The Euler-Mascheroni constant and the Riemann hypothesis

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Abstract

We discuss the relationship between arithmetic and analysis as it has bearing on the nature of the prime number theorem and the Riemann hypothesis.

We use two main components in this discussion:

- An arithmetic with fewer assumptions than real and complex analysis
- An understanding that arithmetic and analysis cannot in combination produce a numerical contradiction.

The conclusions of this discussion are derived from the relationship between arithmetic and analysis. In particular, we focus on the derivation of entities in the language and the assumptions behind the constructions. We refer to a logical and semantic dictionary for arithmetic D1 with a universe of discourse UD1. Correspondingly we have D2 and UD2 for real and complex analysis.

A helpful piece of imagery which plays no part is to see UD1 as the hub of a bicycle wheel and UD2 as the completed wheel with spokes and an outer rim. In UD1 the spokes are removed and we have disconnection between UD1 and UD2. Otherwise we may think freely between the two.

If we talk about unprovability of a proposition in UD1 it means that there cannot be a proof within the words of D1 and the assumptions behind the meaning of those words.

Let $l(x) = \sum_{n \le x} \frac{1}{n}$.

Let $\pi(x) = \sum_{\substack{p \le x \\ p \text{ prime}}} 1$.

Let μ the Möbius function from multiplicative number theory and let $M_1(x) = \sum_{n \le x} \mu(n)$.

Let $M_{k+1}(x) = \sum_{n \le x} M_k(n) \ \ (k \ge 1)$.

Let Θ =lub{σ:$\zeta(s) \neq 0$ for Re{s}> σ. We know $1/2 \le \Theta \le 1$.

Let $A(x) = \sum_{1 \le n \le x} a(n)$.

Let Δ be the lub of numbers θ such that $A(x) = O\left(x^{\theta}\right)$ as $x \to \infty$.

We call Δ the order or precise order of $A(x)$ and denote this by $A(x) = |O\left(x^{\Delta}\right)$ as $x \to \infty$.

Correspondingly, if α is the lub of numbers θ such that $A(x) = \Omega_+(x^\theta)$ as $x \to \infty$ we write

$A(x) = |\Omega_+ (x^\alpha)$ as $x \to \infty$. If β is the glb of numbers θ such that $A(x) = \Omega_- (x^\theta)$ as $x \to \infty$ we write

$A(x) = |\Omega_- (x^\beta)$ as $x \to \infty$. If $\alpha = \beta = \Delta$ we write $A(x) = |\Omega_\pm(x^\Delta)$ as $x \to \infty$.

We note that if the order of A(x) is rational then it clearly has meaning in UD1.

The following definition provides an arithmetic meaning for the big O notation: –

$T(x) = O\left(x^{\frac{p}{q}}\right)$ as $x \to \infty$ means $\exists\, A > 0, x_A > 0 :$ for $x > x_A$; $|T(x)|^q < Ax^p$ as $x \to \infty$.

It is well known that if $\exists$ rational $q \in [\frac{1}{2}, 1)$ such that $M(x) = O(x^q)$ as $x \to \infty$ then ζ is zero free for $\sigma > q$. We show this proposition and the proposition that ζ has zeros arbitrarily close to $\sigma = 1$ are both unprovable in UD1. This leads to the conclusion that all computed zeros of ζ in the critical strip are simple zeros and lie on $\sigma = 1/2$.

Firstly we spent some time reviewing a standard development of some number systems from the point of view of induction from elementary arithmetic.

Focus on the finite

A widely accepted way of formally developing the language and arguments of much mathematics is via the Zermelo-Fraenkel axioms. In particular, the theory allows a unified set-theoretic way of developing the number systems, from defining the natural numbers through to integers and rational numbers, and the completion of the rational numbers to the real numbers and the corresponding complex numbers.

The unifying set theoretic development of mathematics sometimes has an 'arithmetic' as a universe of thought and activity which includes at least a significant chunk of real and complex variable theory. We do not use this meaning for arithmetic. We regard counting and UD1 induction as being a containment strategy for entities under discussion in arithmetic and this will be discussed in the construction of the number systems involved.

We see that arithmetic may be developed in a stronger argument framework than analysis yet the additional assumptions of analysis extend the range of arithmetic in a 'continuous' numerical way, albeit within a rather special logical subset of arithmetic - the real numbers.

We have more words and more mathematics in UD2 compared to UD1, but in logical terms UL2 is a weaker system of 'truths'.

The extent of arithmetic is bounded by inductive argument n⟶n* and the extra assumptions of analysis put certain analytic results beyond the reach of arithmetic. That irrational numbers cannot be described numerically in UD1 provides the most familiar example.

The arithmetic of reference (UD1)

The arithmetic or finite arithmetic discussed here proceeds in a reasoning system in which outward results are fully justified in terms of earlier 'true' results based on the original assumptions and classical laws of inference.

When we talk about arithmetic we refer to the rules and language which enable us to abstract properties of the pattern developing from:

1   1+1   1+1+1   1+1+1+1   1+1+1+1+1   1+1+1+1+1+1   1+1+1+1+1+1+1   ....

The concept of 'unbounded' may be derived from the simple observation that there is no last entity in the above pattern. Indeed, each block is different from all the preceding blocks and the construction is open ended. Thus 'unbounded' is a concept which is understood through recognition compared to the notion of infinite set which cannot be understood in this way.

An underlying set theory here, which is very much in the background, excludes the assumption of the infinite set.

The laws and elementary results of arithmetic are developed using the first four Peano postulates.

Namely:

- 1 is a natural number
- If n is a natural number there exists a unique natural number n* which is a natural number called the successor of n
- If n is a natural number then $n^* \neq 1$
- If n and m are natural numbers and $n^*=m^*$ then $n=m$.

Addition is defined by $n+1 = n^*$ and $n+m^* = (n+m)^*$, and multiplication is defined by $n.1=1$ and $n.m^*=n.m + n$.

We introduce the concept of order using the definition $a<b$ if there exists c such that $a+c=b$.

Familiar laws and theorems of arithmetic may then be derived using rules from classical logic. The argument form, mathematical induction, follows from the repeated use of the mechanism of modus ponens in the form: if P(n) is true and P(n*) implies P(n*) is true then P(n*) is true.

