SELBERG ZETA FUNCTION AND TRACE FORMULA FOR THE BTZ BLACK HOLE

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Abstract. A Selberg zeta function is attached to the three-dimensional BTZ black hole, and a trace formula is developed for a general class of test functions. The trace formula differs from those of more standard use in physics in that the black hole has a fundamental domain of infinite hyperbolic volume. Various thermodynamic quantities associated with the black hole are conveniently expressed in terms of the zeta function.

1. Introduction

The importance of the Selberg zeta function and the Selberg trace formula for a discrete group Γ of isometries of hyperbolic n-space $\mathbb{H}^n$ is fairly well established by now in the Physics literature, where one usually assumes that the fundamental domain F for the action of Γ has finite hyperbolic volume $V(F)$ (cf. for example [3], [17]). On the other hand, the three-dimensional Bañados, Teitelboim, Zanelli (BTZ) black hole [1], [4], [5], [6], which is a solution of Einstein's vacuum equation with a negative cosmological constant, has a Euclidean quotient presentation $\Gamma\backslash\mathbb{H}^3$ for an appropriate Γ where, however, the fundamental domain has *infinite* hyperbolic volume. For the non-spinning black hole, for example, one can choose Γ to be an Abelian group generated by a single hyperbolic element.

More generally, for discrete groups of isometries of $\mathbb{H}^n$ with $V(F) = \infty$ (such groups are called Kleinian groups), Selberg zeta functions and trace formulas exist, due in the surface ($n = 2$) case to Patterson (see [12], esp. paper III) and Guillopé-Zworski [9] and in the case $n \geq 3$ excluding fundamental domains F with cusps to Perry [16] (Perry's work depends on previous work of Patterson [13], [14] and Patterson-Perry [15]). Matters are more difficult in the infinite-volume setting due to the infinite multiplicity of the continuous spectrum and absence of a canonical renormalization for the scattering operator which makes it trace-class. In the special case of the BTZ black hole $B_\Gamma = \Gamma\backslash\mathbb{H}^3$ however, where the structure is relatively simple, one can by-pass much of the general theory and proceed more directly to define a Selberg zeta function Z_Γ attached to B_Γ and establish a trace formula which is a special version of the Poisson formula for resonances derived in [16]. This is done in the present paper.

Actually, the function $\log(Z_\Gamma)$ already appears in the papers [4], [10], [11], for example, although it is not formally identified as such. Thus we shall indicate that quantities such as the effective action of the BTZ instanton, black hole free energy, and certain functional determinants employed in the study of quantum corrections to the entropy, for example, are all expressible in terms of Z_Γ. Moreover, the trace

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formula of the present paper (Theorem 1) complements trace formulas presented in [4], [11]. Here we see, as not seen in [4], [11], that the traditional role played by the eigenvalues of the Laplacian in the trace formula is played in the infinite-volume case by scattering resonances. The latter are poles of a meromorphically continued scattering matrix for the Laplacian on B_Γ. This scattering matrix is a pseudodifferential operator which, unlike that of the Schrödinger operator, is unbounded in general.

2. BTZ Metric and Quotient Structure

We begin our discussion by reviewing how the BTZ metric ds^2 is related to a standard hyperbolic constant curvature metric–in particular, as our approach differs slightly from that in the literature. For thermodynamic reasons we consider the Euclidean form of the metric. It is given by

$$\begin{aligned} ds^2 = &\left[N(r)^2 + r^2 N^\phi(r)^2\right] d\tau^2 + N(r)^{-2}\, dr^2 \\ &+ 2r^2 N^\phi(r)\, d\phi\, d\tau + r^2\, d\phi^2 \end{aligned} \tag{2.1}$$

for

$$N(r)^2 = -M - \Lambda r^2 - J^2/4r^2, \quad N^\phi(r) = -J/2r^2 \tag{2.2}$$

for suitable coordinates (r, ϕ, τ) on a region of anti-deSitter space, where $M > 0$, $J \geq 0$, and $\Lambda < 0$ are the black hole mass, angular momentum, and cosmological constant, respectively [1], [5], [6]. This metric solves the Einstein equations in three dimensions,

$$R_{ij} - \frac{1}{2} g_{ij} R - \Lambda g_{ij} = 0, \tag{2.3}$$

with constant scalar curvature $R = 6/\sigma^2$ for $\sigma = 1/\sqrt{-\Lambda}$. The coordinates (r, ϕ, τ) can be related to the coordinates (x, y, z) on hyperbolic 3-space

$$\mathbb{H}^3 = \left\{(x, y, z) \in \mathbb{R}^3 : z > 0\right\} \tag{2.4}$$

(upper half-space model) as follows. First let $r_+ > 0$ and $r_- \in i\mathbb{R}$ ($i^2 = -1$) be given by

$$\begin{aligned} r_+^2 &= \frac{M\sigma^2}{2}\left[1 + \left(1 + \frac{J^2}{M^2\sigma^2}\right)^{1/2}\right], \\ r_- &= -\frac{\sigma J i}{2r_+}. \end{aligned} \tag{2.5}$$

