Proc. R. Soc. Lond. A 413, 183–198 (1987)
Printed in Great Britain

The Bakerian Lecture, 1987

Quantum chaology

By M. V. Berry, F.R.S.

H. H. Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, U.K.

(*Lecture delivered* 5 *February* 1987 – *Typescript received* 2 *March* 1987)

Bounded or driven classical systems often exhibit chaos (exponential instability that persists), but their quantum counterparts do not. Nevertheless, there are new régimes of quantum behaviour that emerge in the semiclassical limit and depend on whether the classical orbits are regular or chaotic, and this motivates the following definition.

Definition. Quantum chaology is the study of semiclassical, but non-classical, behaviour characteristic of systems whose classical motion exhibits chaos.

This is illustrated by the statistics of energy levels. On scales comparable with the mean level spacing (of order h^N for N freedoms), these fall into universality classes: for classically chaotic systems, the statistics are those of random matrices (real symmetric or complex hermitian, depending on the presence or absence of time-reversal symmetry); for classically regular ones, the statistics are Poisson. On larger scales (of order h, i.e. classically small but semiclassically large), universality breaks down. These phenomena are being explained by representing spectra in terms of classical closed orbits: universal spectral behaviour has its origin in very long orbits; non-universal behaviour depends only on short ones.

In Henry Baker's day, 'chaology' meant 'The history or description of *the* chaos' (O.E.D. 1893). *The* chaos was the state of the world before creation ('without form, and void') so that chaology was a theological term. That area of theology has not been very active for the past two centuries (unless we extend its scope to include some recent speculations in cosmology) and so we are justified in reviving the term chaology, which will now refer to the study of unpredictable motion in systems with causal dynamics, as exemplified by the contributions at the meeting on 'dynamical chaos' of which this lecture is a part.

But what is 'quantum chaology'? One obstacle to a definition is the growing understanding that quantum systems are not chaotic in the way that classical systems are. (I am speaking of unpredictability in the evolution of the expectation values of observable quantities, and not of the quite different randomness unavoidably encoded in the wavefunction.)

As an example, consider ionizing a hydrogen atom by shining microwaves on it. This is well modelled by the quantum mechanics of an electron in two electric fields: Coulomb, from the nucleus, and oscillatory, from the radiation. If the atom is highly excited to begin with, we might be justified, on the basis of the correspondence principle, in thinking of the electron as moving classically. If in addition the illuminating microwaves are intense, the classical progress towards ionization is not a smooth outward spiralling but an erratic diffusion: the fields make the electron orbits chaotic (Leopold & Percival 1979; Jensen 1985). Exactly this behaviour (or rather the ionization probabilities that follow from it) has been observed in experiments (Bayfield *et al.* 1977). (We are here very far from the perturbation régime of one-photon ionization, the photoelectric effect, that was so important at the birth of quantum mechanics.) Surely these experiments illustrate 'quantum chaos'? They do not, because chaos is unpredictability that *persists* (strictly for infinite times) and in these experiments the atoms traverse only a short stretch of microwave field and so diffuse for only a short time.

The surprise comes in quantum calculations for longer times. These show that although initially the highly excited quantum electron absorbs energy in the classical way (that is, diffusively), after a long time there is a transition to a new régime in which the quantum electron absorbs energy more slowly. The first calculations (Casati *et al.* 1979) were for a model system, in which a particle on a ring (a rotator) is kicked periodically with an impulse that depends on where it is. For strong kicks the classical rotator momentum diffuses (energy grows linearly). But the quantum energy almost always eventually stops growing (usually it oscillates quasiperiodically). The analogous régime for the ionization problem (Casati *et al.* 1984; Casati *et al.* 1986; Blümel & Smilansky 1987) has not yet been probed experimentally, although I understand that it soon will be.

These calculations are important because they illustrate a general phenomenon: the quantum suppression of classical chaos (Chirikov *et al.* 1981; Fishman *et al.* 1982; Grempel *et al.* 1984). To see easily that this suppression must occur, observe first that classical chaos can be regarded as the emergence of complexity on infinitely fine scales in classical phase space: smooth curves representing families of orbits develop elaborate convolutions, like cream spreading on coffee. But quantum mechanics involves Planck's constant h, which is an area in phase space (momentum times distance) below which structure is smoothed away (for an illustration see Korsch & Berry 1981).

Although we do not have chaotic quantum evolution, we do have here a *new quantum phenomenon* that emerges in the semiclassical limit in systems that classically *are* chaotic, and this motivates the following definition.

Definition. Quantum chaology is the study of semiclassical, but non-classical, behaviour characteristic of systems whose classical motion exhibits chaos.

'Semiclassical' here means 'as $h \rightarrow 0$'. (Of course Planck's constant is not dimensionless and so can take any value, depending on the choice of units; what is meant is that the ratio of h to some classical quantity with the same dimensions – action – tends to zero.)

Here I will concentrate not on time evolution but on the quantum chaology of *spectra*, that is eigenvalues of the energy operator for systems whose classical counterparts are chaotic. This is important because these eigenvalues are the energies of stationary states, which are the quantum mechanical way of describing *things*, that is persisting objects like atoms and molecules, whose properties do not depend on when we measure them. We will be concerned not with the ground state but with the description of many highly excited states; this is the semiclassical limit.

My main aim is to bring to your attention a remarkable quantum chaotic property of spectra, and describe the first step towards explaining it. Before doing so, I must point out that these semiclassical quantum problems are but one example of the asymptotics of eigenvalues. Essentially the same mathematics describes the modes of vibration of elastic membranes, or sound waves (in a lecture hall my voice excites modes near the 20000th, which is surely 'asymptotic'), and much else. The 'classical limits' of these non-quantum problems involve the 'rays' of elasticity or sound; geometrically the rays are geodesics: straight line trajectories reflected specularly, like billiard balls, at the boundaries of the domain. The two-dimensional billiard domain of figure 1*a* has chaotic geodesics: it is the stadium of Bunimovich. The domain of figure 1*b*, the circle, does not. In mechanical terminology, the stadium orbits are *ergodic* (they possess no constants of motion other than the energy) while the circle orbits are *integrable* (because of symmetry, their angular momentum is conserved as well). For a quantum particle of mass m in a billiard domain D, eigenvalues E are determined by

$$\left.\begin{aligned} \nabla^2\psi + (2mE/\hbar^2)\psi = 0 \text{ in D}, \\ \psi = 0 \text{ on the boundary of D}. \end{aligned}\right\} \tag{1}$$

The remarkable quantum chaotic property is that the distribution of the eigenvalues displays *universality*. This is the slightly pretentious way in which physicists denote identical behaviour in different systems. The most familiar example is thermodynamics near critical points (of, say, fluids and magnets).

To see the universality we need to magnify the spectrum so that the mean spacing of the levels is unity. The required magnification is the mean level density $\langle d \rangle$. What is $\langle d \rangle$? The answer comes from the roughest eigenvalue asymptotics, initiated by Pockels in 1891, developed by Rayleigh and Jeans who needed to count cavity modes for the theory of black-body radiation, and given a firm mathematical foundation by Herman Weyl in 1913 (for a review see Baltes & Hilf 1976). Their result was that if the classical system has N freedoms (e.g. $N = 2$ for billiards) then

$$\langle \mathrm{d}(E) \rangle \rightarrow \frac{\mathrm{d}\Omega(E)/\mathrm{d}E}{h^N} \quad \text{as} \quad h \rightarrow 0, \tag{2}$$

where $\Omega(E)$ is the volume of that part of classical phase space whose points have energies less than E. (These ideas have been refined and extended in several directions: see for example Kac 1966; Simon 1983*a*,*b*; Berry 1987.)

