On Fourier and Zeta(s)

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Abstract

We study some of the interactions between the Fourier Transform and the Riemann zeta function (and Dirichlet-Dedekind-Hecke-Tate L-functions).

Contents

1 Introduction

The zeta function $\zeta(s)$ assumes in Riemann's paper quite a number of distinct identities: it appears as a Dirichlet series, as an Euler product, as an integral transform, as an Hadamard product[1]… We retain three such identities and use them as symbolic vertices for a triangle:

$$\begin{tikzcd}[arrows=dash] & \sum_n \frac{1}{n^s} \arrow[dl] \arrow[dr] & \\ \prod_\rho (1 - \frac{s}{\rho}) \arrow[rr] & & \prod_p \frac{1}{1-p^{-s}} \end{tikzcd}$$

These formulae stand for various aspects of the zeta function (in the bottom left corner we omit the poles and Gamma factor for easier reading; also, as is known, some care is necessary to make the product converge, for example pairing ρ with $1-\rho$ is enough.) For the purposes of this manuscript, we may tentatively name these various aspects as follows:

$$\begin{tikzcd}[arrows=dash] & \text{summations} \arrow[dl] \arrow[dr] & \\ \text{zeros} \arrow[rr] & & \text{primes} \end{tikzcd}$$

A reading, even casual, of Riemann's paper reveals how much Fourier analysis lies at its heart, on a par with the theory of functions of the complex variable. Let us enhance appropriately the triangle:

$$\begin{tikzcd}[arrows=dash] & \sum_n \frac{1}{n^s} \arrow[dl, "\text{Fourier}"'] \arrow[dr, "\text{Fourier}"] & \\ \prod_\rho (1 - \frac{s}{\rho}) \arrow[rr, "\text{Fourier}"] & & \prod_p \frac{1}{1-p^{-s}} \end{tikzcd}$$

Indeed, each of the three edges is an arena of interaction between the Fourier Transform, in various incarnations, and the Zeta function (and Dirichlet L-series, or even more general number theoretical zeta functions.) We specialize to a narrower triangle:

$$\begin{tikzcd}[arrows=dash] & \begin{array}{c} \sum_n \frac{1}{n^s} \\ \text{summations} \end{array} \arrow[dl, "\begin{array}{c}\text{Hilbert spaces}\\ \text{and vectors}\end{array}"'] \arrow[dr, "\begin{array}{c}\text{Adeles}\\ \text{and Ideles}\end{array}"] \arrow[d, phantom, "?"] & \\ \begin{array}{c} \prod_\rho (1 - \frac{s}{\rho}) \\ \text{zeros} \end{array} \arrow[rr, "\text{Explicit Formulae}"] & {} & \begin{array}{c} \prod_p \frac{1}{1-p^{-s}} \\ \text{primes} \end{array} \end{tikzcd}$$

The big question mark serves as a remainder that we are missing the 2-cell (or 2-cells) which would presumably be there if the nature of the Riemann zeta function was really understood.

[1] Riemann explains how $\log \zeta(s)$ may be written as an infinite sum involving the zeros.

It is to be expected that some of the spirit of the well-known ideas from the theory of function fields will at some point be incorporated into the strengthening of the 1-cells, but we shall not discuss this here. These ideas are also perhaps relevant to adding a 2-cell, or rather even a 3-cell, and we have only contributed to some basic aspects of the down-to-earth 1-cells.

We first survey our work [11] [12] [15] on the "Explicit Formulae" and the local scale invariant conductor operators $\log|x|_\nu + \log|y|_\nu$. Following from local to global the idea of multiplicatively analysing the additive Fourier Transform we briefly discuss our work on adeles, ideles, scattering and causality [13] [14].

We then proceed to new material with a reexamination of the Poisson summation formula on adeles and an exposé of the basic properties of a related but subtly distinct co-Poisson summation. The shift from Poisson to co-Poisson is related to our contribution [16] [17] to the criterion of Nyman-Beurling and to the Theorem of Báez-Duarte, Balazard, Landreau and Saias [2], and we give a more detailed study of the aspects of this contribution which are related to the so-called Hilbert-Pólya idea.

The spirit of the co-Poisson summation leads to a simpler approximation of the so-called Hilbert-Pólya idea, and we expand upon our Note [18] on the construction of Hilbert Spaces HP_Λ and Hilbert vectors $Z^\Lambda_{\rho,k}$ associated with the non-trivial zeros of the Riemann zeta function. This takes place within the context of a multiplicative spectral (scattering) analysis of the Fourier cosine transform, a special instance (about which much remains unknown) of the de Branges Theory of Hilbert Spaces of entire functions [7].

We explain how this applies to Dirichlet L-series, and conclude with speculations on the nature of the zeta function, the GUE hypothesis, and the Riemann Hypothesis.