Then

$$\begin{aligned} r_-^2 &= \frac{M\sigma^2}{2}\left[1 - \left(1 + \frac{J^2}{M^2\sigma^2}\right)^{1/2}\right], \\ |r_-| &= \frac{\sigma J}{2r_+} = ir_-. \end{aligned} \tag{2.6}$$

The numbers r_+ and r_- are the black hole event horizon and inner horizon, respectively. Also, let V_+ and V_- be the open subsets of $\mathbb{H}^3$ given by

$$V_\pm = \{(x, y, z) \in \mathbb{H}^3 : \pm x > 0\} \tag{2.7}$$

and let θ be the function defined on $\mathbb{R}^2 - \{(0,0)\}$ by

$$\theta(x,y) = \begin{cases} \frac{\pi}{2}\left(1 - \frac{x}{|x|}\right) + \arctan(y/x) & x \neq 0 \\ \frac{y}{|y|}\frac{\pi}{2} & x = 0 \end{cases}; \tag{2.8}$$

i.e.,

$$\begin{aligned} \cos\theta(x,y) &= \frac{x}{\sqrt{x^2+y^2}}, \\ \sin\theta(x,y) &= \frac{y}{\sqrt{x^2+y^2}}; \end{aligned} \tag{2.9}$$

here $\theta(x,y)$ belongs to the interval $[\pi/2, 3\pi/2)$. To a point $(x,y,z) \in V_\pm$ one associates a point $(r,\phi,\tau) \in \mathbb{R}^3$ by setting

$$\begin{aligned} r &= \left[r_+^2 + (r_+^2 - r_-^2)\left(\frac{x^2+y^2}{z^2}\right)\right]^{1/2} > r_+ \\ \phi &= \frac{\sigma}{r_+^2 + |r_-|^2}\left\{|r_-|\,\theta(x,y) + \frac{r_+}{2}\log(x^2+y^2+z^2)\right\} \\ \tau &= \frac{\sigma^2}{r_+^2 + |r_-|^2}\left\{r_+\theta(x,y) - \frac{|r_-|}{2}\log(x^2+y^2+z^2)\right\}; \end{aligned} \tag{2.10}$$

see (2.8) and note that

$$\theta(x,y) = \begin{cases} \arctan(y/x), & (x,y,z) \in V_+ \\ \pi + \arctan(y/x), & (x,y,z) \in V_- \end{cases}.$$

Conversely, to a point $(r,\phi,\tau) \in \mathbb{R}^3$ one associates the point $(x,y,z) \in \mathbb{H}^3$ given by

$$\begin{aligned} x &= A(r)\,[\cos f(\phi,\tau)]\exp(r_+\phi/\sigma - |r_-|\,\tau/\sigma^2) \\ y &= A(r)\,[\sin f(\phi,\tau)]\exp(r_+\phi/\sigma - |r_-|\,\tau/\sigma^2) \\ z &= B(r)\exp(r_+\phi/\sigma - |r_-|\,\tau/\sigma^2) \end{aligned} \tag{2.11}$$

for

$$A(r) = \left[\frac{r^2 - r_+^2}{r^2 - r_-^2}\right]^{1/2}, \quad B(r) = \left[\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right]^{1/2} \tag{2.12}$$

and for $f : \mathbb{R}^2 \to \mathbb{R}$ the function defined by

$$f(s,t) = \frac{r_+ t}{\sigma^2} + \frac{|r_-|\,s}{\sigma}. \tag{2.13}$$

It can be checked that the correspondence $(x,y,z) \longleftrightarrow (r,\phi,\tau)$ defined by equations (2.10)-(2.13) is a 1-1 mapping of V_+ onto $(r_+,\infty) \times f^{-1}((-\pi/2,\pi/2))$ and of V_- onto $(r_+,\infty) \times f^{-1}((\pi/2,3\pi/2))$ such that the metric in (2.1) is transformed into the standard hyperbolic metric

$$ds^2 = \frac{\sigma^2}{z^2}(dx^2 + dy^2 + dz^2) \tag{2.14}$$

on $\mathbb{H}^3$ restricted to V_+, V_-.

On the other hand, in (2.1) one has periodicity in the Schwarzschild variable ϕ which we take into account as follows. If ϕ in (2.11) is replaced by $\phi + 2\pi n$ where $n \in \mathbb{Z}$, the set of whole numbers, then (x, y, z) there is transformed to the point (x', y', z') where

$$\begin{aligned} x' &= \exp(2\pi n r_+/\sigma) \left[x \cos\left(2\pi n \left| r_- \right| / \sigma\right) - y \sin\left(2\pi n \left| r_- \right| / \sigma\right) \right], \\ y' &= \exp(2\pi n r_+/\sigma) \left[y \cos\left(2\pi n \left| r_- \right| / \sigma\right) + x \sin\left(2\pi n \left| r_- \right| / \sigma\right) \right], \\ z' &= \exp(2\pi n r_+/\sigma) z. \end{aligned} \tag{2.15}$$