The level spacing is thus of order h^N, so that we need a microscope with power h^{-N}. What do we see with it? Of course we see the individual scaled levels, call them x_j, instead of the original levels E_j. Ideally we would have an asymptotic theory to predict these levels with an error that gets semiclassically small in comparison with the mean spacing h^N. For integrable (non-chaotic) classical motion we do have such a theory, in the form of the W.K.B. method and its

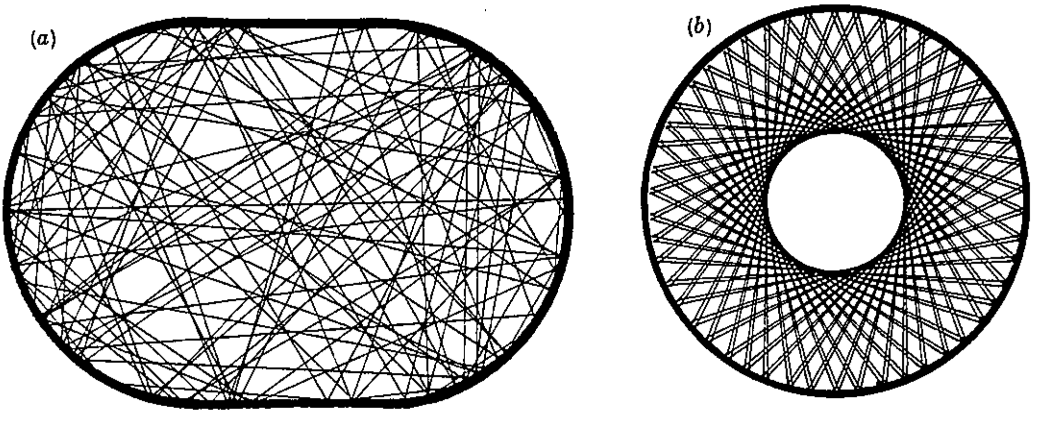

FIGURE 1. Classical orbits (bouncing geodesics) in billiards: (*a*) stadium of Bunimovich (chaotic), (*b*) circle (regular). For more details see, for example, Berry 1981*a*.

refinements and descendants (see, for example, Berry & Mount 1972; Percival 1977; Berry 1983). And these methods can be extended far into the chaotic régime if there is some residual order in phase space ('vague tori') and under not too semiclassical conditions (Reinhardt & Dana, this symposium). But for fully chaotic systems no fully asymptotic eigenvalue theory exists: we must make do with *statistics* of levels, and it is these that exhibit universality.

One such statistic, a short-range one, is the *level spacings probability distribution* $P(S)$, that is, the distribution of $S_j = x_{j+1} - x_j$. Figure 2 shows $P(S)$ computed from several hundred levels of the stadium, superimposed on $P(S)$ for another

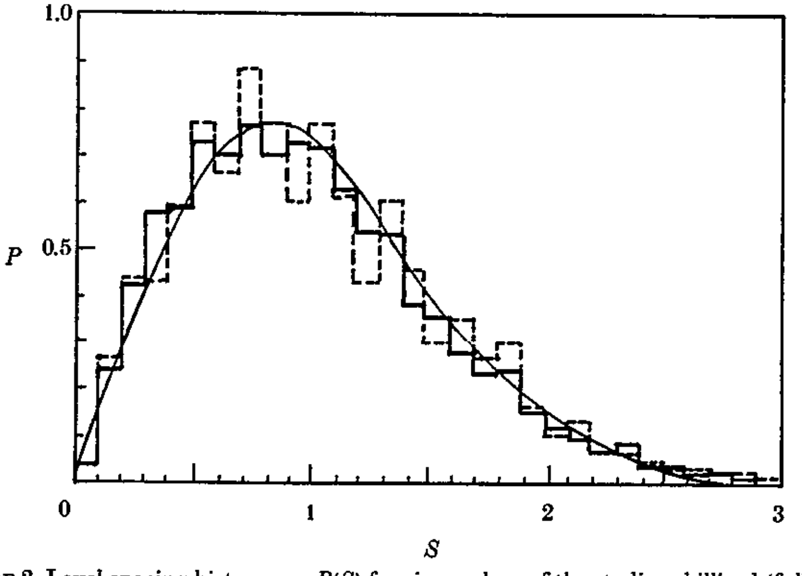

FIGURE 2. Level spacing histograms $P(S)$ for eigenvalues of the stadium billiard (full lines, after Bohigas 1984*a*) and the Sinai billiard (dashed lines, after Bohigas *et al.* 1984*b*), and the level spacings distribution for random real symmetric matrices (smooth curve), closely approximated by $P(S) = (\frac{1}{2}\pi \exp\{-\frac{1}{4}\pi S^2\}$.

classically chaotic billiard system: the billiard of Sinai, which is a square with a circular obstacle at its centre. These are two different systems, but the distributions are evidently the same; this is universality.

Another statistic – a long-range one – is the *spectral rigidity* $\Delta(L)$. This measures the fluctuations of the spectral staircase $\mathcal{N}(x)$, whose treads are at the eigenvalues x_j and whose risers have unit height ($\mathcal{N}(x)$ counts the number of levels below x). The rigidity (Dyson & Mehta 1963) is the mean-square deviation of the staircase from the straight line that best fits it over a range L, that is

$$\Delta(L) = \left\langle \min_{A,B} \frac{1}{L} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} L \, \mathrm{d}x \, [\mathcal{N}(x) - Ax - B]^2 \right\rangle. \tag{3}$$

Figure 3 shows the rigidities for the same two chaotic billiards; again they are the same, illustrating universality.

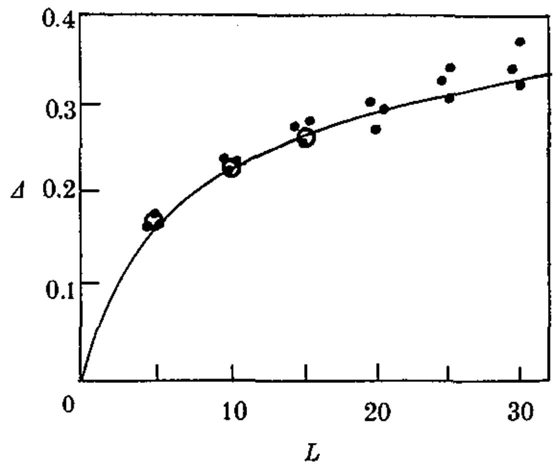

FIGURE 3. Spectral rigidity $\Delta(L)$ for eigenvalues of the stadium billiard (filled circles, after Bohigas *et al.* 1984*a*) and the Sinai billiard (open circles, after Bohigas *et al.* 1984*b*), and the rigidity for random real symmetric matrices (smooth curve), whose asymptote is $\Delta(L) \rightarrow (1/\pi^2) \ln L + \text{const.}$ as $L \rightarrow \infty$.