2 On the Explicit Formulae and $\log|x| + \log|y|$

Riemann discovered the zeros and originated the idea of counting the primes and prime powers (suitably weighted) using them. Indeed this was the main focus of his famous paper. Later a particularly elegant formula was rigorously proven by von Mangoldt:

$$\sum_{1<n<X} \Lambda(n) + \frac{1}{2}\Lambda(X) = X - \sum_\rho \frac{X^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2})$$

Here $X > 1$ (not necessarily an integer) and $\Lambda(Y) = \log(p)$ if $Y > 1$ is a positive power of the prime number p, and is 0 for all other values of Y. The ρ's are the Riemann Zeros (in the critical strip), the sum over them is not absolutely convergent, even after pairing ρ with $1-\rho$. It is defined as $\lim_{T\to\infty} \sum_{|\mathrm{Im}(\rho)|<T} X^\rho/\rho$.

In the early fifties Weil published a paper [32] on this topic, and then another one [33] in the early seventies which considered non-abelian Artin (and Artin-Weil) L-functions. While elucidating already in his first paper new algebraic structure, he did this maintaining a level of generality encompassing in its scope the von Mangoldt formula (although it requires some steps to deduce this formula from the Weil explicit formula.) The analytical difficulties arising are an expression of the usual difficulties with Fourier inversion. The "test-function flavor" of the "Riemann-Weil explicit formula" had been anticipated by Guinand [24] (among other authors too, perhaps.)

So in our opinion a more radical innovation was Weil's discovery that the local terms of the Explicit Formulae acquire a natural expression on the ν-adics, and that this enables to put the real and complex places on a par with the finite places (clearly Weil was motivated by analogies with function fields, we do not discuss that here.) A lot of algebraic number theory [34] is necessary in Weil's second paper to establish this for Artin-Weil L-functions and is related to Weil's questions about a missing "class field theory at infinity" (on this topic, Connes [20] has argued about the relevance of the classification of factors for von Neumann algebras).

We stay here at the much simpler level of Weil's first paper and show how to put all places of the number field at the same level. It had first appeared in Haran's work [25] that it was possible to formulate the Weil's local terms in a more unified manner than had originally been done by Weil. We show that an operator theoretical approach allows, not only to formulate, but also to deduce the local terms in a unified manner.

The starting point is Tate's Thesis [31]. Let K be a number field and K_ν one of its completions. Let $\chi_\nu : K_\nu^\times \to S^1$ be a (unitary) multiplicative character. For $0 < \mathrm{Re}(s) < 1$ both $\chi_\nu(x)|x|^{s-1}$ and $\chi_\nu(x)^{-1}|x|^{-s}$ are tempered distributions on the additive group K_ν and the *Tate's functional equations* are the identities of distributions:

$$\mathcal{F}_\nu(\chi_\nu(x)|x|^{s-1}) = \Gamma(\chi_\nu, s)\chi_\nu(x)^{-1}|x|^{-s}$$

for certain functions $\Gamma(\chi_\nu, s)$ analytic in $0 < \mathrm{Re}(s) < 1$, and meromorphic in the complex plane. This is the local half of Tate's Thesis, from the point of view of distributions. See also [23]. Implicit in this equation is a certain normalized choice of additive Haar measure on K_ν, and $\mathcal{F}_\nu$ is the corresponding additive Fourier transform.

Let us view this from a Hilbert space perspective. The quasi-characters $\chi_\nu(x)^{-1}|x|^{-s}$ are never square-integrable, but for $\mathrm{Re}(s) = 1/2$ they are the generalized eigenvectors arising in the spectral analysis of the unitary group of dilations (and contractions): $\phi(x) \mapsto \phi(x/t)/\sqrt{|t|_\nu}$, $x \in K_\nu$, $t \in K_\nu^\times$. Let I_ν be the unitary operator $\phi(x) \mapsto \phi(1/x)/|x|_\nu$, and let $\Gamma_\nu = \mathcal{F}_\nu \cdot I_\nu$. Then:

$$\Gamma_\nu(\chi_\nu(x)^{-1}|x|^{-s}) = \Gamma(\chi_\nu, s)\chi_\nu(x)^{-1}|x|^{-s}$$

and this says that the $\chi_\nu(x)^{-1}|x|_\nu^{-s}$, for $\mathrm{Re}(s) = 1/2$, are the generalized eigenvectors arising in the spectral analysis of the unitary scale invariant operator $\Gamma_\nu = \mathcal{F}_\nu \cdot I_\nu$.