The transformation $(x, y, z) \mapsto (x', y', z')$ can be viewed group-theoretically. Namely, the complex unimodular group $G = SL(2, \mathbb{C})$ acts on $\mathbb{H}^3$ in a standard way: for $(x, y, z) \in \mathbb{H}^3$, $g \in G$,

$$g \cdot (x, y, z) = (u, v, w) \in \mathbb{H}^3 \tag{2.16}$$

where for $t = x + iy$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{aligned} u + iv &= \frac{(at + b)\overline{(ct + d)} + a\bar{c}z^2}{\left|ct + d\right|^2 + \left|c\right|^2 z^2}, \\ w &= \frac{z}{\left|ct + d\right|^2 + \left|c\right|^2 z^2}, \end{aligned} \tag{2.17}$$

where the bar denotes complex conjugation. Moreover if $\Gamma \subset G$ is the discrete group of G defined by

$$\begin{aligned} \Gamma &= \left\{ \begin{bmatrix} e^{\pi n (r_+ + i |r_-|)/\sigma} & 0 \\ 0 & e^{-\pi n (r_+ + i |r_-|)/\sigma} \end{bmatrix} : n \in \mathbb{Z} \right\} \\ &= \{\gamma^n : n \in \mathbb{Z}\} \text{ for } \gamma = \begin{bmatrix} e^{\pi (r_+ + i |r_-|)/\sigma} & 0 \\ 0 & e^{-\pi (r_+ + i |r_-|)/\sigma} \end{bmatrix} \end{aligned} \tag{2.18}$$

then one has exactly that $\gamma^n \cdot (x, y, z) = (x', y', z')$ by (2.16)-(2.18).

In other words, periodicity in ϕ means, in group-theoretic terms, that we can regard the Euclidean BTZ black hole as the quotient space $B_\Gamma = \Gamma \backslash \mathbb{H}^3$ under the left action of Γ on $\mathbb{H}^3$ given by (2.16), (2.17). The metric on B_Γ pulls back (under the quotient map $\mathbb{H}^3 \mapsto B_\Gamma$) to the metric (2.14). The quotient space presentation has several advantages. For example, the heat kernel of B_Γ is readily obtained from the (well-known) heat kernel of $\mathbb{H}^3$ by averaging the latter over Γ. Also, quotient spaces of hyperbolic space (of an arbitrary dimension) by a discrete group of isometries are objects well-studied by mathematicians. In particular we can apply results in [15], in the next section, especially as the fundamental domain F for the action on $\mathbb{H}^3$ of Γ in (2.18) has an infinite hyperbolic volume:

$$\int_F \frac{dx\, dy\, dz}{z^3} = \infty \tag{2.19}$$

with F given by the region between upper hemispheres of radii $R = 1$ and $R = \exp(2\pi\sqrt{M})$.

3. Scattering Theory on B_Γ

In this section we summarize the basic results of scattering theory for B_Γ in order to illuminate the physical meaning of the scattering resonances which appear in the trace formula that will be derived in the next section. The computations that we will present are well-known in the mathematics literature (see for example Epstein [7], Guillopé-Zworski [8], and Appendix B of Patterson-Perry [15] where more general classes of 'cylindrical manifolds' are treated) but we present the main ideas here for the reader's convenience.

The manifold B_Γ is obtained from the fundamental domain

$$F = \left\{ (x,y,z) \in \mathbb{R}^3 : z > 0,\ 1 \le \sqrt{x^2+y^2+z^2} \le \exp(2\pi\sqrt{M}) \right\} \tag{3.1}$$

by identifying points on the upper hemisphere of radius 1 with their images on the upper hemisphere of radius $\exp(2\pi\sqrt{M})$ under the action of the generator

$$\gamma = \begin{bmatrix} e^{\ell/2+i\theta/2} & 0 \\ 0 & e^{-\ell/2-i\theta/2} \end{bmatrix}$$

of Γ (compare (2.18)), where $\ell = 2\pi r_+/\sigma$ is the length of the geodesic segment which projects to a single closed geodesic on the quotient, and $\theta = 2\pi|r_-|/\sigma$. The manifold B_Γ equipped with the metric induced from $\mathbb{H}^3$ has a single closed geodesic of length ℓ. Intuitively, the scattering resonances of the Laplacian on B_Γ are generated by quasimodes corresponding to this single closed geodesic and its iterates.

To make these notions more precise we compute the resolvent and scattering operator for Δ_Γ, the positive Laplacian on B_Γ, by separation of variables. Referring to (2.14), we shall take $\sigma = 1$ corresponding to the standard constant curvature -1 metric on $\mathbb{H}^3$. Introduce spherical coordinates $(\rho, \vartheta, \varphi)$ on $\mathbb{R}^3$ so that

$$\begin{aligned} x &= \rho \cos\vartheta \cos\varphi, \\ y &= \rho \cos\vartheta \sin\varphi, \\ z &= \rho \sin\vartheta. \end{aligned} \tag{3.2}$$

The fundamental domain F is the region

$$\left\{ (\rho, \vartheta, \varphi) : 1 \le \rho \le \exp(2\pi\sqrt{M}),\ \vartheta \in (0, \pi/2),\ \varphi \in [0, 2\pi) \right\} \tag{3.3}$$

and the hyperbolic metric becomes

$$ds^2 = \frac{1}{\rho^2 \sin^2\vartheta} \left(d\rho^2 + \rho^2\, d\vartheta^2 + \rho^2 \cos^2\vartheta\, d\varphi^2 \right). \tag{3.4}$$

