Stirling's Formula

We will prove Stirling's formula by *Euler-Maclaurin summation*, which may be regarded as a standard method. Sources are Rainville, *Special Functions*, and Edwards' *The Riemann Zeta Function.*

There are four common definitions of the Gamma function. First, it can be defined by means of the Eulerian integral of the second kind:

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^s \,\frac{dt}{t}, \tag{1}$$

valid if $\mathrm{re}(s) > 0$; second, there is Euler's formula

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\, n^s}{s(s+1)\cdots(s+n)}, \tag{2}$$

valid if $\mathrm{re}(s) > 0$; third, there is Weierstrass' definition:

$$\Gamma(s) = e^{-\gamma s}\, s^{-1} \prod_{n=1}^{\infty} \left[\left(1+\frac{s}{n}\right)^{-1} e^{s/n} \right], \tag{3}$$

whe and fourth, there is Euler's definition

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^{s} \left(1+\frac{s}{n}\right)^{-1} \right]. \tag{4}$$

We don't need (3) or (4), but mention them for informational purposes. We will take (1) as the definition of $\Gamma(s)$, and deduce (2) in Proposition 3.

Proposition 1. *The Gamma function, defined by (1) has meromorphic continuation to all of $\mathbb{C}$, and is analytic except at $s = 0, -1, -2, \cdots$, where it has simple poles.*

PROOF. The formula

$$\Gamma(s+1) = s\,\Gamma(s) \tag{5}$$

is easily proved by integration by parts from the definition (1). Since the integral (1) is absolutely convergent for $\mathrm{re}(s) > 0$, the formula

$$\Gamma(s) = \frac{1}{s}\,\Gamma(s+1)$$

gives meromorphic continuation to the domain $\mathrm{re}(s) > -1$, holomorphic except at $s = 0$ where there is a simple pole with residue $\Gamma(1) = 1$, and iterating this process gives the meromorphic continuation to all $\mathbb{C}$, with simple poles at $s = 0, -1, -2, \cdots$. ∎

We will need the Eulerian integral

(6) $$B(p,q) = \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt,$$

convergent when $\mathrm{re}(p), \mathrm{re}(q) > 0$.

Proposition 2. *We have*

(7) $$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)},$$

PROOF. By the variable changes $t = u^2$ and $u = \sin(\theta)$, we arrive at the equivalent forms

(8) $$B(p,q) = 2\int_0^1 u^{2p-1}\,(1-u^2)^{q-1}\,du,$$

and

(9) $$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta)\,\cos^{2q-1}(\theta)\,d\theta.$$

Now using the formula

$$\Gamma(s) = 2\int_0^\infty e^{-r^2}\,r^{2s-1}\,dr,$$

which follows from (1) by the variable change $t = r^2$, we have

$$\begin{aligned}
\Gamma(p+q)\,B(p,q) &= 4\int_0^\infty e^{-r^2}\,r^{2p+2q-1}\,dr \int_0^{\pi/2} \sin^{2p-1}(\theta)\,\cos^{2q-1}(\theta)\,d\theta \\
&= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,\big(r\sin(\theta)\big)^{2p-1}\,\big(r\cos(\theta)\big)^{2q-1}\,r\,dr\,d\theta \\
&= \left[2\int_0^\infty e^{-y^2}\,y^{2p-1}\,dy\right]\left[2\int_0^\infty e^{-x^2}\,x^{2q-1}\,dx\right],
\end{aligned}$$

where we have switched from polar to rectangular coordinates: $x = r\,\cos(\theta)$ and $y = r\,\sin(\theta)$, $dx\,dy = r\,dr\,d\theta$, whence (8). ■

Note that (9) implies that $B\big(\frac{1}{2},\frac{1}{2}\big) = \pi$, so

(10) $$\Gamma\big(\tfrac{1}{2}\big) = \sqrt{\pi}.$$

Lemma 1. *Let $t \in \mathbb{R}$, and let n be a positive integer. Define*

$$f_n(t) = \begin{cases} \left(1 - \frac{t}{n}\right)^n & \text{if } 0 \le t \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then for fixed t, the sequence $\{f_n(t)\}$ is monotone increasing.