Now, it is clear from figures 2 and 3 that these data are accurately fitted by smooth curves representing the eigenvalue statistics of infinite real symmetric *matrices whose elements are random numbers*. Random-matrix theory (Porter 1965) was developed in the 1960s to model the complicated many-body energy operators for atomic nuclei (whose observed spectra they describe very well (Haq *et al.* 1982)). Ten years ago we (Berry & Tabor 1977) began to suspect it might also describe systems which although simple (like billiards) have chaotic classical orbits, and this has turned out to be so (Bohigas & Giannoni 1984).

Contrast this universality class with the spectral statistics of systems whose classical motion is not chaotic. Figure 4*a* shows the spacings distribution, and figure 4*b* the rigidity, for that most humble of regular systems, the particle in a two-dimensional rectangular box. It was surprising (ten years ago) to predict (Berry & Tabor 1977) and then find the statistics to be those of a set of *random*

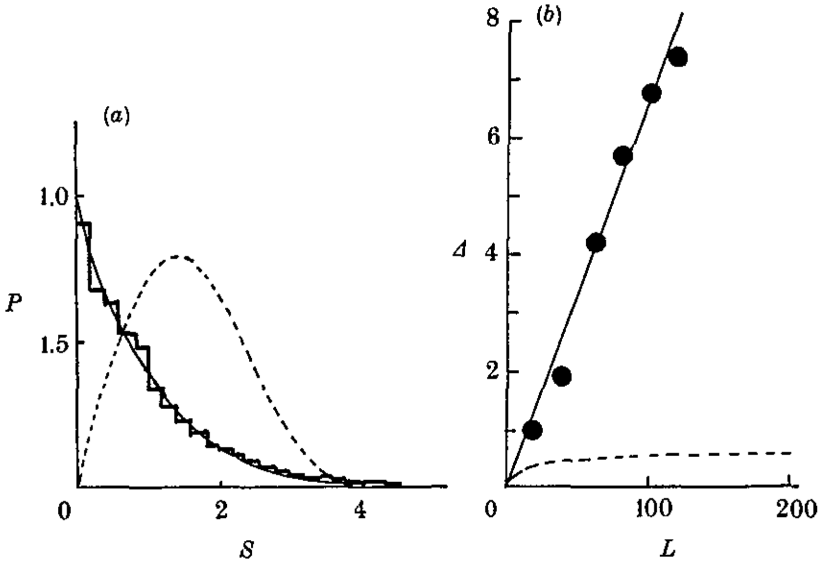

FIGURE 4. Level spacings distribution $P(S)$ (histogram in (*a*) and spectral rigidity $\Delta(L)$ (circles in (*b*)) for eigenvalues of the rectangular billiard. The full curves are the statistics for Poisson-distributed eigenvalues ($P(S) = \exp(-S)$ and $\Delta(L) = \frac{1}{15}L$) and the dashed curves are the statistics for random real symmetric matrices.

numbers (that is, poissonian). These behave very differently from the eigenvalues of random matrices, which are more well ordered in that they repel each other: for example, $P(S)$ vanishes linearly as $S \to 0$ instead of tending to a constant, and the asymptote of $\Delta(L)$ rise only logarithmically rather than linearly.

So far we have two universality classes, one for classically chaotic systems and one for classically regular systems, with spectra generated by random real symmetric matrices and Poisson processes respectively. Now, the matrices of quantum mechanics need not be real symmetric. The most general case is achieved for systems which, unlike billiards (or, more generally, particles in scalar potentials), *do not possess time-reversal symmetry* (T). For these, the energy operators are represented by complex hermitian, rather than real symmetric, matrices. The spectra of such random matrices, and also of the corresponding quantized chaotic systems, fall into a *third universality class.*

To illustrate it we break T by applying an external magnetic field to a charged particle moving chaotically. It is very instructive to concentrate the field into a single line of magnetic flux Φ. This is the chaotic equivalent of the effect discovered nearly thirty years ago in Bristol by Aharonov & Bohm (1959): the flux line does not alter the classical trajectories but does affect the quantum mechanics, in this case by changing the eigenvalues (Berry & Robnik 1986*a*). These are determined not by (1) but by

$$\left.\begin{aligned}(\nabla - \mathrm{i}qA(r)/\hbar)^2\psi + (2mE/\hbar^2)\psi = 0 \text{ in D},\\ \psi = 0 \quad \text{on the boundary of D},\end{aligned}\right\} \tag{4}$$

where $A(r)$ is any vector potential satisfying $\nabla \times A = \Phi\delta(r)$.

Figure 5 shows the spectral statistics of an Aharonov–Bohm billiard ('Africa') with chaotic trajectories (Africa is a cubic conformal image of the unit disc, illustrated in figure 6). Evidently $P(S)$ now vanishes quadratically as $S \to 0$, rather than linearly. The rigidity is different too: its logarithmic asymptote is only half that for chaotic systems with T. Thus T-breaking induces a *spectral phase transition*, to the third universality class. (Additional symmetries can mimic the effect of T, as explained by Robnik & Berry 1986). The Aharonov–Bohm chaotic billiard might appear contrived, but might be capable of realization with a tiny solenoid and the essentially two-dimensional electrons in certain semiconductor interfaces (M. Pepper, personal communication). Exact sum rules for Aharonov–Bohm eigenvalues are given by Berry (1986*a*).

It is instructive to digress and look at the *wavefunctions* of these systems without T, and particularly at their zeros (Berry & Robnik 1986*b*). *With T*, wavefunctions