The base ordering of N-tuples whose elements are natural numbers

We start with the knowledge that $1<2<3<$..........   .

Assume we have an ordering of $\mathrm{n}-$ tuples for $1 \leq \mathrm{n} \leq \mathrm{N}-1$ (fixed $\mathrm{N} \geq 2$) which includes the condition

$$\text{if } \sum_{k\leq n} a_k < \sum_{k\leq n} b_k \text{ then } (a_1, a_2, \ldots a_n) < (b_1, b_2, \ldots b_n).$$

Then we define part of the ordering for N- tuples by

$(a_1, a_2, \dots a_N) < (b_1, b_2, \dots b_N)$ if $\sum_{k \leq N} a_k = \sum_{k \leq N} b_k$.

We focus on the case where $\sum_{k \leq N} a_k = \sum_{k \leq N} b_k$ to complete the ordering for $N -$ tuples.

If $\sum_{i \leq k} a_i = \sum_{i \leq k} b_i$ for $1 \leq k \leq N$, then $a_k = b_k$ for $1 \leq k \leq N$.

Consequently, if $(a_1, a_2, \dots a_n) \neq (b_1, b_2, \dots b_n)$ then there exists greatest k with $1 \leq k \leq N$ such that $(a_1, a_2, \dots a_k) \neq (b_1, b_2, \dots b_k)$.

If $k = N$ and without loss of generality with $a_N < b_N$ we define $(a_1, a_2, \dots a_N) < (b_1, b_2, \dots b_N)$.

Otherwise, without loss of generality, with this choice of $k < N$, if $(a_1, a_2, \dots a_k) < (b_1, b_2, \dots b_k)$ then $(a_1, a_2, \dots a_N) < (b_1, b_2, \dots b_N)$.

These orderings may be used in inductive arguments involving N-tuples as we may count along the ordering to include any nominated case.

In particular, the ordering for ordered pairs in this definition is:

(1,1)<(2,1)<(1,2)<(3,1)<(2,2)<(1,3)<(4,1)<(3,2)<(2,3)<(1,4)<(5,1)<(4,2)<(3,3).....(A).

Thus we can count ordered pairs eventually counting to a nominated pair (N, M). This is not the ordering of the integers or rational numbers but provides a basis for defining the binary operations on these entities inductively.

When we prove inductively that the binary operation of addition and multiplication produce the same results for equivalent entities, all the entities (ordered pairs) being talked about are within counting range. The realm of discussion is accessible inductively. This is important because the constructions deal with unbounded entities and the extent we are convinced about this is aligned with the weight we place on inductive argument in UD1.

Integers

To construct the integers we use the equivalence relation, (a,b)~(c,d) if and only if a+d =c+b. It may be helpful to view (a,b) in a new notation as a-b as all the laws of integer addition and multiplication will follow by inductive arguments.

So a-b is one of the representations of a particular number in much the same way as a/b may be viewed once we define the rational numbers via the (a,b) route.

The positive numbers, zero and negative numbers are identified by n→[(n*,1)] and '0' = [(a,a)] and [(a,b)]+[(b,a)] ='0'.

Importantly, this construction is contained in the language and assumptions of arithmetic and equivalent elements may be listed in an unbounded sequence and the independence of representation in defining the binary operations is established by induction.

For example:

'0' =(1,1) = (2,2) =(3,3) = ….

'1' =(2,1) = (3,2) =(4,3) = …

'2' =(3,1) = (4,2) =(5,3) = …

…………………………………………..

'-1' =(1,2) = (2,3) =(3,4) = …

'-2' =(1,3) = (2,4) =(3,5) = …

……………………………………………

Thus we understand the extent of '0' ,'-1', '1', ….inductively and induction is a fundamental part of defining the integers in arithmetic. They are comprehensible in the language of inductive arithmetic.

Once we have equivalent elements ordered by induction we are able to verify independence of representation and cover any nominated cases inductively.

Rational numbers

Similar techniques for the construction of the rational numbers use the equivalences relation defined by (a,b)~(c,d) iff ad=bc (a,c integers and b and d non-zero integers) and addition and multiplication defined by (a,b)+(c,d)=(ad+bc ,bd) and (a,b).(c,d)=(ab,cd).

Once again equivalent elements are countable and we also have the well known result that the rational numbers are countable (described in order by a prescribed unbounded sequence $\{s_n\}$ (see (A) above).

Further the ordering of rational numbers is defined as with natural numbers.

As mentioned, the notation (a,b) is replaced by a/b and the familiar laws follow inductively as with the integers.

The truth of a proof has the same status as the truth of the result of doing a piece of conventional 'long multiplication' which has been checked as correct, and if necessary traced all the way back to the initial axioms of the arithmetic. There is only ever a finite amount of checking necessary to verify results. The result follows from the rules.

Thus, in arithmetic so far, we only require a finite universe, inductive argument and the correct reasoning of the 'average mathematician in the street'.

We brush over the imbedding of the natural numbers into the integers and the integers into the rational numbers on the grounds that we are still in arithmetic after a bit of fancy footwork.

It should be clear after some work that we may derive integers and rational numbers in arithmetic without any new assumptions or words, the meaning of which cannot be defined in the arithmetic (Ayres [1], Havil [4], Russell [8].

We note that the fundamental theorem about the uniqueness of factorisation of the natural numbers into prime numbers is essentially an inductive result in UD1 ([2] Davenport).

Completion of the rational numbers (real numbers)

The natural numbers provide the most basic notion of an unbounded sequence {n} and with the capacity to form number expressions using addition and multiplication and using subscripting we may define many unbounded sequences $\{s_n\}$. For example $\{s_n\} = \{\frac{1}{2}n(n+1)\}$.

The notion of limit may be formulated in arithmetic using rational sequences:-

$$2 - \frac{1}{2^K} \leq 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^K} < 2 \ \ (K \geq 1) .$$

i.e. the ϵ, δ definitions for the notions of sequence and series limits arise from arithmetic and some of these sequences have limit values in arithmetic. In UD1 limit values are meaningless unless they are proven rational in UD1, and then the definition of limit value remains in UD1.

Two rational Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are equivalent iff $\{ x_n - y_n \}$ is Cauchy convergent to zero

By definition $\{x_{n)}\}+\{y_n\} = \{x_n + y_n\}$ and $\{x_{n)}\}.\{y_n\} = \{x_n . y_n\}$.

