dX \quad \textbf{(17)}$$

then, this is again expressed as conjugate terms;

$$\zeta(z)=\frac{2^{z-1}}{2^{z}-1}\frac{z}{z-1}+\frac{2^{z-1}}{(2^{z}-1)}\left[i\left[\begin{array}{l}-\sum_{n=1}^{\infty}\left[\frac{\Gamma(1-z)}{\Gamma(-z-n+2)(n-1)!}\right]\left[(-i)^{n-1}\int_{0}^{\infty}\frac{(X)^{n-1}}{e^{\pi X}-1}dX\right]\\ +\sum_{n=1}^{\infty}\left[\frac{\Gamma(1-s)}{\Gamma(-z-n+2)(n-1)!}\right]\left[(i)^{n-1}\int_{0}^{\infty}\frac{(X)^{n-1}}{e^{\pi X}-1}dX\right]\end{array}\right]\right] \quad \textbf{(18)}$$

This can be reduced to:

$$\zeta(z)=\frac{2^{(z-1)}z}{(2^{z}-1)(z-1)}-\left(\frac{2^{(z-1)}}{2^{z}-1}\int_{0}^{\infty}\sum_{\nu=1}^{\infty}\left(\frac{2X^{(2\nu-1)}(-1)^{(\nu+1)}\Gamma(-z+1)}{\Gamma(-z-2\nu+2)\Gamma(2\nu)(\mathbf{e}^{(\pi X)}-1)}\right)dX\right) \quad \textbf{(19)}$$

Using the integral:

$$\int_{0}^{\infty}\frac{x^{2\nu-1}}{e^{\mu x}-1}=\left[\frac{1}{\mu^{2\nu}}\right]\Gamma(2\nu)\zeta(2\nu) \quad \textbf{(20)}$$

For ***Re(μ) > 0, (μ =π)*** , and ***Re(2ν) > 1,*** equation (20) becomes*:*

$$\zeta(z)=\frac{2^{(z-1)}z}{(2^{z}-1)(z-1)}-\frac{2^{(z-1)}\left(\sum_{\nu=1}^{\infty}\left(\frac{2\zeta(2\nu)(-1)^{(\nu+1)}\Gamma(-z+1)}{\pi^{(2\nu)}\Gamma(-z-2\nu+2)}\right)\right)}{2^{z}-1} \quad \textbf{(21)}$$

Noting that**:**

$$\zeta(2\,\nu) = \frac{2^{(2\,\nu-1)}\,\pi^{(2\,\nu)}\,(-1)^{(\nu+1)}\,\mathrm{bernoulli}(2\,\nu)}{(2\,\nu)!} \qquad \textbf{(22)}$$

then,

$$\zeta(z) = \frac{2^{(z-1)}\,z}{(2^z-1)\,(z-1)} - \frac{2^{(z-1)}\left(\sum_{\nu=1}^{\infty} \frac{2^{(2\,\nu)}\,\mathrm{bernoulli}(2\,\nu)\,\Gamma(-z+1)}{\Gamma(2\,\nu+1)\,\Gamma(-z-2\,\nu+2)}\right)}{2^z-1} \qquad \textbf{(23)}$$

$$\frac{\zeta(z)\,(2^z-1)}{2^{(z-1)}} + \left(\sum_{\nu=1}^{\infty} \frac{2^{(2\,\nu)}\,\mathrm{bernoulli}(2\,\nu)\,\Gamma(-z+1)}{\Gamma(2\,\nu+1)\,\Gamma(-z-2\,\nu+2)}\right) = \frac{z}{z-1} \qquad \textbf{(24)}$$

We can also express the function as;

$$\frac{\zeta(1-z)\,(2^{(1-z)}-1)}{2^{(-z)}} + \left(\sum_{\nu=1}^{\infty} \frac{2^{(2\,\nu)}\,\mathrm{bernoulli}(2\,\nu)\,\Gamma(z)}{\Gamma(2\,\nu+1)\,\Gamma(1+z-2\,\nu)}\right) = \frac{-1+z}{z} \qquad \textbf{(25)}$$