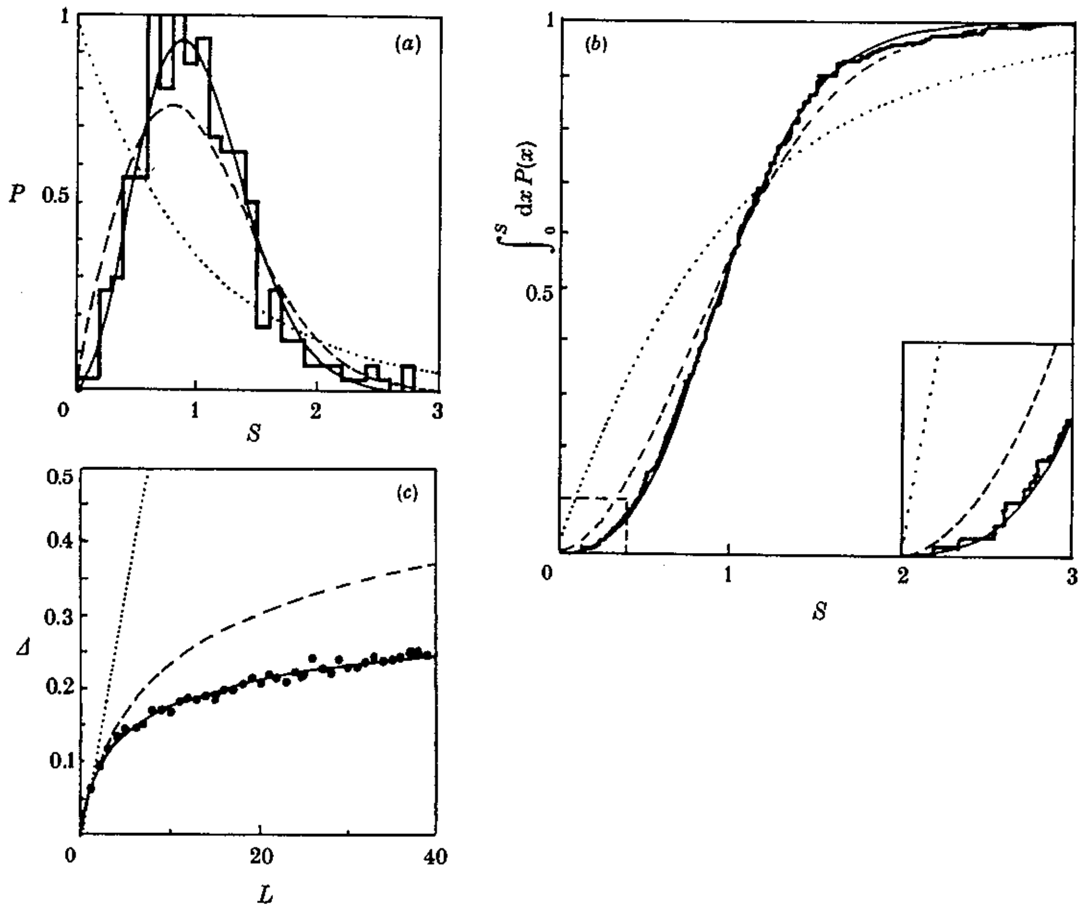

FIGURE 5. Level spacings distribution $P(S)$ (histogram in (*a*)), cumulative level spacings distribution $\int_0^S \mathrm{d}x P(x)$ (histogram in (*b*)), and spectral rigidity $\Delta(L)$ (circles in (*c*)) for eigenvalues of the Aharonov–Bohm 'Africa' billiard with flux $q\Phi/h = \frac{1}{2}(\sqrt{5}-1)$. The full curves are the statistics for random complex hermitian matrices, for which $P(S) \approx (32/\pi^2)\exp(-4S^2/\pi)$ and $\Delta(L) \to (1/2\pi^2)\ln L + \text{const.}$ as $L \to \infty$. The dashed curves are the statistics for random real symmetric matrices and the dotted curves are Poisson statistics.

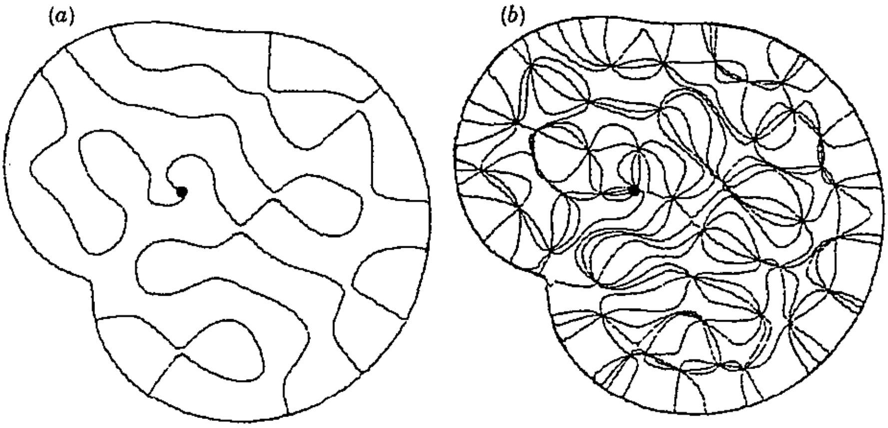

FIGURE 6. 50th eigenstate ψ of 'Africa' Aharonov–Bohm billiard with flux $q\Phi/h = \frac{1}{2}(\sqrt{5}-1)$. (a) Nodal lines of Re ψ, which are very similar to the nodal lines of ψ with zero flux; (b) wavefronts (contours of phase of ψ) at intervals of $\frac{1}{4}\pi$.

are real and so in two dimensions their zeros are the familiar *nodal lines* ($\psi(x,y) = 0$), which in quantum chaotic systems wander irregularly (figure 6a) with average spacing equal to the de Broglie wavelength $\lambda = \hbar/\sqrt{2mE}$ (McDonald & Kaufman 1979; Berry 1983; see also Heller 1984, 1986). Without T, wavefunctions are inescapably complex and so their zeros are points ($\mathrm{Re}\,\psi(x,y) = 0$, $\mathrm{Im}\,\psi(x,y) = 0$). Each of these points is a singularity of the wavefronts (contours of the phase of ψ) (figure 6b), which radiate from it like spokes from an axle (Nye & Berry 1974; Berry 1981b). Classical waves, like those on the surface of the sea, are of course real, but share the properties of 'inescapably complex' ones if their patterns are stationary but not standing, that is $\mathrm{Re}\,\psi$, where

$$\psi(\boldsymbol{r}, t) = F(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{5}$$

with $F(\boldsymbol{r})$ complex; thus the nodal lines of $\mathrm{Re}\,\psi$ move. The *tide waves* are like this, because of the symmetry-breaking caused by the Earth's rotation (relative to the Moon), and the phase singularities are the *amphidromic points* where the cotidal lines (wavefronts) meet (figure 7), as described by Whewell (1833, 1836) (see also Defant 1961).

Back now to eigenvalues. There is a set of numbers of great mathematical importance whose statistics precisely mimic the energy levels of a quantum chaotic system without T, namely the imaginary parts of the *zeros of Riemann's zeta function*. This function is defined (Edwards 1974) by analytically continuing to the whole complex z-plane Euler's product over primes p:

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}}. \tag{6}$$

Riemann showed that the zeros of $\zeta(z)$ determine the fluctuations in the density of primes (that is their importance) and conjectured that they all have real part $\frac{1}{2}$; thus

$$\zeta(\tfrac{1}{2}+\mathrm{i}E_j) = 0, \tag{7}$$

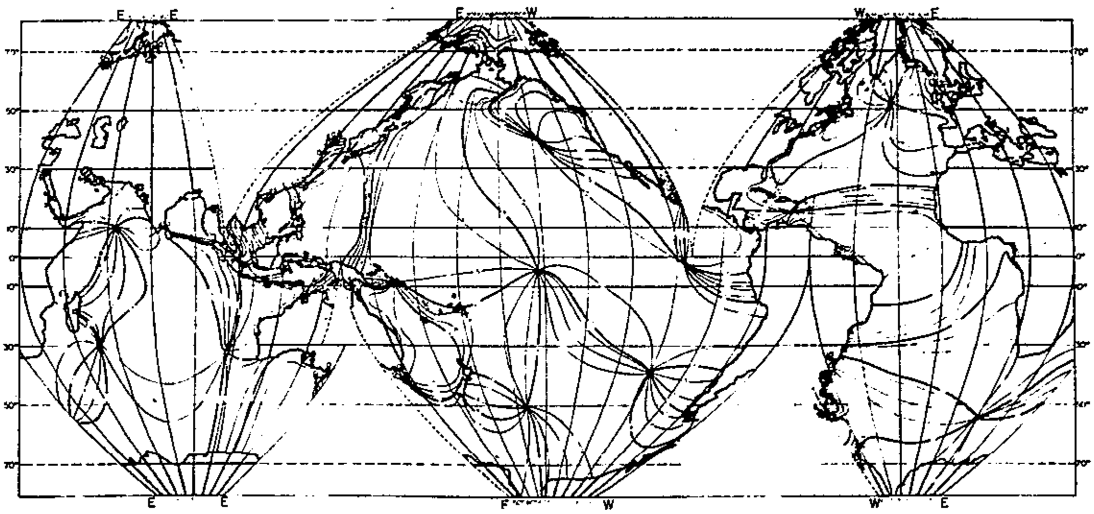

FIGURE 7. Cotidal lines in the oceans. These are wavefronts of the 12h tide wave, a forced vibration of the water of the whole earth; each line connects points where the tide is high at a given time. The singularities are amphidromic points, where there is no tide. (From Defant 1961.)

where $\{E_j\}$ are real. This conjecture has been verified by computation for the first 1.5×10^9 zeros (Van de Lune *et al.* 1986). It is an old idea (going back at least to Hilbert & Polya) that the Riemann conjecture would be confirmed if it could be shown that $\{E_j\}$ are the eigenvalues of some hermitian operator, but this has not been found.