PROOF. It is clearly sufficient to show that if $0 < t < x$ then

$$\frac{d}{dx}\left(1 - \frac{t}{x}\right)^x > 0.$$

We compute easily the derivative

$$\frac{d}{dx}\left(1 - \frac{t}{x}\right)^x = \frac{1}{x}\left(1 - \frac{t}{x}\right)^{x-1}\left[t + (x-t)\,\log\left(1 - \frac{t}{x}\right)\right].$$

Thus we need to know that

$$t + (x-t)\,\log\left(1 - \frac{t}{x}\right) > 0. \tag{11}$$

We note that if $0 < u < 1$ then

$$\frac{u}{1-u} > -\log(1-u),$$

as is easily seen by comparing their Taylor series:

$$u + u^2 + u^3 + \ldots > u + \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 + \ldots .$$

Applying this with $u = t/x$ gives us (11). □

Proposition 3. *Define the Gamma function by the Eulerian integral (1). Then (2) is valid if* $\mathrm{re}(s) > 0$.

PROOF. We note that the functions defined in Lemma 1 converge to $f_n(t) \to e^{-t}$. By the Monotone Convergence Theorem,

$$\Gamma(s) = \lim_{n\to\infty} \int_0^\infty f_n(t)\, t^{s-1}\, dt.$$

It is sufficient therefore to show that

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1}\, dt = \frac{n!\, n^s}{s(s+1)\cdots(s+n)}.$$

The variable change $u = t/n$ transforms this integral into

$$n^s \int_0^1 u^{s-1}\,(1-u)^n\, du = n^s\, B(s, n+1) = \frac{n^s\,\Gamma(s)\,\Gamma(n+1)}{\Gamma(s+n+1)},$$

which has the advertised value. ■

Next, we recall the basics of *Euler-Maclaurin summation.* Let $B_1(x) = x - \frac{1}{2}$. This is the first *Bernoulli polynomial.* Let $x \to [x]$ be the *greatest integer function.*

Proposition 4. *Let f be a function with continuous first derivative on the interval $[0, n]$. Then*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \tfrac{1}{2} f(0) + \tfrac{1}{2} f(n) + \int_0^n B_1(x - [x])\, f'(x)\,dx. \tag{12}$$

PROOF. Integrating by parts, with $a \in \mathbb{Z}$, we have

$$\int_a^{a+1} B_1(x - [x])\, f'(x)\,dx = \int_a^{a+1} (x - a - \tfrac{1}{2})\, f'(x)\,dx$$

$$= (x - a - \tfrac{1}{2})\, f(x)\big]_a^{a+1} - \int_a^{a+1} f(x)\,dx = \tfrac{1}{2} f(a+1) + \tfrac{1}{2} f(a) - \int_a^{a+1} f(x)\,dx.$$

Summing over $0 \le a < n$ we obtain (12). ■

The function $\Gamma(s)$ is never zero, though it has poles on the negative real axis. Thus there is a well-defined branch of $\log \Gamma(s)$ on the complement of the negative real axis, which is real on the positive real axis.

Lemma 2. *Suppose that s is not on the negative real axis. Then the integral*

$$\int_0^\infty \frac{B_1(x - [x])}{x + s}\,dx \tag{13}$$

is conditionally convergent. Suppose that $\epsilon > 0$ is given. Then there exists a constant C depending on ϵ such that for all $s = re^{i\theta}$ such that $|\theta| < \pi - \epsilon$, the integral (13) is bounded by $C\,|s|^{-1}$.

PROOF. The main property of $B_1(x)$ which we need is that $\int_0^1 B_1(x)\,dx = 0$. Then

$$\int_n^{n+1} \frac{B_1(x - [x])}{x + s}\,dx = \int_n^{n+1} B_1(x - n) \left[\frac{1}{x + s} - \frac{1}{n + s}\right] dx$$

$$= \int_n^{n+1} \frac{B_1(x - n)\,(n - x)}{(x + s)(n + s)}\,dx.$$

Since on the interval $[n, n+1]$ we have $|B_1(x - n)| < \frac{1}{2}$, $|n - x| < 1$, we have

$$\Big|\int_n^{n+1} \frac{B_1(x - [x])}{x + s}\,dx\Big| \le \tfrac{1}{2} \int_n^{n+1} \frac{1}{|(x + s)(n + s)|}\,dx.$$