When we define equivalence here the notion is at quite a level of abstraction and closer examination reveals a weakness or level of uncertainty we have to live with if we wish to do arithmetic using the entities of the analytic system.

As with the earlier constructions, we need to 'prove' that the operations of addition and multiplication are independent of the sequences representing the new entities.

A standard approach is to show that if $\{x_{n)}\} \sim \{x'_n\}$ and $\{y_{n)}\} \sim \{y'_n\}$ then $\{x_n + y_n\} \sim \{x'_n + y'_n\}$ and

$\{x_n . y_n\} \sim = \{x'_n . y'_n\}$ but this is only part of what needs to be understood: $-$

As happens in creative work, constructions of various kinds are described through some activity within a base system and it is only after an initial prototype is being considered that we may discover restrictions or qualifications or new assumptions are necessary to understand the new operating environment.

It is convenient but not necessary to use hindsight to see why some qualification is needed here.

We take it as understood we can assume that the real number system we are talking about is that indicated above (see Havil [4]).

The real numbers which come out of this construction cannot be listed or counted by an unbounded sequence in arithmetic (Cantor's diagonal argument). Further, and most importantly, the rational sequences equivalent to each other are uncountable (see for example, Internet [7]; compare this to the construction of the integers and the rational numbers).
Then, after constructing the real numbers we need to re-examine the affirmation that the addition and multiplication defined on equivalent rational Cauchy sequences are independent of the choice of sequences. It cannot be established inductively in arithmetic since we have an uncountable number of equivalent sequences. All we are ever able to offer in UD1 for this affirmation is a 'weak' inductive argument which only ever covers a countable number of sequences. One additional assumption in the construction of the real numbers by this method is thus that this 'weak' inductive argument has validity over all the members of the uncountable class. We see below that this 'weak' induction is sufficiently strong to ensure there is no possibility of numerical contradiction between numerical arithmetic results (all known results about numbers) and analytic numerical results.

Note also that the definition in arithmetic that a<b if and only if there exists c>0 such that a+c=b extends to the real numbers. Similarly, the 'truth' of the ordering of the real numbers uses a countable sample of real numbers in a similar weak inductive argument.

We can only ever imagine the ordering of a countable number of real numbers since otherwise the rational numbers would also be uncountable as there is at least one rational number between any two real numbers. We make the additional assumption in the construction of the real numbers that the defined ordering does not lead to contradiction.

These assumptions mark a crucial turning point in thought in the construction of the real number system. We could reject the uncertainty of the real numbers and stay in arithmetic as a closed axiomatic system. This would be somewhat akin to living in a small village and not knowing what is behind the hill on the boundary of the village, but it is a device in the logical and mathematical context which bears fruit.
The crucial new assumptions which allow us to consider analysis as a special part of arithmetic do not need to impinge on the normal manipulative, rule following theory around specific functions for interesting reasons.

Clearly, no matter how many centuries of arithmetic pass we will always have in writing or on record a countable number of prescribed rational Cauchy sequences of natural numbers. We know with hindsight that if we consider the more abstract notion of having the property of being a number sequence then we have an uncountable number of sequences. There is an increase in the level of abstraction when we are dealing with entities like the real numbers, and with this there is a corresponding difference in the type of certainty.

In the deriving of relationships in analysis – doing mathematics in a rule following way – the arguments often include relationships between prescribed rational Cauchy convergent series and although some arguments and values of the series may not be in UD1 the actual arithmetic criteria of being a prescribed convergent rational series is something which is inevitably argued out inductively in UD1. This 'on the ground' sort of mathematics is thus not concerned with the arguments producing the system of complex analysis because it is justified by induction in UD1. The convergent series are there in UD1 by inductive argument, and it then is a question of what sort of entity values are substituted in the series which determines whether we are in UD1 or UD2. This helps mathematicians not to be too worried about the foundations of mathematics because in their own thoughts they are doing nothing 'wrong' and a consensus judgment will confirm that the rules are being followed!

In UD2 we could restrict the ϵ, δ definitions involving limits to cases where limit values are rational and then we are in UD1. We imagine a 'shadow' arithmetical analysis inside/outside the more general real analysis. With hindsight of the development of real analysis and the inter-related closeness of rational and irrational numbers, it is then obvious that any analytical result about numbers will not find numerical contradiction in arithmetic. In the manipulation of relationships between analytic functions to derive point-wise results we substitute rational approximations to any required degree of accuracy. Thus we see continuity between arithmetic and analysis in results about numbers.

Since the merging of analysis and arithmetic cannot produce contradiction about numerical results when we derive results about numbers we may freely move and think in UD1 and UD2 without any chance of locating a numerical contradiction. If we run into a contradiction following some assumptions, we follow the normal approach and look to explain the contradiction within the relevant universe of discourse.

As a result, any numerical results gained by the use of computers confined to binary logic and using at most UD2 theory are results which cannot contradict results in UD1. We could perhaps go further and say such results are in UD1 but the interpretation of results does require reference to UD2.

When we are doing 'practical' analysis – (e.g. the classical theory of the Riemann zeta function) – we have the inductive strength of arithmetic, but we need to keep in mind that at a purely theoretical level we are arguing in UD2.

We cannot be so certain about results when we start to use inductive arguments in UD2. In Appendix 2 Lemma 7, we use an inductive argument based on analytic entities in UD2 to produce the result-
for each $k \geq 1, \forall \epsilon > 0;\ \ M_k(x) = \Omega_{\pm}\left(x^{k-1+\Theta-\epsilon}\right)$ as $x \to \infty$. This looks to be too strong a theorem for UD1.

We have argued in earlier discussions that the weakest collection, with $\Theta = \dfrac{1}{2}$, is unprovable in UD1.

We do not pursue this further in this discussion but detail is provided in Appendix 2.

We take the complex numbers as a simple field extension of the real numbers and assume for convenience that complex analysis may be developed in UL2 without further assumptions.

In summary, UD2 is more prolific than UD1 because there are more words in D2 than D1 but the extra assumptions concerning uncountable entities make it a weaker logical system and in a logical sense it is a special part of arithmetic.