Recently Odlyzko (1987) has computed some statistics for spectacularly high E_j. Figure 8*a* shows the spacings distribution for 10^5 zeros near the 10^{12}th; agreeing very closely with $P(S)$ for random complex hermitian matrices and so with that of some unknown quantum system without T whose unknown classical limit is chaotic. He also computed the number variance (figure 8*b*), a quantity closely related to the rigidity, and discovered that the three- and four-zero correlations (figure 8*c*, *d*) agrees perfectly with the corresponding complex random-matrix statistics. Riemann's conjecture thus acquires, in addition to its number-theoretic importance, a further significance (Berry 1986*b*): when (if) the operator with eigenvalues E_j is found, it will surely be simple, and will provide a paradigm for quantum chaology comparable with the harmonic oscillator for quantum non-chaology.

Here is a way of breaking T without magnetic fields, in *relativistic quantum chaology*. Take a massless particle ('neutrino') moving in the plane and described by the equation of Dirac (who gave this lecture in the year of my birth), but with a four-scalar potential $V(x,y)$ rather than the usual electric potential. For such a particle, the wave is a two-component spinor satisfying (Berry & Mondragon 1987)

$$\begin{pmatrix} V(x,y) & -\mathrm{i}\hbar c(\partial_x - \mathrm{i}\partial_y) \\ -\mathrm{i}\hbar c(\partial_x + \mathrm{i}\partial_y) & -V(x,y) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{8}$$

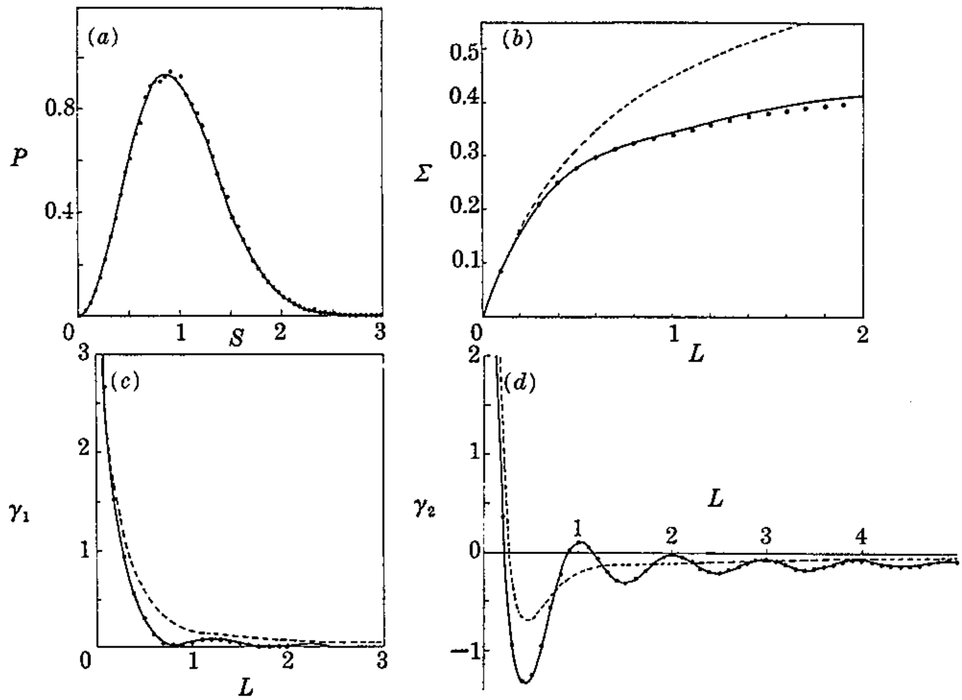

FIGURE 8. Statistics of imaginary parts of Riemann zeros, computed by A. M. Odlyzko. (*a*) $P(S)$ (from Odlyzko 1987); (*b*) number variance $\Sigma(L) \equiv \langle (n-L)^2 \rangle$, where n is the actual number of zeros in an interval where the average number is L (the interval is $2\pi L/\ln(E/2\pi e)$) (from data kindly supplied by A. M. Odlyzko); (*c*) Skewness $\gamma_1(L) \equiv \langle (n-L)^3 \rangle / \langle (n-L)^2 \rangle^{\frac{3}{2}}$ (kindly supplied by O. Bohigas); (*d*) excess $\gamma_2(L) \equiv \langle (n-L)^4 \rangle / \langle (n-L)^2 \rangle^2 - 3$ (kindly supplied by O. Bohigas). Full curves, random complex hermitian matrices; dashed curves, random real symmetric matrices.

This equation does not possess time-reversal invariance. Figure 9 shows the spectral statistics when $V(x, y)$ represents a hard wall (neutrino billiards), showing once again the statistics of complex hermitian random matrices if the billiard is classically chaotic, and Poisson statistics if it is regular.

Originally I hoped, following a suggestion of Professor Atiyah, that this kind of relativistic quantum chaology might help in the search for the elusive Riemann operator, but this has not yet proved to be so. However, Volkov & Pankratov (1985) and Pankratov *et al.* (1987) have recently discovered that an equation very similar to (7) appears to describe peculiar electron states localized in the interface between certain pairs of semiconductors (e.g. PbTe and SnTe, and HgTe and CdTe).

There is a *fourth* universality class, associated with chaotic systems that have time-reversal symmetry and also half-integer total spin (Porter 1965), but I will not speak about it.

So far we have seen that on fine scales the statistics of spectra fall into universality classes that depend on whether the classical motion is regular or chaotic, and on the symmetry of the energy operator. Now I have to explain how this universality is compromised in two ways.