This is approximately $\frac{1}{2}\,(n+s)^{-2}$, and if s lies in the sector described in the Lemma, it is bounded by an absolute constant times $|n + s|^{-2}$. Thus the integral is of the same order as $\sum_{n=1}^{\infty} (n + s)^{-2}$. It is now easy to see that the integral (13) is bounded by $C\,|s|^{-1}$ in the sector described in the Lemma. ■

Proposition 5. *We have*

$$\log\ \Gamma(s) = -s + \left(s - \tfrac{1}{2}\right) \log(s) + \tfrac{1}{2}\log(2\pi) - \int_0^\infty \frac{B_1(x-[x])}{x+s}\,dx. \tag{14}$$

The remainder term is estimated in Lemma 2.

PROOF. The indefinite integral $\int \log(x)\,dx = x\log x - x$. Thus Proposition 4 implies

$$\sum_{k=0}^{n} \log(s+k)$$
$$= [(x+s)\log(x+s) - (x+s)]_0^n + \tfrac{1}{2}\log(s) + \tfrac{1}{2}\log(n+s) + \int_0^n \frac{B_1(x-[x])}{x+s}\,dx,$$

or (simplifying)

$$\sum_{k=0}^{n} \log(s+k) = (s+n+\tfrac{1}{2})\log(s+n) - (s-\tfrac{1}{2})\log(s) - n + \int_0^n \frac{B_1(x-[x])}{x+s}\,dx. \tag{15}$$

Now since $\lim_{n\to\infty} n/(n+1) = 1$, (2) implies

$$\Gamma(s) = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)^{s-1}}{s(s+1)\cdots(s+n)}, \tag{16}$$

so

$$\log\ \Gamma(s) = \lim_{n\to\infty} \sum_{k=0}^{n} \log(1+k) + (s-1)\log(n+1) - \sum_{k=0}^{n}\log(s+k).$$

Using (15), this can be rewritten

$$\lim_{n\to\infty}(s+n+\tfrac{1}{2})\Big\{\log(n+1) - \log(n+s)\Big\} + (s-\tfrac{1}{2})\log(s)$$
$$+ \int_0^n \frac{B_1(x-[x])}{x+1}\,dx - \int_0^n \frac{B_1(x-[x])}{x+s}\,dx.$$

Since for large n we have $\log(n+1) - \log(n+s) \sim \frac{1-s}{n}$,

$$\lim_{n\to\infty}(s+n+\tfrac{1}{2})\Big\{\log(n+1) - \log(n+s)\Big\} = 1-s,$$

and since the integrals

$$\int_0^\infty \frac{B_1(x-[x])}{x+1}\,dx, \qquad \int_0^\infty \frac{B_1(x-[x])}{x+s}\,dx$$

are conditionally convergent by Lemma 2, we may let $n \to \infty$ to obtain

$$\log\ \Gamma(s) = 1 - s + \left(s - \tfrac{1}{2}\right)\ \log(s) + \int_0^\infty \frac{B_1(x-[x])}{x+1}\,dx - \int_0^\infty \frac{B_1(x-[x])}{x+s}\,dx.$$

We will thus obtain (13) provided we prove

$$\int_0^\infty \frac{B_1(x-[x])}{x+1}\,dx = -1 + \tfrac{1}{2}\log(2\pi). \tag{17}$$

By (16) and (10), we have

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)^{-1/2}}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ldots\cdot\frac{2n+1}{2}} = \lim_{n\to\infty} \frac{2^{2n+1}\left((n+1)!\right)^2(n+1)^{-3/2}}{(2n+1)!}.$$

By (15),

$$\log\ (n+1)! = \left(n + \tfrac{3}{2}\right)\ \log(n+1) - n + \int_0^n \frac{B_1(x-[x])}{x+1}\,dx.$$

Thus

$$\log\ \sqrt{\pi} = \lim_{n\to\infty} \log\left(\frac{2^{2n+1}\,(n+1)^{2n+3/2}}{(2n+1)^{2n+3/2}}\right) + 2\int_0^n \frac{B_1(x-[x])}{x+1}\,dx - \int_0^{2n} \frac{B_1(x-[x])}{x+1}\,dx.$$