For all $z \in C$, if the ***ζ(z)*** vanishes *anywhere, then by the reflection formula,* ***ζ(1-z)*** also vanishes. Thus, multiplying (24) and (25), gives,

$$\left(\sum_{\nu=1}^{\infty} \frac{2^{(2\,\nu)}\,\mathrm{bernoulli}(2\,\nu)\,\Gamma(1-z)}{\Gamma(2\,\nu+1)\,\Gamma(-z-2\,\nu+2)}\right)\left(\sum_{\nu=1}^{\infty} \frac{2^{(2\,\nu)}\,\mathrm{bernoulli}(2\,\nu)\,\Gamma(z)}{\Gamma(2\,\nu+1)\,\Gamma(1+z-2\,\nu)}\right) = 1 \qquad \textbf{(26)}$$

Thus, the zeta function only vanishes if these two relations are *conjugate* with absolute value unity, or if they are *reciprocal relations*:

$$\sum_{\nu=1}^{\infty} \frac{2^{(2\nu)}\text{bernoulli}(2\nu)\,\Gamma(1-z)}{\Gamma(2\nu+1)\,\Gamma(-z-2\nu+2)} = \frac{1}{\displaystyle\sum_{\nu=1}^{\infty} \frac{2^{(2\nu)}\text{bernoulli}(2\nu)\,\Gamma(z)}{\Gamma(2\nu+1)\,\Gamma(1+z-2\nu)}} \qquad \textbf{(27)}$$

In the case when they are reciprocal relations, they could be represented by the form:

$$\sum_{\nu=1}^{\infty} \frac{2^{(2\nu)}\text{bernoulli}(2\nu)\,\Gamma(1-z)}{\Gamma(2\nu+1)\,\Gamma(-z-2\nu+2)} = e^{(-2k)} \qquad \textbf{(28)}$$

and,

$$\sum_{\nu=1}^{\infty} \frac{2^{(2\nu)}\text{bernoulli}(2\nu)\,\Gamma(z)}{\Gamma(2\nu+1)\,\Gamma(1+z-2\nu)} = e^{(2k)} \qquad \textbf{(29)}$$

Either sum could take the positive or negative values, provided they are of opposite sign. Then, we could write for the conjugates:

$$\sum_{\nu=1}^{\infty} \frac{2^{(2\nu)}\text{bernoulli}(2\nu)\,\Gamma(-z+1)}{\Gamma(2\nu+1)\,\Gamma(-z-2\nu+2)} = [-\cos(2k) + i\sin(2k)] \qquad \textbf{(30)}$$

and

$$\sum_{\nu=1}^{\infty} \frac{2^{(2\nu)}\text{bernoulli}(2\nu)\,\Gamma(z)}{\Gamma(2\nu+1)\,\Gamma(z-2\nu+1)} = [-\cos(2k) - i\sin(2k)] \qquad \textbf{(31)}$$

Relations (28), (29), (31) and (32) could be written in the form:

$$\sum_{\nu=1}^{\infty} \frac{2^{(2\nu)}\,\mathrm{bernoulli}(2\nu)\,\Gamma(-z+1)}{\Gamma(2\nu+1)\,\Gamma(-z-2\nu+2)} = -\frac{\cot(k)+i}{\cot(k)-i} \qquad \textbf{(32)}$$

where, the ***k***, could take on the real positive, negative or complex values.

Note that ***no assumption has been made about the half-line***, and the only condition I have placed on the relations, is that the ***zeta vanishes***.