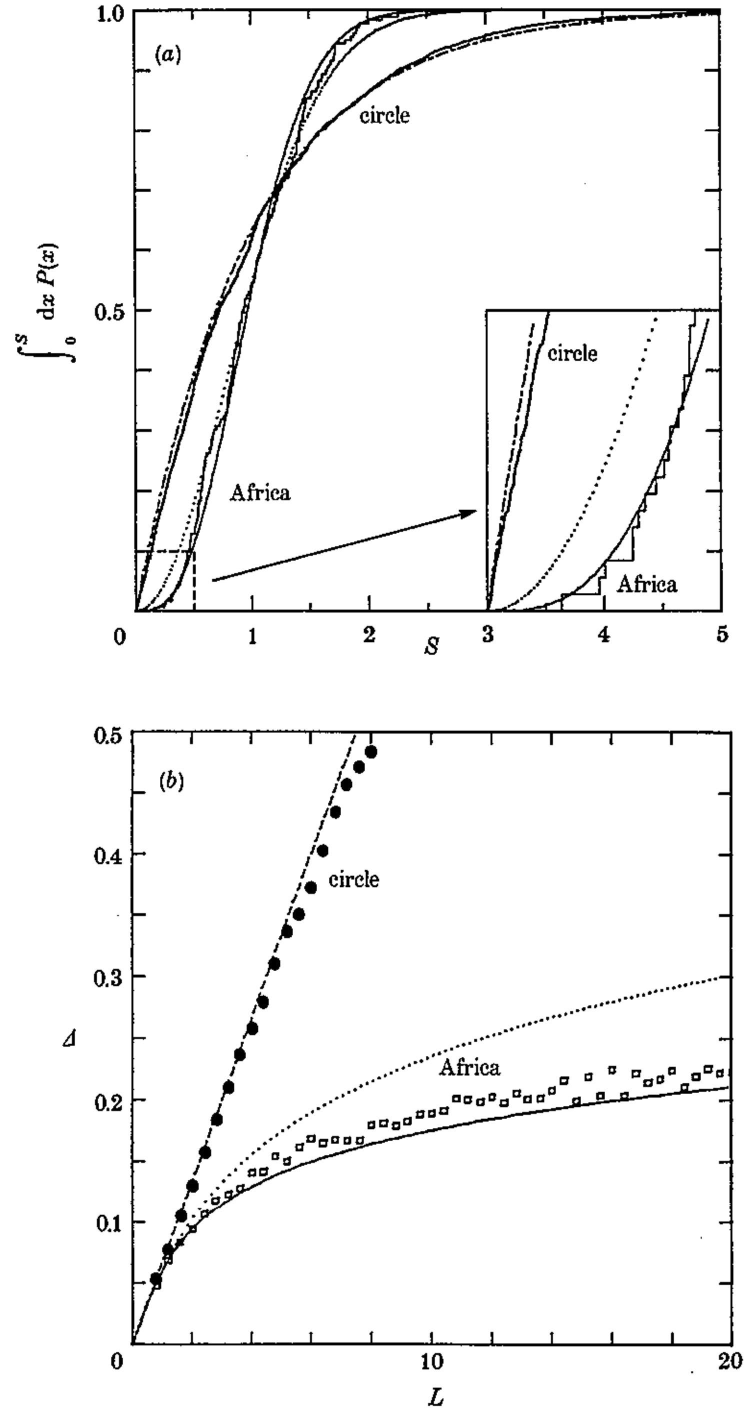

FIGURE 9. (*a*) Cumulative level spacings distribution $\int_0^S dx P(x)$ and (*b*) spectral rigidity $\Delta(L)$, for neutrino 'Africa' and neutrino circle billiards. Full curves, random complex hermitian matrices; dotted curves, random real symmetric matrices; dashed and chain curves, Poisson statistics.

First, some very important systems are partly regular and partly chaotic in their classical motion; vibrating molecules, for example. Their spectral statistics can be understood as those of a *superposition* of spectra from different universality classes, each spectrum being associated with a different chaotic or regular region in classical phase space (Berry & Robnik 1984). Figure 10 shows some recent calculations by Wunner *et al.* (1986), of the spacings distribution of the zero-angular-momentum, even-parity electron levels of a hydrogen atom in a very strong magnetic field (6T), in three different energy ranges. The point is that the

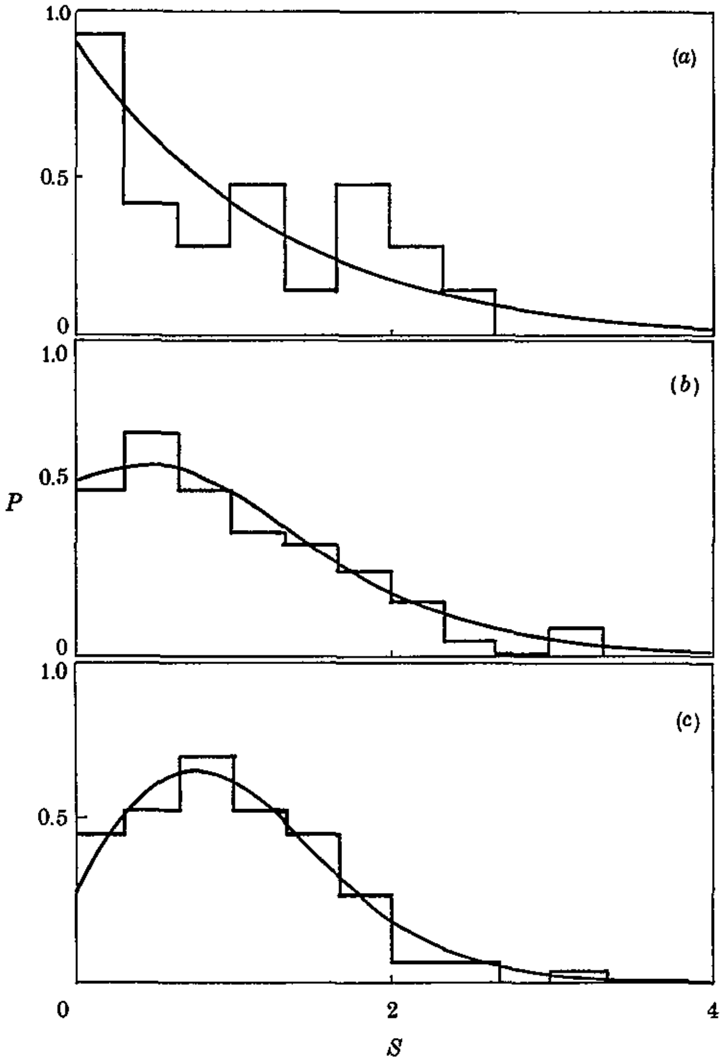

FIGURE 10. Level spacings distributions $P(S)$ for even-parity, zero-angular momentum energy levels of a hydrogen atom in a 6T uniform magnetic field, for three different energy ranges with different phase-space fractions q of regular orbits: (*a*) $-130\ \text{cm}^{-1} < E < -100\ \text{cm}^{-1}$ ($q = 0.71$, 47 levels); (*b*) $-100\ \text{cm}^{-1} < E < -70\ \text{cm}^{-1}$ ($q = 0.32$, 71 levels); (*c*) $-70\ \text{cm}^{-1} < E < -40\ \text{cm}^{-1}$ ($q = 0.16$, 116 levels); the smooth curves are $P(S)$ for superpositions of Poisson and random real symmetric matrix spectra. (From Wunner *et al.* 1986.)

corresponding classical motion gets more chaotic as the energy increases. These régimes are now within the reach of experiment (and are of course far removed from the familiar low-field 'perturbation' domain of the Zeeman effect).

The second compromise, of deep theoretical significance, is that universality is only local: for correlations involving very many levels, it breaks down. Recall the $\hbar^{-N}$ microscope that magnified the energies E_j to the numbers x_j with mean spacing unity, and note that N, the number of classical freedoms, is at least two for non-trivial cases. Now reduce the microscope's power to $\hbar^{-1}$. (These gedankenmagnifications are strongly reminiscent of the 'non-standard analysis' used nowadays to describe infinitesimals (Harnik 1986).) We will see energy ranges that are still *classically* small (of order $\hbar$) but *semiclassically large* in that they include many levels (a number of order $\hbar^{-(N-1)}$). At these magnifications, energy-level statistics are not universal: they depend on classical details.