It will be useful to set

$$\begin{aligned} \rho &= \exp(u) \\ \vartheta &= 2\arctan(e^{-t}) \end{aligned} \tag{3.5}$$

so that

$$\begin{aligned} u &= \log\rho \\ t &= -\log\tan(\vartheta/2) \end{aligned} \tag{3.6}$$

and

$$\sin\vartheta = \operatorname{sech}(t) \tag{3.7}$$
$$\cos\vartheta = -\tanh(t)$$

and the metric becomes

$$ds^2 = dt^2 + \cosh^2 t\; du^2 + \sinh^2 t\; d\varphi^2. \tag{3.8}$$

In these coordinates we have $(t, u, \varphi) \in (0, \infty) \times (0, \ell) \times (0, 2\pi)$, and the group action induces the identification

$$(t, 0, \varphi) \sim (t, \ell, \varphi + \theta) \tag{3.9}$$

of the two boundary components. The positive Laplacian is

$$\Delta_\Gamma = -\frac{1}{\sinh(t)\cosh(t)}\frac{\partial}{\partial t}\left(\sinh(t)\cosh(t)\frac{\partial}{\partial t}\right) - \frac{1}{\cosh^2 t}\frac{\partial^2}{\partial u^2} - \frac{1}{\sinh^2 t}\frac{\partial^2}{\partial \varphi^2} \tag{3.10}$$

acting on $\mathcal{H} = L^2((0, \infty) \times (0, \ell) \times (0, 2\pi), \sinh(t)\cosh(t)\, dt\, du\, d\varphi)$.

Owing to the group identifications, the operator Δ_Γ carries periodic boundary conditions

$$f(t, 0, \varphi) = f(t, \ell, \varphi + \theta). \tag{3.11}$$

Let us write

$$f(t, u, \varphi) = \sum_{m=-\infty}^{\infty} f_m(t, u)e^{im\varphi}. \tag{3.12}$$

The boundary conditions (3.11) on f then imply that the boundary conditions

$$f_m(t, \ell) = f_m(t, 0)e^{-im\theta} \tag{3.13}$$
$$\frac{\partial f_m}{\partial u}(t, \ell) = \frac{\partial f_m}{\partial u}(t, 0)e^{-im\theta}$$

hold for f_m. The eigenvectors of the one-dimensional problem

$$-\psi'' = \lambda\psi$$
$$\psi(\ell) = \psi(0)e^{-im\theta} \tag{3.14}$$
$$\psi'(\ell) = \psi'(0)e^{-im\theta}$$

are functions ψ_{mn} associated to eigenvalue k_{mn}^2 where

$$\psi_{mn}(u) = \exp(ik_{mn}u)$$

and

$$k_{mn} = -m\theta/\ell + 2\pi n/\ell. \tag{3.15}$$

Note that the indices m and n run over all integers. Thus

$$f(t, u, \varphi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_{mn}(t)\psi_{mn}(u)\exp(im\varphi). \tag{3.16}$$

This decomposition induces a decomposition of the Hilbert space

$$\mathcal{H} = \bigoplus_{m=-\infty}^{\infty} \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_{mn} \tag{3.17}$$

with $\mathcal{H}_{mn} \simeq L^2(\mathbb{R}^+, \sinh(t)\cosh(t)\ dt\,)$, and a corresponding decomposition $\Delta_\Gamma \simeq \bigoplus_{m=-\infty}^{\infty} \bigoplus_{n=-\infty}^{\infty} \hat{L}_{mn}$ where

$$\hat{L}_{mn} = -\frac{1}{\sinh t \cosh t}\frac{\partial}{\partial t}\left(\sinh t \cosh t \frac{\partial}{\partial t}\right) + k_{mn}^2 \operatorname{sech}^2 t + m^2 \operatorname{csch}^2 t. \tag{3.18}$$

We can reduce this to a Schrödinger-type scattering problem by the unitary transformation $U : L^2(\mathbb{R}^+, \sinh t\, \cosh t\ dt\,) \to L^2(\mathbb{R}^+, dt)$ given by

$$(Uf)(t) = (\sinh(t)\cosh(t))^{1/2}\ f(t). \tag{3.19}$$

Writing

$$L_{mn} = U\hat{L}_{mn}U^{-1} \tag{3.20}$$

we have

$$L_{mn} = -\frac{\partial^2}{\partial t^2} + 1 + \left(k_{mn}^2 + \frac{1}{4}\right)\operatorname{sech}^2 t + \left(m^2 - \frac{1}{4}\right)\operatorname{csch}^2 t. \tag{3.21}$$

This is a Schrödinger-type operator whose potential has a t^{-2} singularity at $t = 0$ with coefficient $(m^2 - 1/4)$ but is otherwise a 'short-range' perturbation of $-\partial^2/\partial t^2$. For $|m| \geq 1$, the formal differential operator L_{mn} is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^+ \setminus \{0\})$. In the case $m = 0$, the operator L_{0n} is not essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^+ \setminus \{0\})$ and the case must be treated separately. The differential operator is in the limit-circle case at $t = 0$ which means we must impose a boundary condition in order to insure self-adjointness. We do this effectively by taking the Friedrichs extension of the formal differential operator L_{mn} on $\mathcal{C}_0^\infty(\mathbb{R}^+ \setminus \{0\})$.