Now we note that

$$\log\left(\frac{2^{2n+1}\,(n+1)^{2n+3/2}}{(2n+1)^{2n+3/2}}\right) = -\tfrac{1}{2}\ \log(2) + \log\left(\left(1+\tfrac{1}{n}\right)^{2n+3/2}\right) - \log\left(\left(1+\tfrac{1}{2n}\right)^{2n+3/2}\right),$$

which has $-\frac{1}{2}\ \log(2) + 1$ as its limiting value. Taking the limit, we obtain (17). ■

Proposition 5 is a form of Stirling's formula. We would like to know that the definite integral on the right is small. In fact, it is possible to estimate it effectively by integrating one more time to make the integral absolutely convergent. (As it stands, it is only *conditionally convergent.*) To this end, we introduce the second Bernoulli polynomial $B_2(x) = x^2 - x + \frac{1}{6}$. It has the properties that $B_2' = 2B_1$, $\int_0^1 B_2(x)\,dx = 0$, and $B_2(0) = B_2(1) = \frac{1}{6}$.

Theorem 1. *If s is in the complement of the negative real axis, then*

$$\log\ \Gamma(s) = -s + \left(s - \tfrac{1}{2}\right)\ \log(s) + \tfrac{1}{2}\ \log(2\pi) + \tfrac{1}{12s} - \tfrac{1}{2}\int_0^\infty \frac{B_2(x-[x])}{(x+s)^2}\,dx. \tag{18}$$

The integral

$$\tag{19} \tfrac{1}{2}\int_0^\infty \frac{B_2(x-[x])}{(x+s)^2}\,dx$$

is absolutely convergent, and if $\epsilon > 0$ is given, then for all $s = re^{i\theta}$ such that $|\theta| < \pi - \epsilon$, the remainder (19) is bounded by a constant (depending on ϵ) times $|s|^{-2}$.

Actually, we will see in Theorem 2 that the remainder is actually bounded by a constant times $|s|^{-3}$. However $|s|^{-2}$ is what we can conveniently prove without further integrations by parts.

PROOF. If a is a nonnegative integer, integrating by parts, we have

$$\begin{aligned}\int_a^{a+1} \frac{B_1(x-[x])}{x+s}\,dx &= \int_a^{a+1} \frac{B_1(x-a)}{x+s}\,dx \\ &= \left[\tfrac{1}{2}\frac{B_2(x-a)}{x+s}\right]_a^{a+1} + \tfrac{1}{2}\int_a^{a+1} \frac{B_2(x-a)}{(x+s)^2}\,dx \\ &= \tfrac{1}{12}\left(\frac{1}{a+1+s} - \frac{1}{a+s}\right) + \tfrac{1}{2}\int_a^{a+1} \frac{B_2(x-[x])}{(x+s)^2}\,dx.\end{aligned}$$

Summing over a gives

$$\int_0^\infty \frac{B_1(x-[x])}{x+s}\,dx = -\frac{1}{12s} + \tfrac{1}{2}\int_0^\infty \frac{B_2(x-[x])}{(x+s)^2}\,dx,$$

so (13) implies (18). The estimation of the integral (19) may be accomplished by emulating the proof of Lemma 2—we leave this to the reader, mentioning only that the key point is to use the fact that $\int_0^1 B_2(x)\,dx = 0$. ■

The formula (18) is the beginning of the *asymptotic expansion* (at ∞) for the Gamma function. In order to proceed further, we need the basic properties of the Bernoulli numbers and Bernoulli polynomials. We note that for fixed x, the function $t\,e^{tx}/(e^t-1)$ is analytic in the disk of radius 2π with center at $t=0$, hence has a Taylor expansion. Let $B_n(x)$ be the coefficients in this expansion:

$$\tag{20} \frac{t\,e^{tx}}{e^t-1} = \sum_{n=0}^\infty \frac{B_n(x)\,t^n}{n!}.$$

We define $B_n = B_n(0)$.