To illustrate the breakdown of universality at long range, figure 11*a* shows the rigidity for the (classically regular) particle in a rectangular box, computed by Casati *et al.* (1985). When L is not too large we see the straight line of the 'universal' Poisson statistics (this was figure 4*b*), but when L approaches the square root of the number of the highest level included in the calculation (which for this case corresponds to an energy range of order $\hbar$), $\Delta(L)$ oscillates and then saturates at a value that depends on this number and also on the aspect ratio of the rectangle; that is, non-universally. Figure 11*b* shows the number variance for the Riemann zeros (underlying which there appears to be a chaotic classical system). When L is not too large we see the logarithmic curve of the 'universal' statistic for random complex matrices (this was figure 8*b*), but for larger L the variance oscillates about a value which depends on the number of zeros, that is, non-universally.

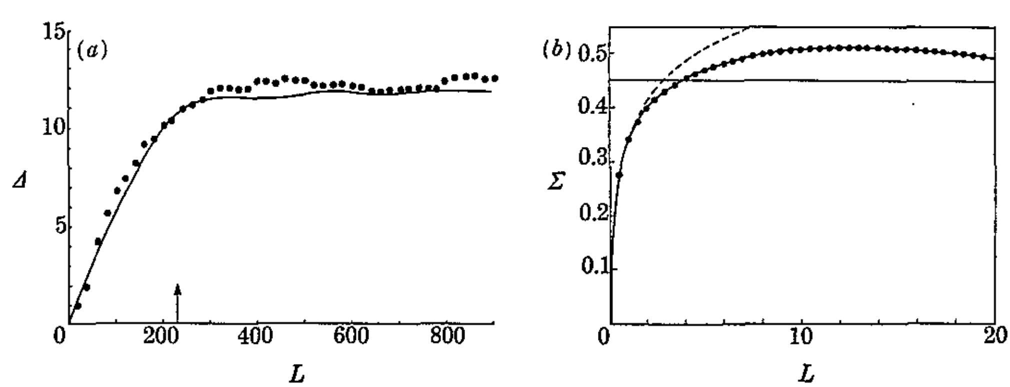

FIGURE 11. (*a*) Spectral rigidity $\Delta(L)$ for rectangular billiard, continuing figure 4*b* to larger L. The circles were computed from the eigenvalues near the 20000th Casati *et al.* 1985); the smooth curve was obtained from the sum over closed orbits by Berry (1985); the arrow is the L corresponding to an energy range $\hbar/T_{\min}$, where $T_{\min}$ is the period of the shortest closed orbit. (*b*) Number variance $\Sigma(L)$ for 10^5 Riemann zeros near the 10^{12}th, continuing figure 8*b* to larger L; the circles are plotted from data kindly supplied by A. M. Odlyzko; the smooth curve is the 'semiclassical' theory (adapted from Berry 1985), which predicts oscillation about the horizontal line $\Sigma(\infty) = 0.4518$ $(= [\ln \ln (E/2\pi) + 1.2615]/\pi^2$, where $(E/2\pi) \ln (E/2\pi e) = 10^{12})$; the dashed curve is for random complex matrices.

Until now I have spoken of spectral universality as an unexplained observation based on numerical experiments inspired by guesses. And so it was until recently, but now we have the beginnings of a theory (Berry 1985). Because of the h-magnifications involved, the theory has to be semiclassical: we must 'sew the quantum flesh on the classical bones'.

What are these bones? According to a beautiful picture developed by Gutzwiller (1971, 1978) and by Balian & Bloch (1972), they are the *classical closed orbits*, in terms of which an asymptotic formula can be given for the density of quantum eigenvalues (for a review, see Berry 1983; for the application to integrable systems, see Berry & Tabor 1976). These ideas can be traced back to de Broglie who in 1923 conceived of quantization as the constructive self-inteference of waves accompanying orbiting particles (think of Ouroboros, the mythical self-swallowing snake). For some mathematical systems (the Laplace–Beltrami operator on surfaces of constant negative curvature), the relation between spectra and closed geodesics is exact rather than asymptotic, and is called the Selberg trace formula (McKean 1972; Hejhal 1976; Balazs & Voros 1986; Series, this symposium.)

With the exception of some simple cases, the quantum levels are not in one-to-one correspondence with closed orbits (for an illustration, see Keating & Berry 1988; if they were, we would have a general formula for semiclassical quantization. Instead, each classical orbit describes an *oscillatory clustering* of the levels on a scale ΔE determined by its period T_n: this scale is just what would be expected from the uncertainty principle:

$$\Delta E = h/T_n. \tag{9}$$

Thus longer orbits give spectral information on finer scales, and it is this observation that gives the key to understanding the universality of the statistics (Berry 1985). With the h^{-N} microscope, we are concerned with the finest scales of spectral structure, of the order of the mean level spacing, so $\Delta E \sim h^N$. These scales depend on classical orbits with periods $T_n \sim h/\Delta E \sim 1/h^{(N-1)}$, that is, on *extremely long orbits*. Now, the distribution of these long orbits in phase space is very different for integrable and chaotic systems. For integrable systems, the orbits form continuous families whose number grows with period as T^N. For chaotic systems, the orbits are isolated and unstable and their number proliferates exponentially (as $\exp(HT)/HT$ where H is the Kolmogorov entropy – instability exponent of the orbit). In an important paper, Hannay & Ozorio de Almeida (1984) have shown that the way these long orbits contribute to one form of the asymptotic spectral formula is *universal*: it depends only on whether the orbits are chaotic or not, and on no other feature of the classical motion. It is this *classical* universality that begets the *quantum* universality, for it is possible to employ it as one ingredient in a derivation (Berry 1985) of the spectral rigidity $\Delta(L)$ (but not, so far, the spacings distribution), yielding precisely the Poisson and random-matrix formulae that so accurately fit the numerical computations.

The same arguments explain why universality breaks down at the larger energy scales $\Delta E \sim h$: the spectral fluctuations in this range are determined by orbits with period $T \sim h/\Delta E \sim h^0$, which are not long and so differ from system to system. The quantitative theory of this breakdown of universality (Berry 1985) works rather well, as figure 11 shows.

In summary, the vigorous development of quantum chaology during the last decade has been stimulated by the interplay of two factors: the realization that chaotic motion is ubiquitous in classical mechanics, and the discovery of associated new régimes of quantum behaviour. But the mathematical difficulties in understanding these régimes are severe, fundamentally because the semiclassical limit $h \rightarrow 0$ is highly singular. At the risk of sounding slightly paradoxical, I would say that we are discovering the connections between classical mechanics and quantum mechanics to be richer and more subtle than either mechanics is when considered on its own.

References

Aharonov, Y. & Bohm, D. 1959 *Phys. Rev.* **115**, 485–491.

Balazs, N. L. & Voros, A. 1986 *Physics Rep.* **143**, 109–240.

Balian, R. & Bloch, C. 1972 *Ann. Phys.* **69**, 76–160.

Baltes, H. P. & Hilf, E. R. 1976 *Spectra of finite systems.* Mannheim: B. I. Wissenschaftsverlag.

Bayfield, J. E., Gardner, L. D. & Koch, P. M. 1977 *Phys. Rev. Lett.* **39**, 76–79.