To analyze this operator, we recall from the appendix to [8] (see Lemma A.2) the following useful result concerning Pöschel-Teller potentials.

Lemma 1. *Let*

$$V_{\mu,\nu}(t) = \mu(\mu+1)\operatorname{csch}^2 t - \nu(\nu+1)\operatorname{sech}^2 t$$

where $\mu, \nu \in -\frac{1}{2} + i\mathbb{R}^+ \cup [-\frac{1}{2}, \infty)$. *Let*

$$H_{\mu,\nu} = -\frac{d^2}{dt^2} + V_{\mu,\nu}(t)$$

be the symmetric operator on $\mathcal{C}_0^\infty(\mathbb{R}^+)$ *and let* $\overline{H}_{\mu,\nu}$ *denote its Friedrichs extension. Then* $\overline{H}_{\mu,\nu}$ *has continuous spectrum of multiplicity one with scattering matrix*

$$s(\overline{H}_{\mu,\nu})(k) = -2^{-2ik}\frac{\Gamma(ik)}{\Gamma(-ik)}\frac{\Gamma((\mu+\nu-ik+2)/2)}{\Gamma((\mu+\nu+ik+2)/2)}\frac{\Gamma((\mu-\nu-ik+1)/2)}{\Gamma((\mu-\nu+ik+1)/2)}.$$

For the proof see [8].

Remark 1. *It follows that* $s(\overline{H}_{\mu,\nu})$ *has simple poles at:*

(1) $ik = 0, 1, 2, \cdots$;
(2) $ik = \mu + \nu + 2j + 2,\ j = 0, 1, 2, \cdots$;
(3) $ik = \mu - \nu + 2j + 1,\ j = 0, 1, 2, \cdots$.

and simple zeros at

(1) $ik = 0, -1, -2, \cdots;$
(2) $ik = -\mu - \nu - 2j - 2$, $j = 0, 1, 2, \cdots;$
(3) $ik = -\mu + \nu - 2j - 1$, $j = 0, 1, 2, \cdots.$

In our case,

$$\begin{aligned} \mu(\mu+1) &= m^2 - 1/4 \\ \nu(\nu+1) &= -k_{mn}^2 - 1/4 \end{aligned}$$

so the solutions lying in the specified domain for μ and ν are

$$\begin{aligned} \mu &= |m| - 1/2 \\ \nu &= -1/2 + i\,|k_{mn}|\,. \end{aligned}$$

It will be helpful to recast the formulas for the scattering matrix in terms of $s = 1 - ik$. Writing $\mathfrak{s}_{mn}(s)$ for the scattering matrix associated to L_{mn}, we have

$$\mathfrak{s}_{mn}(s) = 2^{2s-2}\frac{\Gamma(s-1)}{\Gamma(1-s)}\frac{\Gamma((\mu+\nu+1+s)/2)}{\Gamma((\mu+\nu+3-s)/2)}\frac{\Gamma((\mu-\nu+s)/2)}{\Gamma((\mu-\nu+2-s)/2)}.$$

Note that

$$\mathfrak{s}_{mn}(s)\mathfrak{s}_{mn}(2-s) = 1$$

We note that the initial Γ-factors $\Gamma(s-1)/\Gamma(1-s)$ generate 'trivial' poles and zeros. As shown in [15], it is the poles of

$$\mathcal{S}_{mn}(s) = 2^{2-2s}\frac{\Gamma(s-1)}{\Gamma(1-s)}\mathfrak{s}_{mn}(s)$$

which determine the singularities of the zeta function. As

$$\mathcal{S}_{mn}(s) = \frac{\Gamma((\mu+\nu+1+s)/2)}{\Gamma((\mu+\nu+3-s)/2)}\frac{\Gamma((\mu-\nu+s)/2)}{\Gamma((\mu-\nu+2-s)/2)},$$

$$\mu + \nu + 1 = |m| + i\,|k_{mn}|\,,$$

and

$$\mu - \nu = |m| - i\,|k_{mn}|\,,$$

it follows from Lemma 1, Remark 1 and (3.15) that $\mathcal{S}_{mn}(s)$ has poles at the points

$$s_{m,n,j} = -2j - |m| \pm i\left(\frac{2\pi n - m\theta}{\ell}\right) \tag{3.22}$$

where $j = 0, 1, 2, \cdots$.

In the next section we shall show by explicit computation that the poles of the scattering matrix coincide with zeros of the zeta function.

