

# PATH INTEGRAL REPRESENTATION OF THE RIEMANN ZETA FUNCTION

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The Riemann zeta function  $\zeta(s)$  for real  $s$  is equivalent to the partition function of the system with energy eigenvalues  $E_n = \ln n$  and inversion temperature  $\beta$ . Analytic continuation of Feynman path integral representation of this partition function to the complex plane give us a new formula for  $\zeta(s)$ .

The Riemann zeta function is defined as a Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

or as a product over primes

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad (2)$$

and also as an integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx. \quad (3)$$

In all these representations  $s = \sigma + it$  and  $\sigma > 1$ . Zeta function may be analytically continued to the whole complex plane (except  $s = 1$ , where it has a simple pole), and all nontrivial zeros according to the celebrated Riemann hypothesis lie on the critical line  $\sigma = \frac{1}{2}$  [1,2]. Riemann zeta function and especially his hypothesis about zeros have interesting connections with physics, e.g. with cosmology [3], quantum scattering [4] and chaotic phenomena [5-7].

Dirichlet series (1) can be rewritten in the form

$$\zeta(s) = \sum_{n=1}^{\infty} \exp(-s \ln n). \quad (4)$$

If we take  $s$  real and put  $\beta = s$  and  $E_n = \ln n$  we get

$$\zeta(s) = \sum_{n=1}^{\infty} \exp(-\beta E_n), \quad (5)$$

which is exactly the partition function for the system with dimensionless Hamiltonian

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x) \quad (6)$$

with inversion temperature  $\beta$  and eigenvalues  $E_n = \ln n$  (potential  $V(x)$  may be approximately determined using e.g. some inversion techniques).

Expression for the partition function in terms of the path integral is given by [8]:

$$Z(\beta) = \int_{-\infty}^{\infty} \rho(x, x) dx, \quad (7)$$

where

$$\rho(x_2, x_1) = \int_{x_1}^{x_2} \exp \left\{ - \int_0^\beta \left[ \frac{1}{2} \left( \frac{dx(u)}{du} \right)^2 + V(x(u)) \right] du \right\} D[x(u)], \quad (8)$$

where  $x_2 = x(\beta)$  and  $x_1 = x(0)$ .

In [9,10] the partition function has been analytically continued to the complex  $\beta$ -plane in order to investigate critical behaviour of some systems (different systems seem to have the same number and location of zeros in the  $\beta$ -plane). Owing to the availability of the renormalization group method [11], this approach has not been further developed. Only recently some works appeared which discovered fractal features of phase boundaries in complex  $\beta$ -plane [12].

Analytic continuation of partition function (7) to the complex  $\beta(s)$  plane yields exactly the Riemann zeta function.

Critical phenomena are now largely studied using the renormalization group analysis of the path integral representation of partition function [13]. Using path integral representation of the Riemann zeta function and applying powerful techniques of statistical mechanics and field theory may shed new light on the nature of the distribution of zeros of the Riemann zeta function.

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