Proposition 6. *$B_n(x)$ is a polynomial of degree n. We have $B_n' = n\,B_{n-1}$ and*

$$\int_0^1 B_n(x)\,dx = 0$$

if $n > 0$. If $n \neq 1$, then $B_n(1) = B_n(0) = B_n$. If n is odd and $n > 1$ then $B_n = 0$.

The polynomials $B_n(x)$ are called *Bernoulli polynomials*; B_n are called *Bernoulli numbers.* We note that two distinct conventions are in existence for the Bernoulli numbers. We have

$$B_0 = 1, \qquad B_1 = -\tfrac{1}{2}, \qquad B_2 = \tfrac{1}{6}, \qquad B_3 = 0, \qquad B_4 = -\tfrac{1}{30},$$

$$B_0(x) = 1, \qquad B_1(x) = x - \tfrac{1}{2}, \qquad B_2(x) = x^2 - x + \tfrac{1}{6}, \qquad B_3(x) = x^3 - \tfrac{3x^2}{2} + \tfrac{x}{2}.$$

PROOF. We note that

$$\frac{d}{dx}\frac{t\,e^{tx}}{e^t - 1} = t\,\frac{t\,e^{tx}}{e^t - 1},$$

which implies that $B_n' = n\,B_{n-1}$. Since $B_0(x)$ is the constant term in the taylor expansion of $t\,e^{tx}/(e^t - 1)$ at $t = 0$, we have $B_0(x) = 1$, and the relation $B_n' = n\,B_{n-1}$ now implies that $B_n(x)$ is a polynomial of degree n. Since

$$\int_0^1 \frac{t\,e^{tx}}{e^t - 1}\,dx = 1,$$

we have $\int_0^1 B_n(x)\,dx = 0$ if $n > 0$. Because

$$\sum_{n=0}^{\infty}\left[\frac{B_n(1)}{n!} - \frac{B_n(0)}{n!}\right] = \frac{t\,e^t}{e^t - 1} - \frac{t}{e^t - 1} = t,$$

we have $B_n(1) = B_n(0)$ if $n \neq 1$. We have

$$\sum \frac{B_n\,t^n}{n!} = \frac{t}{e^t - 1},$$

and since

$$\frac{t}{e^t - 1} + \frac{t}{2}$$

is easily checked to be an even function, it follows that all B_n with n odd and > 1 are zero. ∎

Theorem 2. *If s is in the complement of the negative real axis, then*

$$\log\,\Gamma(s) = -s + \left(s - \tfrac{1}{2}\right)\log(s) + \tfrac{1}{2}\log(2\pi) + \sum_{n=1}^{N} \frac{B_{2n}}{2n(2n-1)\,s^{2n-1}} - \tfrac{1}{2N+1}\int_0^\infty \frac{B_{2N+1}(x-[x])}{(x+s)^{2N+1}}\,dx. \tag{21}$$

The integral

$$\tfrac{1}{2}\int_0^\infty \frac{B_{2N+1}(x-[x])}{(x+s)^{2N+1}}\,dx \tag{22}$$

is absolutely convergent, and if $\epsilon > 0$ is given, then for all $s = re^{i\theta}$ such that $|\theta| < \pi - \epsilon$, the remainder (22) is bounded by a constant (depending on ϵ) times $|s|^{-2N-1}$.

PROOF. We obtain

$$\log\,\Gamma(s) = -s + \left(s - \tfrac{1}{2}\right)\log(s) + \tfrac{1}{2}\log(2\pi) + \sum_{n=2}^{M} \frac{B_n}{n(n-1)\,s^{n-1}} - \tfrac{1}{M}\int_0^\infty \frac{B_M(x-[x])}{(x+s)^M}\,dx$$

from Theorem 1 by successive integrations by parts. (Details are left to the reader.) In this expansion, we take $M = 2N+1$. Note that since the odd Bernoulli numbers vanish, we may omit them from the summation. The estimation of the remainder follows the proof of Lemma 2. ∎

As an example, let us show how Stirling's formula would be used to compute the Gamma function numerically. The following routine for Wolfram Mathematica tabulates the terms $B_{2n}/(2n\,(2n-1)\,s^{2n-1})$ in the asymptotic form (21) of Stirling's formula:

```
Terms[s_]:=Table[N[BernoulliB[2n]/(2n*(2n-1)*s^(2n-1)),20],{n,1,10}]
```