The question [11] which leads from Tate's Thesis (where the zeros do not occur at all) to the topic of the Explicit Formulae is: *what happens if we take the derivative with respect to s in Tate's functional equations?* Proceeding formally we obtain:

$$-\Gamma_\nu(\log|x|_\nu\, \chi_\nu(x)^{-1}|x|_\nu^{-s}) = \Gamma'(\chi_\nu, s) \cdot \chi_\nu(x)^{-1}|x|_\nu^{-s} - \Gamma(\chi_\nu, s)\log|x|_\nu \cdot \chi_\nu(x)^{-1}|x|_\nu^{-s}$$

$$\log(|x|_\nu)\cdot\chi_\nu(x)^{-1}|x|_\nu^{-s} - \Gamma_\nu\left(\log|x|_\nu \frac{\chi_\nu(x)^{-1}|x|_\nu^{-s}}{\Gamma(\chi_\nu,s)}\right) = \left(\frac{d}{ds}\log\Gamma(\chi_\nu,s)\right)\cdot\chi_\nu(x)^{-1}|x|_\nu^{-s}$$

$$\left(\log|x|_\nu - \Gamma_\nu\cdot\log|x|_\nu\cdot\Gamma_\nu^{-1}\right)\cdot(\chi_\nu(x)^{-1}|x|_\nu^{-s}) = \left(\frac{d}{ds}\log\Gamma(\chi_\nu,s)\right)\cdot\chi_\nu(x)^{-1}|x|_\nu^{-s}$$

Let H_ν be the scale invariant operator $\log|x|_\nu - \Gamma_\nu\cdot\log|x|_\nu\cdot\Gamma_\nu^{-1} = \log|x|_\nu + \mathcal{F}_\nu\cdot\log|x|_\nu\cdot\mathcal{F}_\nu^{-1}$, which we also write symbolically as:

$$H_\nu = \log|x|_\nu + \log|y|_\nu$$

then we see that the conclusion is:

2.1. Theorem ([11] [12]). *The generalized eigenvalues of the* conductor operator H_ν *are the logarithmic derivatives of the Tate Gamma functions:*

$$H_\nu(\chi_\nu(x)^{-1}|x|_\nu^{-s}) = \left(\frac{d}{ds}\log\Gamma(\chi_\nu,s)\right)\cdot\chi_\nu(x)^{-1}|x|_\nu^{-s}$$

Let $g(u)$ be a smooth function with compact support in $\mathbb{R}_+^\times$. Let $\widehat{g}(s) = \int g(u)u^{s-1}\,du$ be its Mellin transform. Let χ be a unitary character on the idele class group of the number field K, with local components χ_ν. Let $Z(g,\chi)$ be the sum of the values of $\widehat{g}(s)$ at the zeros of the (completed) Hecke L-function $L(\chi,s)$ of χ, minus the contribution of the poles when χ is a principal character ($t\mapsto|t|^{-i\tau}$).

Using the calculus of residues we obtain $Z(g,\chi)$ as the integral of $\widehat{g}(s)(d/ds)\log L(\chi,s)$ around the contour of the infinite rectangle $-1\leq\mathrm{Re}(s)\leq 2$. It turns out that the compatibility between Tate's Thesis (local half) and Tate's Thesis (global half) allows to use the functional equation without having ever to write down explicitely all its details (such as the discriminant of the number field and the conductor of the character), and leads to:

2.2. Theorem ([11]). *The explicit formula is given by the logarithmic derivatives of the Tate Gamma functions:*

$$Z(g,\chi) = \sum_\nu \int_{\mathrm{Re}(s)=\frac{1}{2}} \left(\frac{d}{ds}\log\Gamma(\chi_\nu,s)\right)\widehat{g}(s)\,\frac{|ds|}{2\pi}$$

At an archimedean place the values $\widehat{g}(s)$ on the critical line give the multiplicative spectral decomposition of the function $g_{\chi,\nu} := x\mapsto\chi_\nu(x)^{-1}g(|x|_\nu)$ on (the additive group) K_ν, and, after checking normalization details, one finds that the local term has exact value $H_\nu(g_{\chi,\nu})(1)$. At a non-archimedean place, one replaces the integral on the full critical line with an integral on an interval of periodicity of the Tate Gamma function, and applying Poisson summation (in the vertical direction) to $\widehat{g}(s)$ to make it periodical as well it is seen to transmute into the multiplicative spectral decomposition of the function $g_{\chi,\nu} := x\mapsto\chi_\nu(x)^{-1}g(|x|_\nu)$ on K_ν! So we jump directly from the critical line to the completions of the number field K, and end up with the following version of the explicit formula:

2.3. Theorem ([11] [12]). *Let at each place ν of the number field K:*

$$g_{\chi,\nu} = x \mapsto \chi_\nu(x)^{-1} g(|x|_\nu)$$

on K_ν ($g_{\chi,\nu}(0) = 0$). Then

$$Z(g,\chi) = \sum_\nu H_\nu(g_{\chi,\nu})(1)$$

where H_ν is the scale invariant operator $\log|x|_\nu + \log|y|_\nu$ acting on $L^2(K_\nu, dx_\nu)$.