Section 2

The natural logarithm (ln) and the inverse function (exp) are not in UD1

Indeed, note that $\exp(q)$ is irrational for non zero rational q (Hardy and Wright [3]).
Hence $\ln(q)$ is also irrational for every positive non zero rational.
Thus, $\ln(n)$ has no meaning in UD1. The same comments apply to exp.

On the face of it, $\exp(x)$ may be constructed in arithmetic as a convergent series but it is a null construction.

It is hoped that this simple observation will soon provide a Eureka moment for the reader!

Section 3

The extended prime number theorem

The prime number theorem proves that $\pi(x)$ (the number of primes less than or equal to x) is asymptotically equal to $\frac{x}{\ln(x)}$.

The extended prime number theorem proves without qualification, that for k=1,2,3 ....

$$\pi(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \frac{2!\,x}{\ln^3(x)} + \cdots + \frac{(k-1)!\,x}{\ln^k(x)} + O_k\left(\frac{x}{\ln^{k+1}(x)}\right) \text{ as } x \to \infty.$$

See for example Ingham [6] Chapter 3, Hardy and Wright [3] Chapter 20, Tenenbaum [9] and Appendix 1).

This extraordinary result seems to take third place behind the prime number theorem and the Riemann hypothesis! The more familiar form is:

$\exists\, c > 0$ such that $\pi(x) - \mathrm{li}(x) = O\left(xe^{-c\sqrt{\ln(x)}}\right)$ as $x \to \infty$,

where $\mathrm{li}(x) = \int_2^x \frac{dt}{\ln(t)}$.

Here we see an unbounded number of different asymptotic terms each of which has order 1. Thus we see an unbounded number of logically distinct terms of order 1 involved in describing

$\pi(x)$ prior to trying to derive a remainder term for $\pi(x) - \mathrm{li}(x)$ of order less than unity.
No doubt series could be constructed whose distribution fits for K terms but fall over on the (K+1)th term. This starts to hint at the unprovability of the order of the error term $\pi(x) - \mathrm{li}(x)$ in UD1.

This is not quite water-tight yet as the above expression for $\pi(x)$ uses $\ln(x)$ which is not in UD1,

and so the above formula as it stands is meaningless in UD1.

We are able to reframe the extended prime number theorem in terms of $\mathrm{l}(x)$ with the possibility that the reformed expression only uses UD1 entities. Using the relationship $\ln(x) = \mathrm{l}(x) - \gamma + O(1/x)$ as $x \to \infty$ we see:-

$\exists$ defined polynomials $f_1, f_2, f_3 \ldots$ with integer coefficients such that for $k \geq 1$

$$\pi(x) = \left\{\frac{f_1(\gamma)x}{\mathrm{l}(x)} + \frac{f_2(\gamma)x}{\mathrm{l}^2(x)} + \frac{f_3(\gamma)x}{\mathrm{l}^3(x)} + \frac{f_4(\gamma)x}{\mathrm{l}^4(x)} + \cdots + \frac{f_k(\gamma)x}{\mathrm{l}^k(x)}\right\} + O_k\left(\frac{x}{\mathrm{l}^{k+1}(x)}\right) \text{ as } x \to \infty \ldots\ldots(0),$$

where γ is Euler's constant (see Appendix 1).

For example:

$f_1(\gamma) = 1, \quad f_2(\gamma) = 1 + \gamma, \quad f_3(\gamma) = \gamma^2 + 2\gamma + 2, \quad f_4(\gamma) = \gamma^3 + 3\gamma^2 + 6\gamma + 6,$

$f_5(\gamma) = \gamma^4 + 4\gamma^3 + 12\gamma^2 + 24\gamma + 24,$ and so on.

The expressions in (0) are in UD1 provided γ is rational. Fortunately, we do not have to get a headache trying to prove in UD1 that γ is rational, and we suspect that such attempts would forever involve inescapable circularity.

Explanation of the σ=1/2 phenomenon (the Riemann hypothesis)

It is well known in the theory of the Riemann zeta function that:

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} \ln(p) = x + |O(x^{\Theta}) \text{ as } x \to \infty$$

and

$$M(x) = |O(x^{\Theta}) \text{ as } x \to \infty$$

and

$$\mathrm{li}(x) - \pi(x) = |O(x^{\Theta}) \text{ as } x \to \infty,$$

and these three propositions are logically equivalent in UD2.

Firstly consider equivalent propositions H_1 and H:

$$H \equiv \forall \epsilon > 0, \exists \text{ rational } q \in [\frac{1}{2}, 1) \text{ such that } \sum_{\substack{p \le x \\ p \text{ prime}}} \ln(p) = x + O(x^{q+\epsilon}) \text{ as } x \to \infty$$

and

$$H_1 \equiv \forall \epsilon > 0, \exists \text{ rational } q \in [\frac{1}{2}, 1) \text{ such that } \sum_{\substack{p \le x \\ p \text{ prime}}} l(p) = x + \gamma\pi(x) + O(x^{q+\epsilon}) \text{ as } x \to \infty \ldots\ldots (1).$$

We have noted above, the actual result:

$$\forall \text{ numbers } k \ge 1, \pi(x) = \left\{\frac{f_1(\gamma)x}{l(x)} + \frac{f_2(\gamma)x}{l^2(x)} + \frac{f_3(\gamma)x}{l^3(x)} + \cdots + \frac{f_k(\gamma)x}{l^k(x)}\right\} + O_k\left(\frac{x}{l^{k+1}(x)}\right) \text{ as } x \to \infty \ldots\ldots (2).$$

The immediate goal (G1) is to show that H_1 is unprovable in UD1.

We consider two cases here:-

Case 1: γ is irrational.

In UD1, isolated from the extra assumptions in UD2, the number γ do not exist. Since $f_1(\gamma) = 1$ and $f_2(\gamma) = 1 + f_2(\gamma)$ the strongest possibe derivable theoretical result is: −

$$\pi(x) = \frac{x}{l(x)} + O\left(\frac{x}{l^2(x)}\right) \text{ as } x \to \infty.$$

The strongest theoretical result for the RHS of (1) in UD1 is thus $\sum_{\substack{p \le x \\ p \text{ prime}}} l(p) = x + O(\frac{x}{l(x)})$ as $x \to \infty$.

i. e. H_1 is unprovable in UD1.