It follows that when the ***function vanishes***, relations (28), (29), (31) and (32) could all be put in the form:

$$\sum_{\nu=1}^{\infty} f(1-z)\frac{2^{2\nu}B_{2\nu}}{\Gamma(2\nu+1)} = -\frac{\cot k + i}{\cot k - i} \qquad \textbf{(33)}$$

provided

$$z = \frac{1}{2} + \frac{1}{2}\, i \tan(k) \qquad \textbf{(34)}$$

and ***k***, could ***take on real or complex values***.

When the *k* is *real,* then we obtain *complex roots* on the half-line for positive and negative values of *k* proving that the roots must lie on the half-line, When $k = \boldsymbol{iK}$ is complex,

$$z = \frac{1}{2} + \frac{1}{2} i \tan(iK) \qquad \textbf{(35)}$$

we obtain real roots :

$$z = \frac{1}{2} - \frac{e^{2K} - 1}{e^{2K} + 1} \qquad \textbf{(36)}$$

Although the analysis does not give us the roots, it is indeed possible to find all the solutions of the Riemann-zeta function on the half-line provided the real solutions are treated as placed on the extended half-line for which the root vectors angles, $\boldsymbol{k}$ are complex.

Then, one sees that *all the possible roots of the function must lie on the half-line* with the exception that, for the real negative roots, the functional root-vector arguments are imaginary, while for the non-trivial roots, these arguments are real. Some of the solutions for the negative line are given by the complex values of $\boldsymbol{K}$;

$z = -2, K = 1.570796327 + 0.2027325541\ I$

$z = -4, K = 1.570796327 + 0.1115717757\ I$

$z = -6, K = 1.570796327 + 0.07707533991\ I$

$z = -8, K = 1.570796327 + 0.05889151783\ I$

$z = -10, K = 1.570796327 + 0.04765508990\ I$

$z = -12, K = 1.570796327 + 0.04002135384\ I$

$z = -14, K = 1.570796327 + 0.03449643574\ I$

$z = -16, K = 1.570796327 + 0.03031231091\ I$

$z = -18, K = 1.570796327 + 0.02703361064\ I$

which are all rotated and shifted from the half-line by $\pi/2$, and $2n\pi$, $n = 1,2,3,\ldots\infty$, respectively. This can be understood if for the complex solutions, we consider the argument of the r^{th} rotation of the root vector, σ^r_s, as expressed by the argument of the root at infinity σ_∞:

$$s = \frac{1}{2} + \frac{1}{2}i\frac{\arg \sigma_s^r}{\arg \sigma_\infty}, \quad \tan(k) = \frac{\sigma_s^r}{\sigma_\infty} = \frac{\left[\frac{\pi}{2}r\right]}{\left[\frac{\pi}{2}\right]} = 2y \tag{37}$$

It follows that for the complex roots, k is the angle formed by the root vector with the real line by r right angle rotations of the root vector.

For the real solutions, we consider the n^h rotations of the root vectors at infinity, σ^n_ξ as expressed by rational functions the argument of the complex ½ - line root at infinity σ_∞, for k complex,

$$s = \frac{1}{2} + \frac{1}{2}i\frac{\arg \sigma_\infty^n}{\arg \sigma_\infty}, \quad \tan(K) = i\frac{\arg \sigma_\infty^n}{\arg \sigma_\infty} = i\frac{\left[\frac{\pi}{2} + 2\pi n\right]}{\left[\frac{\pi}{2}\right]} = (1+4n)i \tag{38}$$

so that, $s = -2n$ are solutions! It follows that for the real roots, k is the angle formed by the root vector with the real line by n^{th} full rotations of the infinite root vector. So that, the function solutions can be written as rational functions of the argument of the root vectors and the root at infinity. The real roots are a continuation from the complex root at infinity over $2\pi n$ cycles.

5. Conclusions.

In accordance with the above findings, one sees that all roots of the Riemann-zeta function must lie on the line, ***$z = \frac{1}{2} + iy$, where 2y is the ratio of rational functions of the arguments of the root vector and the root vector at infinity.***

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