4. The Zeta Function and the Trace Formula

We now define Selberg's zeta function for the group $\Gamma = \{\gamma^n : n \in \mathbb{Z}\}$ generated by a single hyperbolic element of the form

$$\gamma = \begin{bmatrix} e^z & 0 \\ 0 & e^{-z} \end{bmatrix} \tag{4.1}$$

where $z = a + ib$ for $a > 0$ and $b \geq 0$. In practice we will take $a = \pi r_+$ and $b = \pi |r_-| / \sigma$ in the notation of section 2 of the paper. For the standard action of $SL(2,\mathbb{C})$ on $\mathbb{H}^3$ in (2.16), one has

$$\gamma \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $\theta = 2b$. That is, γ is the composition of a rotation in $\mathbb{R}^2$ with complex eigenvalues $\exp(\pm i\theta)$ and a dilation e^{2a}. The zeta function for Γ is then given by

$$Z_\Gamma(s) = \prod_{k_1=0}^{\infty} \prod_{k_2=0}^{\infty} \left(1 - (e^{i\theta})^{k_1} (e^{-i\theta})^{k_2} e^{-(k_1+k_2+s)\ell}\right) \tag{4.2}$$

where $\ell = 2a$, $\theta = 2b$, and the infinite product converges for all s. Thus $Z_\Gamma(s)$ is entire and its zeros come from those of the factors, namely the numbers

$$\zeta_{n,k_1,k_2} = -(k_1+k_2) + i(k_1-k_2)\theta/\ell + 2\pi i n/\ell \tag{4.3}$$

where $k_1 \geq 0$, $k_2 \geq 0$, and $n \in \mathbb{Z}$. We denote the set of such numbers by $\mathcal{R}$. Note that if $N(r)$ denotes the number of points ζ_{n,k_1,k_2} in a disc of radius r in the complex plane, we have $N(r) \leq Cr^3$.

Let us compare these zeros with the scattering poles (3.22). Solving the equations

$$\begin{aligned} k_1 + k_2 &= 2j + m \\ k_1 - k_2 &= \mp m \end{aligned}$$

if $m \geq 0$ and

$$\begin{aligned} k_1 + k_2 &= 2j - m \\ k_1 - k_2 &= \mp m \end{aligned}$$

if $m < 0$, we see that $(k_1, k_2) = (j, j + |m|)$ or $(j + |m|, j)$ so that as n and m range over all integers, the corresponding (k_1, k_2) range over nonnegative integers. Thus the sets $\{\zeta_{n,k_1,k_2} : n \in \mathbb{Z},\ k_1, k_2 \in \mathbb{N}_0\}$ and $\{s_{m,n,j} : n, m \in \mathbb{Z},\ j \in \mathbb{N}_0\}$ (where $\mathbb{N}_0$ denotes the nonnegative integers) coincide.

It is easy to see that the zeta function is an entire function of order 3 (see, for example Boas [2] for a discussion of the relevant theory) and finite type. It is clearly bounded in absolute value for $\Re(s) \geq 0$ and for $\Re(s) \leq 0$ we may estimate

$$\begin{aligned} |Z_\Gamma(s)| &\leq \left(\prod_{k_1+k_2 \leq |s|} e^{|s|\ell} \right) \cdot \left(\prod_{k_1+k_2 \geq |s|} (1 - \exp\left[(|s| - k_1 - k_2)\,\ell\right]) \right) \\ &\leq C_1 \exp\left(C_2 |s|^3\right) \end{aligned} \tag{4.4}$$

(the first factor on the right-hand side of the first line gives the exponential growth, and the second factor is bounded) which proves the required growth estimate. It follows that we can also write $Z_\Gamma(s)$ as a Hadamard product

$$Z_\Gamma(s) = e^{Q(s)} \prod_{\zeta \in \mathcal{R}} (1 - s/\zeta) \exp\left[(s/\zeta) + \frac{1}{2}(s/\zeta)^2 + \frac{1}{3}(s/\zeta)^3\right] \tag{4.5}$$

where the infinite product goes over zeros of $Z_\Gamma(s)$ of the form (4.3), and Q is a polynomial of degree at most 3. The coefficients of the polynomial can be computed, in principle at least, from the asymptotic condition that $\ln Z_\Gamma(s)$ and its derivatives in s vanish rapidly as $s \to \infty$ along the positive real axis, as easily follows from (4.2). These coefficients will not, however, appear in the trace formula, owing to our choice of testing function.

We will derive the trace formula by integrating the logarithmic derivative of the zeta function against a test function. The key idea is to compute the logarithmic derivative in two different ways: first, using the Euler product (4.2) to obtain the trace in terms of ℓ and φ, and secondly using the Hadamard product (4.5) to obtain the trace in terms of the resonances ζ.

We claim that for $\Re(s)$ large and positive, the formulas

$$\begin{aligned}\log Z_\Gamma(s) &= -\frac{1}{4}\sum_{n=1}^{\infty} \frac{\exp(-n\ell(s-1))}{n\left[\sinh^2(\ell n/2)+\sin^2(\theta n/2)\right]} \\ &= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{\exp(-n\ell(s-1))}{n\left[\cosh(\ell n)-\cos(\theta n)\right]}\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} &= \frac{\ell}{4}\sum_{n=1}^{\infty} \frac{\exp(-n\ell(s-1))}{\left[\sinh^2(\ell n/2)+\sin^2(\theta n/2)\right]} \\ &= \frac{\ell}{2}\sum_{n=1}^{\infty} \frac{\exp(-n\ell(s-1))}{\left[\cosh(\ell n)-\cos(\theta n)\right]}\end{aligned} \tag{4.7}$$

hold. Note that this series converges absolutely and uniformly for $\Re(s) > 0$. To prove (4.6) one uses (4.2) together with the identity