Berry, M. V. 1981*a* *Eur. J. Phys.* **2**, 91–102.

Berry, M. V. 1981*b* Singularities in waves and rays. In *Physics of defects* (*Les Houches Lectures* 34) (ed. R. Balian, M. Kleman & J.-P. Poirier), pp. 453–543. Amsterdam: North Holland.

Berry, M. V. 1983 Semiclassical mechanics of regular and irregular motion. In *Chaotic behaviour of deterministic systems* (*Les Houches Lectures* 36) (ed. G. Iooss, R. H. G. Helleman & R. Stora), pp. 171–271. Amsterdam: North Holland.

Berry, M. V. 1985 *Proc. R. Soc. Lond.* A **400**, 229–251.

Berry, M. V. 1986*a* *J. Phys.* A **19**, 2281–2296.

Berry, M. V. 1986*b* In *Quantum chaos and statistical nuclear physics* (ed. T. H. Seligman & H. Nishioka) (Springer lecture notes in physics no. 263), pp. 1–17.

Berry, M. V. 1987 *J. Phys. A* **20**, 2389–2403.

Berry, M. V. & Mondragon, R. J. 1987 *Proc. R. Soc. Lond.* A **412**, 53–74.

Berry, M. V. & Mount, K. E. 1972 *Rep. Prog. Phys.* **35**, 315–397.

Berry, M. V. & Robnik, M. 1984 *J. Phys. A* **17**, 2413–2421.

Berry, M. V. & Robnik, M. 1986*a* *J. Phys. A* **19**, 649–668.

Berry, M. V. & Robnik, M. 1986*b* *J. Phys. A* **19**, 1365–1372.

Berry, M. V. & Tabor, M. 1976 *Proc. R. Soc. Lond.* A **349**, 101–123.

Berry, M. V. & Tabor, M. 1977 *Proc. R. Soc. Lond.* A **356**, 375–394.

Blümel, R. & Smilansky, U. 1987 *Z. Phys.* (In the press.)

Bohigas, O. & Giannoni, M. J. 1984 Chaotic motion and random-matrix theories. In *Mathematical and computational methods in nuclear physics* (ed. J. S. Dehesa, J. M. G. Gomez & A. Polls) (Lecture Notes in Physics 209), pp. 1–99. New York: Springer Verlag.

Bohigas, O., Giannoni, M. J. & Schmit, C. 1984*a* *J. Phys. Lett.* **45**, L1015–L1022.

Bohigas, O., Giannoni, M. J. & Schmit, C. 1984*b* *Phys. Rev. Lett* **52**, 1–4.

Casati, G., Chirikov, B. V., Ford, J. & Izraelev, F. M. 1979 In *Stochastic behaviour in classical and quantum hamiltonian systems* (ed. G. Casati & J. Ford) (Springer lecture notes in physics 93), pp. 334–352.

Casati, G., Chirikov, B. V. & Guarneri, I. 1985 *Phys. Rev. Lett.* **54**, 1350–1353.

Casati, G., Chirikov, B. V. & Shepelyansky, D. L. 1984 *Phys. Rev. Lett.* **53**, 2525–2528.

Casati, G., Chirikov, B. V., Shepelyansky, D. L. & Guarneri, I. 1986 *Phys. Rev. Lett.* **57**, 823–826.

Chirikov, B. V., Izraelev, F. M. & Shepelyansky, D. L. 1981 *Soviet Sci. Rev.* C **2**, 209–267

Defant, A. 1961 *Physical oceanography*, vol. 2. London: Pergamon.

Dyson, F. J. & Mehta, M. L. 1963 *J. math. Phys.* **4**, 701–712.

Edwards, H. M. 1974 *Riemann's zeta function*, New York and London: Academic Press

Fishman, Shmuel, Grempel, D. R. & Prange, R. E. 1982 *Phys. Rev. Lett.* **49**, 509–512

Grempel, D. R., Fishman, Shmuel & Prange, R. E. 1984 *Phys. Rev.* A **29**, 1639–1647.

Gutzwiller, M. C. 1971 *J. math. Phys.* **12**, 343–358.

Gutzwiller, M. C. 1978 In *Path integrals and their applications in quantum statistical and solid-state physics* (ed. G. J. Papadopoulos & J. T. Devreese), pp. 163–200. New York: Plenum.
Hannay, J. H. & Ozorio de Almeida, A. M. 1984 *J. Phys.* A **17**, 3429–3440.
Haq, R. U., Pandey, A. & Bohigas, O. 1982 *Phys. Rev. Lett.* **48**, 1086–1089.
Harnik, V. 1986 *Math. Intell.* **8**, 41–47, 63.
Hejhal, D. A. 1976 *Duke math. J* **43**, 441–482.
Heller, E. J. 1984 *Phys. Rev. Lett.* **53**, 1515–1518.
Heller, E. J. 1986 In *Quantum chaos and statistical nuclear physics* (ed. T. H. Seligman & H. Nishioka) (Springer lecture notes in physics no. 263), pp. 162–181.
Jensen, R. V. 1985 In *Chaotic behaviour in quantum systems* (ed. G. Casati), pp. 171–186. New York: Plenum.
Kac, M. 1966 *Am. Math. Mon.* **73**, no. 4, part II, 1–23.
Keating, J. P. & Berry, M. V. 1988 (In preparation.)
Korsch, H. J. & Berry, M. V. 1981 *Physical* **3D**, 627–636.
Leopold, J. G. & Percival, I. C. 1979 *J. Phys.* B **12**, 709–721.
McDonald, S. W. & Kaufman, A. N. 1979 *Phys. Rev. Lett.* **42**, 1189–1191.
McKean, H. P. 1972 *Communs pure appl. Math.* **25**, 225–246.
Nye, J. F. & Berry, M. V. 1974 *Proc. R. Soc. Lond.* A **336**, 165–90.
Odlyzko, A. M. 1987 *Maths Comput.* **48**, no. ~~4,~~ 273–308. 177
O.E.D. 1893 *A new English dictionary on historical principles*, vol. 2 (ed. J. A. H. Murray). Oxford: Clarendon Press.
Pankratov, O. A., Pakhomov, S. V. & Volkov, B. A. 1987 *Solid St. Commns* **61**, 93–96.
Percival, I. C. 1977 *Adv. chem. Phys.* **36**, 1–61.
Porter, C. E. 1965 *Statistical theories of spectra: fluctuations*, New York: Academic Press.
Robnik, M. & Berry, M. V. 1986 *J. Phys.* A **19**, 669–682.
Simon, B. 1983*a* *J. funct. Anal.* **53**, 84–98.
Simon, B. 1983*b* *Ann. Phys.* **146**, 209–220.
Van de Lune, J., te Riele, H. J. J. & Winter, D. T. 1986 *Maths Comput.* **46**, no. 74, 667–681.
Volkov, B. A. & Pankratov, O. A. 1985 *JETP Lett* **42**, 178–181.
Whewell, W. 1833 *Phil. Trans. R. Soc. Lond.* **123**, 147–236.
Whewell, W. 1836 *Phil. Trans. Roy. Soc. Lond.* **126**, 289–307.
Wunner, G., Woelk, U., Zech, I., Zeller, G., Ertl, T., Geyer, F., Schweizer, W. & Ruder, H. 1986 *Phys. Rev. Lett.* **57**, 3261–3264.