The function `N[ ,20]` computes its argument *numerically* to *20 digits.*

```
In[6]:= Terms[1/2]

Out[6]= {0.16666666666666666667, -0.022222222222222222222,
  0.025396825396825396825, -0.076190476190476190476,
  0.43097643097643097643, -3.9270951270951270951, 52.512820512820512821,
  -968.31581699346405229, 23546.347175127360859, -730035.50213716108453}
```

Note that the terms decrease for a while, then begin to get larger. In practice, we can only take two or three terms, and Stirling's formula, applied naively, will only give $\Gamma(\frac{1}{2})$ to very poor accuracy. In order to get better accuracy, we make use of the fact that Stirling's formula is more accurate for larger $|s|$. Thus what we must do is to use the formula $\Gamma(s+1) = s\,\Gamma(s)$ to shift the argument into a region where Stirling's formula is more accurate. To begin with,

$$\log \Gamma(\tfrac{1}{2}) = \log \Gamma(\tfrac{3}{2}) + \log(2).$$

So how well can we compute $\log \Gamma(\frac{3}{2})$?

```
In[11]:= Terms[3/2]

Out[11]= {0.05555555555555555556, -0.00082304526748971193416,
  0.000104513684760598340845, -0.000034837894920199446948,
  0.000021895871105849259586, -0.000022168566936471558057,
  0.000032937378757516834933, -0.000067483594185498871258,
  0.000182331713296097983753, -0.00062811641172404202619}
```

Thus we should be able to take five or six terms and obtain four digits of accuracy to the right of the decimal point. That's still not very good. How about $\Gamma(\frac{5}{2})$?

```
In[12]:= Terms[5/2]

Out[12]= {0.03333333333333333333, -0.00017777777777777777778,
  8.1269841269841269841*10^-6, -(9.7523809523809523810*10^-7),
  2.20659932659932659933*10^-7, -(8.0426908202908202908*10^-8),
  4.3018502564102564103*10^-8, -(3.1729772691241830065*10^-8),
  3.08626681693829344603*10^-8, -(3.8274885334448791069*10^-8)}
```

We see that if we take 8 or 9 terms we should get $\log \Gamma(\frac{1}{2})$ to 7 places right of the decimal point. Here is a Mathematica routine which implements Stirling's formula:

```
Stirling[s_,k_]:=N[-s+(s-1/2)Log[s]+(1/2)Log[2Pi]+
  Sum[BernoulliB[2n]/(2n(2n-1)s^(2n-1)),n,1,k],20]

In[16]:= Stirling[5/2,8]

Out[16]= 0.2846828557763245036

In[17]:= N[Log[Gamma[5/2]],20]

Out[17]= 0.28468287047291915963
```

Of course, we get the same accuracy for $\log \Gamma(\frac{1}{2}) = \log \Gamma(\frac{5}{2}) + \log(4/3)$:

```
In[18]:= N[Stirling[5/2,8]+Log[4/3],20]
```

```
Out[19]= 0.5723649282281054311

In[20]:= N[Log[Gamma[1/2]],20]

Out[20]= 0.57236494292470008707
```

Of course if we needed more accuracy we could work instead with $\Gamma(\frac{7}{2})$ or $\Gamma(\frac{9}{2})$.

In conclusion, Stirling's formula is *not* a convergent series for $\Gamma(s)$. Instead, it is an *asymptotic expansion*. The higher terms are significant for large values of s. Still, Stirling's formula can be used to compute $\Gamma(s)$ to any degree of accuracy, by using the formula $\Gamma(s+1) = s\,\Gamma(s)$ to shift the argument into a region where it is highly accurate.

Often we want to estimate the Gamma function along *vertical lines*. For this, the following statment is convenient:

Proposition 7. *Let $\epsilon > 0$ be given, and let a be a complex number. Then*

$$\log \Gamma(s+a) = (s + a - \tfrac{1}{2}) \log(s) - s + R(s),$$

where $R(s)$ is bounded in the region where the arguments of s and $s+a$ are both less than $\pi - \epsilon$ in absolute value.

PROOF. This is an easy deduction from Proposition 5 and Lemma 2. ■