$$\log(1+\alpha) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\alpha^n}{n} \tag{4.8}$$

to conclude that

$$\log Z_\Gamma(s) = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k_1,\,k_2\geq 0} e^{in\theta k_1} e^{-in\theta k_2} e^{-nk_1\ell} e^{-nk_2\ell} e^{-ns\ell} \tag{4.9}$$

and then sums the geometric series. Using the fact that

$$\begin{aligned}\frac{1}{1-e^{-(-i\theta+\ell)n}}\frac{1}{1-e^{-(i\theta+\ell)n}} &= \frac{1}{2}\frac{e^{\ell n}}{(\cosh(\ell n)-\cos(\theta n))} \\ &= \frac{1}{4}\frac{e^{\ell n}}{\left(\sinh^2(\ell n/2)+\sin^2(\theta n/2)\right)}\end{aligned} \tag{4.10}$$

we conclude that (4.6) holds. The identity (4.7) then follows by differentiation.

If $\Phi(t) = Z_\Gamma'(1+it)/Z_\Gamma(1+it)$ we have, setting $s = 1+it$, that

$$\Phi(t) = \frac{\ell}{4}\sum_{m=1}^{\infty} \frac{e^{-im\ell it}}{\left[\sinh^2(m\ell/2)+\sin^2(\theta n/2)\right]}. \tag{4.11}$$

On the other hand, it follows from the Hadamard product representation of $Z_\Gamma(s)$ (4.5) that

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} = Q'(s) + \sum_{\zeta\in\mathcal{R}} \frac{(s/\zeta)^3}{s-\zeta} \tag{4.12}$$

so that, writing $\zeta \in \mathcal{R}$ as $\zeta = 1 + i\tau$,

$$\Phi(t) = Q'(1+it) + i^{-1} \sum_{\zeta = 1 + i\tau \in \mathcal{R}} \frac{(1+it)^3/(1+i\tau)^3}{t - \tau}. \tag{4.13}$$

Let

$$\Psi(t) = \Phi(t) + \Phi(-t). \tag{4.14}$$

and let $\varphi \in \mathcal{C}_0^\infty(0,\infty)$. Let

$$\psi(t) = \int_{-\infty}^{\infty} \exp(i\xi t)\, \varphi(\xi)\, d\xi \tag{4.15}$$

and note that, by construction, ψ is an entire function with rapid decay on the real line or any line of the form $t = \tau + i\zeta$ with $\zeta \geq 0$. We will now derive the trace formula by computing the integral $\int \psi(t)\, \Psi(t)\, dt$ in two different ways.

Theorem 1. *Let $\varphi \in \mathcal{C}_0^\infty(0,\infty)$ and let ψ be given as in (4.15). Then the formula*

$$\frac{\ell}{4} \sum_{m=1}^{\infty} \frac{\varphi(m\ell)}{\left[\sinh^2(m\ell/2) + \sin^2(\theta n/2)\right]} = \sum_{n=-\infty}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \psi(t_{n,k_1,k_2}) \tag{4.16}$$

holds, where $\zeta_{n,k_1,k_2} = 1 + it_{n,k_1,k_2}$.

Proof. We proceed along the lines of [16]. On the one hand, the sum (4.11) is uniformly and absolutely convergent for $t \in \mathbb{R}$ so that we may calculate $\int \psi(t)\Psi(t)\, dt$ by interchanging orders of summation and differentiation, yielding the left-hand side of (4.16). On the other hand, we can compute the integral by closing the contour in the half-plane $\Im(t) > 0$ and using the expression (4.13) for $\Phi(t)$ which is convergent away from the zeros of the zeta function. As shown in [16], there is a sequence of radii $\{R_j\}$ so that if C_j denotes a semicircular contour of radius R_j in the upper half-plane, centered at 0, one has

$$\operatorname{dist}(t_{n,k_1,k_2}, C_j) \leq C R_j^{-2}$$

so that $\Psi(t)$ is regular on each C_j and $|\Psi(t)| \leq C R_j^5$ there. On the other hand, ψ is an entire function obeying the estimates

$$|\psi(z)| \leq C_N (1 + |z|)^{-N}$$

uniform in z with $\Re(z) \geq 0$, for any positive integer N and a positive constant C_N, as may easily be seen from (4.15). These facts allow us to compute $\int \psi(t)\Psi(t)\, dt$ as a limit of integrals $\int_{C_j} \psi(t)\Psi(t)\, dt$ and obtain the right-hand side of (4.16) using the calculus of residues. □

5. Remarks on the BTZ Effective Action and Free Energy

As noted in the introduction, there are cases in the physical literature (in [4], [10], and [11] for example) where the function $\log(Z_\Gamma)$ actually appears although it is not formally identified as such. We point out here, for the record, that quantities such as the effective action $W_{\text{eff}}[B_3]$ of the BTZ instanton $B_\Gamma = B_3$ computed in [11] (where we employ the notation used there) or the free energy at the Hawking inverse temperature computed in [10], indeed are expressible in terms of Z_Γ. In considering quantum corrections to BTZ entropy, the authors in [4] compute among other things certain functional determinants, such as a regularized version of $\det \Delta_\Gamma$ for Δ_Γ the Laplace-Beltrami operator on B_3 induced by the metric (2.14), for example. By a quick glance at equations (57), (60), and (61) of [4] (where only the non-spinning black hole is considered) we see that the logarithms of these determinants can be expressed, similarly, in terms of $\log Z_\Gamma$. Compare also equation (5.3) below.