Case 2: γ is rational.

A proof in UD1 of (1) necessarily implies a proof of (2) in UD1.

The pattern of the proposition (2) includes proving an unbounded number of logically distinct propositions:

$$'1' \equiv \exists \text{ rational } q \in [\frac{1}{2}, 1) \text{ such that } \pi(x) = \left\{\frac{f_1(\gamma)x}{l(x)} + O(\frac{x}{l^2(x)})\right\} \text{ as } x \to \infty.$$

$'2' \equiv \exists$ rational $q \in [\frac{1}{2}, 1)$ such that $\pi(x) = \left\{\frac{f_1(\gamma)x}{l(x)} + \frac{f_2(\gamma)x}{l^2(x)} + O(\frac{x}{l^3(x)})\right\}$ as $x \to \infty$.

$'3' \equiv \exists$ rational $q \in [\frac{1}{2}, 1)$ such that $\pi(x) = \left\{\frac{f_1(\gamma)x}{l(x)} + \frac{f_2(\gamma)x}{l^2(x)} + \frac{f_3(\gamma)x}{l^3(x)} + O(\frac{x}{l^3(x)})\right\}$ as $x \to \infty$.

and so on.

A proof of H_1 in UD1 then implies a proof of unbounded precise, logically independent, asymptotic expressions - '1', '2', '3', …… each of order 1.

However, in UD1 there is nothing beyond the pattern

1 1+1 1+1+1 1+1+1+1 1+1+1+1+1 1+1+1+1+1+1 1+1+1+1+1+1+1 ………… .

In inductive argument in UD1 we cannot get beyond the unbounded pattern without formal definition of new entities (real numbers) but here we are faced with trying to talk about a logical law beyond bounded argument –a meaningless notion in the finite universe of UD1.

Thus the proposition H_1 is unprovable in UD1.

This then achieves the goal G1.

The next goal G2 is to exclude the possibility that we can prove Θ=1 in UD1.

In the case just discussed we noted that we cannot reach Error(x) in the relationship

$$H_1 \equiv \forall\epsilon > 0, \exists \text{ rational } q \in [\frac{1}{2}, 1) \text{ such that } \sum_{\substack{p \le x \\ p \text{ prime}}} l(p) = x - \gamma\pi(x) + \text{Error}(x) \text{ as } x \to \infty$$

to discuss the order of Error(x) in UD1. We thus cannot exclude the possibility

$\text{Error}(x) = |O(x^1)$ as $x \to \infty$. i.e. $\Theta = 1$ is also undecidable in UD1.

This completes Goal 2.

Goal 3 (G3) is to link up with the Riemann hypothesis.

It is convenient to consider the proposition
$H_2 \equiv \forall\epsilon > 0, \exists$ rational $q \in [\frac{1}{2}, 1)$ such that $M(x) = O(x^{q+\epsilon})$as $x \to \infty$.

Since $H_2 \equiv H_1$ in UD2 and H_1 is unprovable in UD1 it follows without contradiction that H_2 is unprovable in UD1 and hence $M(x) = O\left(x^{1/2}\right)$ as $x \to \infty$ is unprovable in UD1.

Since this result cannot be contradicted by numerical results derived assuming theory in UD2 we see that all computed zeros of ζ in the critical strip will be simple and lie on σ=1/2.

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Appendix 1

Section 1

$$I_k(x) = \int_2^x \frac{dt}{\ln^k(t)} = \int_{e^{2k}}^x \frac{dt}{\ln^k(t)} + O_k(1) \text{ as } x \to \infty$$

$$I_k(x) = \frac{x}{\ln^k(x)} + k \int_{e^{2k}}^x \frac{dt}{\ln^{k+1}(t)} + O_k(1) \text{ as } x \to \infty \ldots\ldots (1)$$

For $t \geq e^{2k}$, $\quad \ln^{k+1}(t) \geq 2k\ln^k(t)$

Hence $k\int_{e^{2k}}^{x}\frac{dt}{\ln^{k+1}(t)} \leq \frac{1}{2}\ \ I_k(x) + O_k(1)$ as $x \to \infty$

Hence from (1), $\quad I_k(x) = O_k\left(\frac{x}{\ln^k(x)}\right)$ as $x \to \infty$.

By repeated integration by parts, we thus have

$$\mathrm{li}(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \cdots + (k-1)!\frac{x}{\ln^k(x)} + (k)!\, I_{k+1}(x)$$

Hence

$$\mathrm{li}(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \cdots + (k-1)!\frac{x}{\ln^k(x)} + O_k\left(\frac{x}{\ln^{k+1}(x)}\right) \text{ as } x \to \infty.$$

See [1] Ingham Chapter 3.

Section 2

From the classical theory of the Riemann zeta function,Tenenbaum [9], there exists c>0 such that

$$\pi(x) = \mathrm{li}(x) + O(x\exp(-c(\ln(x))^{\frac{1}{2}}).$$

Noting that $x\exp(-c\left(\ln(x)\right)^{\frac{1}{2}})$ is asymptotically smaller than $\frac{x}{\ln^k(x)}$ it follows that for $k \geq 1$,

$$\pi(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \cdots + (k-1)!\frac{x}{\ln^k(x)} + O_k\left(\frac{x}{\ln^{k+1}(x)}\right) \text{ as } x \to \infty.$$

This form of the prime number theorem shows the remarkable (unconditional) asymptotic properties of the prime number counting function and how it decomposes into unbounded different asymptotic terms.

Section 3

Let $f(x) = (1-x)^{-k}$. Then $f^{(K)}(x) = \dfrac{(k+K-1)!}{(k-1)!}(1-x)^{-(k+K)}$.

Hence

$$f(x) = 1 + kx + \frac{k(k+1)}{2!}x^2 + \cdots + \frac{(k+K-1)!}{(k-1)!}x^{k+K} + (1-\theta(x)x)^{-(k+K+1)}x^{(k+K+1)},$$

where $0 < \theta(x) < 1$.

Hence, for x<1/2,

$$f(x) < 1 + kx + \frac{k(k+1)}{2!}x^2 + \cdots + \frac{(k+K-1)!}{(k-1)!}x^{k+K} + 2^{-(k+K+1)}x^{(k+K+1)}.$$

We thus see

$$f(x) = 1 + kx + \frac{k(k+1)}{2!}x^2 + \cdots + \frac{(k+K-1)!}{(k-1)!}x^{k+K} + O_K\left(x^{(k+K+1)}\right) \text{ as } x \to 0.$$

li(x) in terms of l(x):

$$\frac{x}{\ln^k(x)} = \frac{x}{\{l(x) - \gamma + O(\frac{1}{x})\}^k}$$

$$= \frac{x}{l^k(x)}\{1 - \frac{(\gamma - \epsilon(x))}{l(x)}\}^{-k}$$

$$= \frac{x}{l^k(x)}\{1 + k\frac{(\gamma - \epsilon(x))}{l(x)} + \cdots + \frac{(k+K-1)!}{(k-1)!}\left(\frac{(\gamma - \epsilon(x))}{l(x)}\right)^{k+K} + O_K\left(\frac{(\gamma - \epsilon(x))}{l(x)}\right)^{k+K+1}$$

as $l(x) \to \infty$, where $\epsilon(x) = O\left(\frac{1}{x}\right)$ as $x \to \infty$.

Substituting these expressions into

$$li(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \cdots + (k-1)!\frac{x}{\ln^k(x)} + O_k\left(\frac{x}{\ln^{k+1}(x)})\right) \text{ as } x \to \infty,$$

we see

$$li(x) = \left\{\frac{x}{l(x)} + \frac{f_2(\gamma)x}{l^2(x)} + \frac{f_3(\gamma)x}{l^3(x)} + \cdots + \frac{f_k(\gamma)x}{l^k(x)}\right\} + O_k\left(\frac{x}{l^{k+1}(x)})\right) + O_K\left(\frac{1}{xl(x)}\right) \text{ as } x \to \infty,$$

$$= \left\{\frac{x}{l(x)} + \frac{f_2(\gamma)x}{l^2(x)} + \frac{f_3(\gamma)x}{l^3(x)} + \cdots + \frac{f_k(\gamma)x}{l^k(x)}\right\} + O_k\left(\frac{x}{l^{k+1}(x)}\right) \text{as } x \to \infty,$$

where $f_k$is a polynomial of degree $k - 1$ with integer coefficients.

$f_2(\gamma) = 1 + \gamma, f_3(\gamma) = \gamma^2 + 2\gamma + 2, f_4(\gamma) = \gamma^3 + 3\gamma^2 + 6\gamma + 6, f_5(\gamma) = \gamma^4 + 4\gamma^3 + 12\gamma^2 + 24\gamma + 24,$

and so on.

Appendix 2

Notes on basic analytical results about Dirichlet series and the Riemann zeta function

Let $f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ $(s = \sigma + it)$, $\quad A(x) = \sum_{1 \le n \le N} a(n)$, $\quad A(0) = 0$.

Let γ be the lub of numbers θ such that $A(x) = O(x^{\theta})$. Then we write $A(x) = |O(x^{\gamma})$ and call γ the order or precise order of $A(x)$.

Corresponding, if α is the glb of numbers θ such that $A(x) = \Omega_{+}(x^{\theta})$ we write
$A(x) = |\Omega_{+}(x^{\alpha})$. Similarly if β is the glb of numbers θ such that $A(x) = \Omega_{+}(x^{\theta})$ we write
$A(x) = |\Omega_{-}(x^{\beta})$.

If $\alpha=\beta = \Delta$, we write $A(x) = |\Omega_{+-}(x^{\Delta})$.

Lemma 1

If $\forall\ \epsilon > 0$: $A(N) = O\left(N^{\Delta+\epsilon}\right)$ as $N \to \infty$ $(\Delta > 0)$

then $\sigma > \Delta + 1$: $\sum_{n=1}^{\infty} \frac{A(n)}{n^s} = \frac{1}{(s-1)} f(s-1) + k_{\Delta}(s)$

where $k_{\Delta}(s)$ is analytic for $\sigma > \Delta$.

Proof of lemma 1

Assume $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is convergent for $\sigma > \Delta$.

Then $\forall \epsilon > 0, A(x) = O\left(x^{\Delta+\epsilon}\right)$ as $x \to \infty$.

Hence, $\sum_{n \le x} A(n) = \sum_{n \le x} \sum_{k \le n} a(k)$

$= \sum_{k \le x} a(k)\{[x] - k + 1\}$

$= [x] \sum_{k \le x} a(k) - \sum_{k \le x} ka(k) - \sum_{k \le x} a(k)$

$= x \sum_{k \le x} a(k) - \sum_{k \le x} ka(k) + O(x^{\Delta+\epsilon})$ as $x \to \infty$.

Consequently,

$$s\int_1^\infty \{\sum_{n\le x} A(n)\}/x^{s+1}\,dx = s\int_1^\infty \{\sum_{n\le x} a(n)\}/x^{s}\,dx - s\int_1^\infty \{\sum_{n\le x} na(n)\}/x^{s+1}\,dx + k_\Delta(s)$$

where $k_\Delta(s)$ is analytic in $\sigma > \Delta$.

Hence,

$$s\int_1^\infty \{\sum_{n\le x} A(n)\}/x^{s+1}\,dx = \frac{s}{s-1}f(s-1) - f(s) + k_\Delta(s)$$

$$= \frac{1}{s-1}f(s-1) + k_\Delta(s).$$

Lemma 2

If the Dirichlet series for $f(s)$ is convergent at $s = \Delta$ ($\Delta > 0$) then $A(N) = o(N^\Delta)$ as $N \to \infty$.

Proof of lemma 2

Let $\sum_{1\le n\le N} \frac{a(n)}{n^\Delta} = L(\Delta) + \epsilon(N)$ where $\epsilon(N) \to 0$ as $N \to \infty$ and let $b(N) = \frac{A(N)}{(N)^\Delta}$ $(N \ge 1)$.