By equation (4.5) of [11], $W_{\text{eff}}[B_3]$ is given as a sum $W_{\text{div}}[B_3] + W_{\text{nondiv}}[B_3]$ consisting of a divergent part $W_{\text{div}}[B_3]$ and a non-divergent part $W_{\text{nondiv}}[B_3]$ where

$$W_{\text{nondiv}}[B_3] = -\sum_{n=1}^{\infty} \frac{1}{4n} \frac{\exp(\sqrt{\mu} n\ell_+)}{\left[\sinh^2(n\ell_+/2) + \sin^2(n\ell_-/2)\right]}$$

for $\ell_+ = 2\pi r_+/\sigma$, $\ell_- = 2\pi \left|r_-\right|/\sigma$, and for μ a suitable constant in terms of which the heat kernel on $\mathbb{H}^3$ is expressed; see section III-C of [11] and equation (3.21) there in particular. Given equation (4.6), we clearly have:

Proposition 1. *The identity*

$$W_{nondiv}[B_3] = \log Z_\Gamma(1 + \sqrt{\mu})$$

holds.

The effective action is computed, moreover, in [11], in the case when the black hole has a conical singularity at the horizon (in which case the discrete group of isometries contains an elliptic element). Differentiation of that action with respect to the angular deficit at the conical singularity yields an expression (equation (5.3) of [11])

$$S = \frac{2\pi r_+}{4G_{\text{ren}}} + \sum_{n=1}^{\infty} S_n \tag{5.1}$$

for the quantum entropy S, where G_{ren} is a renormalized Newton constant, and where the sum $\sum_{n=1}^{\infty} S_n$ is a suitable quantum correction to the classical Beckenstein-Hawking entropy $2\pi r_+/4G_{\text{ren}}$. In (5.1) the calculation done for the Euclidean metric (2.1) has been analytically continued to the corresponding Lorentzian metric. Thus S is the entropy of the Lorentzian black hole.

Consider, for example, the case $J = 0$: note that $J = 0$ implies that $r_- = 0$ and so $\ell_- = 0$. Then S_n assumes the form

$$\begin{aligned} S_n &= \frac{1}{2n} \frac{\exp(-\sqrt{\mu} n\ell_+)}{(\cosh(n\ell_+) - 1)} \left[1 + n\ell_+ \coth(n\ell_+) - \frac{n\ell_+ \sinh(n\ell_+)}{(\cosh(n\ell_+) - 1)}\right] \\ &= \frac{1}{2n} \frac{\exp(-\sqrt{\mu} n\ell_+)}{(\cosh(n\ell_+) - 1)} \left[1 - \frac{n\ell_+}{\sinh(n\ell_+)}\right]. \end{aligned} \tag{5.2}$$

Comparing (5.2) with equation (4.6) we see that for the non-spinning black hole the one-loop quantum correction in (5.1) can be expressed as

$$\begin{aligned} \sum_{n=1}^{\infty} S_n &= -\log(Z_\Gamma(1 + \sqrt{\mu})) \\ &\quad - \frac{\ell_+}{2} \sum_{n=1}^{\infty} \frac{\exp(-\sqrt{\mu} n\ell_+)}{(\cosh(n\ell_+) - 1)\sinh(n\ell_+)}. \end{aligned} \tag{5.3}$$

Here Γ is generated by the hyperbolic element

$$\gamma = \begin{bmatrix} e^{\ell_+/2} & 0 \\ 0 & e^{-\ell_+/2} \end{bmatrix}$$

in contrast to a more general loxodromic element.

In section 5.2 of [10], the following expression for black hole free energy $F(\beta)$ at the Hawking inverse temperature

$$\beta = \beta_H = \frac{2\pi\sigma^2}{r_+} \tag{5.4}$$

is derived (see equation (5.25) there), where we assume here that $J = 0$:

$$\beta F(\beta)|_{\beta=\beta_H} = -\frac{r_+\beta_H}{4\pi\sigma^2}\sum_{n=1}^{\infty}\frac{e^{-2\pi(\lambda-1)nr_+/\sigma}}{n(c_n^+-1)} + \text{constant} \tag{5.5}$$

for $c_n^+ = \cosh(2\pi n r_+\sigma)$, and for $\lambda = 1+\sqrt{1+\mu}$ with μ a constant similar in nature to the constant μ above. For example, $\mu = -3/4$ implies $\lambda = 3/2$ in the case of a conformally coupled massless scalar field. That is equation (5.5) can be written

$$\beta F(\beta)|_{\beta=\beta_H} = \log(Z_\Gamma(\lambda)) + \text{constant} \tag{5.6}$$

in view of equation (4.6), since $\theta = 0$ and $r_+\beta_H/2\pi\sigma^2 = 1$.

These brief examples ilustrate how the logarithm of the zeta function $\log(Z_\Gamma)$ appears naturally in three-dimensional black-hole thermodynamics.

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