Since

$$\frac{a(N+1)}{(N+1)^\Delta} = \epsilon(N+1) - \epsilon(N).$$

Hence

$$\frac{A(N+1) - A(N)}{(N+1)^\Delta} = \epsilon(N+1) - \epsilon(N).$$

Thus

$$b(N+1) - (\frac{N}{N+1})^\Delta b(N) = \epsilon(N+1) - \epsilon(N).$$

That is,

$$b(N+1) = (\frac{N}{N+1})^\Delta b(N) + \epsilon(N+1) - \epsilon(N).$$

Applying the last relationship iteratively,

$$b(N+1) = (\frac{M}{N+1})^{\Delta} b(M) + \sum_{M \le i \le N} (\frac{i+1}{N+1})^{\Delta} \left(\epsilon(i+1) - \epsilon(i)\right).$$

Hence

$$b(N+1) = (\frac{M}{N+1})^{\Delta} b(M+1) + \sum_{M \le i \le N} (\frac{i}{N+1})^{\Delta}\, \epsilon(i+1) - \sum_{M \le i \le N} (\frac{i}{N+1})^{\Delta}\, \epsilon(i)$$

$$= (\frac{M}{N+1})^{\Delta} b(M+1) + \sum_{M+1 \le i \le N+1} (\frac{i}{N+1})^{\Delta}\, \epsilon(i) - \sum_{M \le i \le N} (\frac{i+1}{N+1})^{\Delta}\, \epsilon(i)$$

$$= (\frac{M}{N+1})^{\Delta} b(M+1) + \sum_{M \le i \le N} \{(\frac{i}{N+1})^{\Delta} - (\frac{i+1}{N+1})^{\Delta}\} \epsilon(i) + \epsilon(N+1) - (\frac{M}{N+1})^{\Delta}\, \epsilon(M).$$

$$= (\frac{M}{N+1})^{\Delta} b(M+1) - \sum_{M \le i \le N} \{(\frac{i+1}{N+1})^{\Delta} - (\frac{i}{N+1})^{\Delta}\} \epsilon(i) \; + \epsilon(N+1) + (\frac{M}{N+1})^{\Delta}\, \epsilon(M) \ldots \ldots (1).$$

With $|\epsilon(i)| < \epsilon$ for $M > M_0 = M_0(\epsilon)$ we thus have

$$|b(N+1)| \le= |(\frac{M_0}{N+1})^{\Delta} b(M_0+1)| + \epsilon \sum_{M \le i \le N} \{(\frac{i+1}{N+1})^{\Delta} - (\frac{i}{N+1})^{\Delta}\} \; + 2\epsilon.$$

$$|b(N+1)| < (\frac{M_0}{N+1})^{\Delta} |b(M_0+1)| + 3\epsilon\,.$$

Hence for all sufficiently large N we have $|b(N+1)| < 4\epsilon$.

i.e. $\lim_{N \to \infty} b(N) = 0$.

Notes: it is only at (1) and the ensuring argument that we use the property that Δ is real.

e.g. $I = \sum_{M \le i \le N} \{(\frac{i+1}{N+1})^{\Delta} - (\frac{i}{N+1})^{\Delta}\} > 0.$

With Δ = a+ib (a>0), we have the well-known estimate

$$|I| \le |\Delta + 1|| \sum_{M \le i \le N} \int_i^{i+1} x^{\Delta - 1}\, dx|$$

$$\le |\Delta + 1| \sum_{M \le i \le N} \int_i^{i+1} x^{a-1}\, dx$$

$$\leq \frac{|\Delta+1|}{a} \sum_{M \leq i \leq N} (i+1)^a - i^a.$$

Thus lemma 1 is also true with $\mathrm{Re}\{\Delta\}>0$. The case $\Delta=0$ is trivial.

Lemma 3 (Landau)

If the a(n) are real and eventually of one sign, then f(s) has a singularity at the real point on the line of convergence of the series.

Proof of lemma 3

See for example Titchmarsh [10] pages 294-295.

Notes:

There is no corresponding real variable result. $\sum_{n=1}^{\infty} \frac{1}{(\log^2 n) n^{\sigma}}$ is well behaved at $\sigma = 1$.

Lemma 4

For $\sigma > \sigma_0$ let f(s) be analytic and $f(s) = O(t^{\epsilon})$ $(\forall \epsilon > 0)$.

Then $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is convergent for $\sigma > \sigma_0$.

Notes: This result is an application of Perron's formula which is also found in

Titchmarsh [11] pages 300-301.

Let μ denote the Möbius function and let $M_{k+1}(x) = \sum_{n \le x} M_k(n)$ where $M_1(x) = \sum_{n \le x} \mu(n)$.

Lemma 5

For k ≥1 and $\sigma > k+1$;

$$\sum_{n=1}^{\infty} \frac{M_k(n)}{n^s} = \frac{1}{(s-1)(s-2)\ldots.(s-k)\zeta(s-k)} + E_k(s)$$

where $E_k(s)$is analytic for $\sigma > k$ and ζ is the Riemann zeta function.

Proof of lemma 5

Since

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \ (\sigma > 1),$$

the case k=1 follows from lemma1 directly.

We may then use induction and lemma1 to establish the general case.

Lemma 6

For k≥1,

$$\forall \epsilon > 0;\ M_k(x) = \Omega_{\pm}\left(x^{k-\frac{1}{2}-\epsilon}\right) \text{as } x \to \infty.$$

Proof of lemma 6

Suppose we have $1/2 > \Delta > 0$ and $k > 0$ such that $M_k(x) + Ax^{k-\frac{1}{2}-\Delta} \ge 0$.

Then using lemma 4, for σ>k,

$$\sum_{n=1}^{\infty} \frac{M_k(n) + Ax^{k-\frac{1}{2}-\Delta}}{n^s} = \frac{1}{(s-1)(s-2)\ldots.(s-k)\zeta(s-k)} + \zeta(s-k+\frac{1}{2}+\Delta) + E_k(s)$$

where $E_k(s)$ is analytic for $\sigma > k$.

From lemma 3, the Dirichlet series on the LHS of this last equation has a singularity at the real point on its line of convergence.

$E_k(s)$ is analytic for σ> k and the other possibilities for the value of such a point are

σ∈{1,2, .., k, (k+1/2-Δ)}. Thus the RHS is analytic for σ> k+1/2-Δ.

However, ζ(s) has a zero on the line σ=1/2 and hence the RHS has a singularity on the line at

σ = k+1/2.

The lemma now follows.

Recall, Θ =lub{σ:ζ(s)≠0 for Re{s}> σ }.

Lemma 7

For k≥1,

$\forall \epsilon > 0;\ M_k(x) = \Omega_{\pm}\left(x^{k-1+\Theta-\epsilon}\right)$ as $x \to \infty$.

Proof of lemma 7

Suppose we have $\Theta > \Delta > 0$ and $k \geq 1$ such that $M_k(x) + Ax^{k-1+\Theta-\Delta} \geq 0$.

Then using lemma 4, for σ>k,

$$\sum_{n=1}^{\infty} \frac{M_k(n) + Ax^{k-1+\Theta-\Delta}}{n^s} = \frac{1}{(s-1)(s-2)\ldots.(s-k)\zeta(s-k)} + \zeta(s-k+1-\Theta+\Delta) + E_k(s)$$

where $E_k(s)$is analytic for $\sigma > k$.

From lemma 3, the Dirichlet series on the LHS of this last equation has a singularity at the real point on its line of convergence.

The largest real singularity on the RHS is at σ= k+ Θ -Δ. Thus the RHS is analytic for σ> k+ Θ -Δ.

However, ζ(s) has a zero on the line σ= Θ and or arbitrarily close to that line hence the RHS has a singularity either on the line σ = k+ Θ or arbitrarily close to that line.

The lemma now follows.

Logical equivalences to $\sigma = \Theta$.

We know from the theory of the Riemann zeta function that $\frac{1}{2} \leq \Theta \leq 1$.

We can be more precise about the relationship between the critical line and the order and the oscillatory order of the $M_k(x)$ as $x \to \infty$.

Let $RH(\Theta)$ be the statement $\sigma = \Theta$ $(\frac{1}{2} \leq \Theta \leq 1)$.

Theorem 1:

For each integer $k \geq 1$;

(i) $M_k(x) = |O(x^{k-1+\Theta})$ as $x \to \infty \equiv RH(\Theta)$

and

(ii) $M_k(x) = |\Omega_\pm(x^{k-1+\Theta})$ as $x \to \infty \equiv RH(\Theta)$.

Proof of (i):

Suppose, for a given k,

$\forall \epsilon > 0, M_k(x) = |O(x^{k-1+\Theta+\epsilon})$ as $x \to \infty$.

Then from lemma 4 and the theory of Dirichlet series we have

$\sum_{n=1}^{\infty} \frac{M_k(n)}{n^s}$ is convergent for $\sigma > k - 1 + \Theta$ and hence $\frac{1}{\zeta(s-k)}$ is analytic for $\sigma > k + \Theta$.

i. e. $\zeta(s)$ is zero free for $\sigma > \Theta$.

Conversely, suppose $\zeta(s)$ is zero free for $\sigma > \Theta$ $(1/2 \leq \Theta \leq 1)$.

Then for $\sigma > \Theta$ and $\forall \epsilon > 0$; $\frac{1}{\zeta(s)} = O(t^\epsilon)$ as $t \to \infty$.

(See Titchmarsh [11] for the case $\Theta = 1/2$. the argument for any Θ $(1/2 \leq \Theta \leq 1)$ follows similarly.

It then follows immediately from lemma 4 that $\sum_{n=1}^{\infty} \frac{M_k(n)}{n^s}$ is convergent for $\sigma > k - 1 + \Theta$.

Proof of (ii)

From lemma 7, for $k \geq 1$ and $\forall \epsilon > 0$; $M_k(x) = \Omega_\pm(x^{k-1+\Theta-\epsilon})$ as $x \to \infty$.

If for some $k \geq 1$ and some $\Delta > 0$:– $\forall \epsilon > 0$; $M_k(x) = \Omega_\pm(x^{k-1+\Theta+\Delta-\epsilon})$ as $x \to \infty$.

Then the order for $M_k(x)$ is greater than or equal to $k-1+\ \Theta+\Delta$.

But from lemma 8(i), the order of $M_k(x)$ is precisely k-1+ Θ which contradicts the definition of Θ.

Notes

This proof hinges on Landau's theorem and the fact that ζ has a zero on σ=1/2. It also uses an inductive argument P(1),P(2) ……. where P(1) is not in UD1.

In Titchmarsh [11], page 336 he writes, in the introductory paragraph (assuming the Riemann hypothesis):-
'It will be seen that a perfectly coherent theory can be constructed on this basis, which perhaps gives some support to the view that the hypothesis is true'.

Clearly 'perhaps gives some support' is somewhat tentative but at least is a comment about not running into any contradictions in the development of the theory around the Riemann zeta function. It should at this stage be clear to the reader that the same comment could be applied to the hypothesis RH(Θ) but the theory instead of pointing to RH(1/2) as true it points to RH(Θ) unprovable in [1/2,1]. The strongest proof is that RH(1/2) can never be contradicted in calculation and all the calculated zeros will be simple. We have had to employ the additional assumptions of analysis and the small sacrifice is the loss of the sort of truth strength available in an inductive argument in UD1.

Unbounded logic?

An implication of lemma 5 is the possible unprovability of the collection

$\forall\ K\geq 1, \forall\epsilon>0;\ M_k(x)=\Omega_{\pm}\left(x^{k-1+\Delta-\epsilon}\right)$ as $x\rightarrow\infty$.

for any fixed Δ satisfying $0\leq\ \Delta\ \leq 1/2$.

Indeed, if we define the proposition P(K) for integer K≥1 by

$P(K)\equiv\forall\epsilon>0;\ M_k(x)=\Omega_{\pm}\left(x^{k-\Delta-\epsilon}\right)$ as $x\rightarrow\infty$,

we see that P(K) follows from P(K+1) is trivial but the step P(K+1) follows from P(K) is not trivial.

At each stage, as a proposition in arithmetic, some additional property of the Möbius function is needed in a theorem which proved P(K+1) follows from P(K).

How in finite arithmetic would we be able to locate a finite logical argument which established the truth of P(K)→P(K+1) for all K?

On the surface there are an unbounded number of logical cases to consider and the only way to account for this in a finite argument is to use some sort of inductive property of the Möbius function.

We see in the main discussion that this is not possible and the simple assertion

$M(x)=M_1(x)=\Omega\left(x^{\Delta}\right)$ as $x\rightarrow\infty$ is unprovable in UD1 for $\Delta\ \geq 1/2$.