As we evaluate at 1, the "$\log|x|_\nu$" half of H_ν could be dropped, and we could sum up the situation as follows: *Weil's local term is the (additive) Fourier transform of the logarithm!* This is what Haran had proved ([25], in the case of the Riemann zeta function), except that he formulated this in terms of Riesz potentials $|y|_\nu^{-s}$, and did a case-by-case check that Weil local terms may indeed be written in this way. The explicit formula as stated above with the help of the operator $\log|x|_\nu + \log|y|_\nu$ incorporates in a more visible manner the compatibility with the functional equations. Indeed we have

2.4. Theorem ([11] [12]). *The conductor operator H_ν commutes with the operator I_ν:*

$$H_\nu \cdot I_\nu = I_\nu \cdot H_\nu$$

or equivalently as $I_\nu \cdot \log|x|_\nu \cdot I_\nu = -\log|x|_\nu$:

$$I_\nu \cdot \log|y|_\nu \cdot I_\nu = 2\log|x|_\nu + \log|y|_\nu$$

To see abstractly why this had to be true, one way is to observe that H_ν and Γ_ν are simultaneously diagonalized by the multiplicative characters, hence they commute. But obviously H_ν commutes with $\mathcal{F}_\nu$ so it has to commute with I_ν.

Let us suppose χ_ν to be ramified (which means not trivial when restricted to the ν-adic units) and let $f(\chi_\nu)$ be its *conductor exponent*, e_ν the number field *differental exponent* at ν, and q_ν the cardinality of the residue field. In Tate's Thesis [31], one finds for a ramified character

$$\Gamma(\chi_\nu, s) = w(\chi_\nu) q_\nu^{(f(\chi_\nu)+e_\nu)s}$$

where $w(\chi_\nu)$ is a certain non-vanishing complex number, quite important in Algebraic Number Theory, but not for us here. Indeed we take the logarithmic derivative and find:

$$\frac{d}{ds}\log\Gamma(\chi_\nu, s) = (f(\chi_\nu) + e_\nu)\log q_\nu$$

So that:

$$H_\nu(\chi_\nu^{-1}(x)\mathbf{1}_{|x|_\nu=1}(x)) = (f(\chi_\nu) + e_\nu)\log(q_\nu)\chi_\nu^{-1}(x)\mathbf{1}_{|x|_\nu=1}(x)$$

which says that ramified characters are eigenvectors of H_ν with eigenvalues $(f(\chi_\nu) + e_\nu)\log q_\nu$. Hence the name "conductor operator" for H_ν. We note that this contribution of the differental exponent is there also for a non-ramified character and explains why in our version of the Explicit Formula there is no explicit presence of the discriminant of the number field.

If we now go through the computation of the distribution theoretic additive Fourier transform of $\log|x|_\nu$ and compare with the above we end up with a proof [11] of the well-known Weil integral formula [32] [33] [34] (Weil writes $d^\times t$ for $\log(q_\nu)d^*t$):

$$f(\chi_\nu)\log q_\nu = \int_{K_\nu^\times} \mathbf{1}_{|t|_\nu=1}(t)\frac{1-\chi_\nu(t)}{|1-t|_\nu}\,d^\times t$$

In Weil's paper [32] we see that this formula's rôle has been somewhat understated. Clearly it was very important to Weil as it confirmed that it was possible to express similarly all contributions to the Explicit Formula: from the infinite places, from finite unramified places, and from finite ramified places. Weil leaves establishing the formula to the attentive reader. In his second paper [33] he goes on to extend the scope to Artin L-function, and this is far from an obvious thing. One could say that we have replaced Weil's explicit evaluation of an integral by Tate's explicit evaluation of another integral, however the understanding provided by the conductor operator $\log|x|_\nu + \log|y|_\nu$ of the character's logarithmic conductor as an eigenvalue, is to us sufficient credence for its relevance.

An operator theoretical interpretation is also available [11] for the higher derivatives of the logarithms of the Gamma functions: they are the eigenvalues of the commutators one can build with $\log|x|_\nu$ and $\log|y|_\nu$. All these commutators are bounded Hilbert space operators. The conductor operators H_ν are not bounded. They are self-adjoint and bounded on the left, which seems to go in the direction of providing some positivity to the Explicit Formula. As is well-known Weil formulated the Riemann Hypothesis as the positive type property of the distribution for which the explicit formula gives two expressions (the poles have to shifted to the same side as the primes for this), and using the unbounded positive half of the spectrum of H_ν when ν is archimedean we have checked [12] the positivity when the test function $g(u)$ has its support restricted to a small interval $[1/c, c]$, $c > 1$.

3 On adeles, ideles, scattering and causality

We go anti-clockwise around the triangle from the Introduction and consider the zeta and L-functions from the point of view of Adeles and Ideles. Again a major input is Tate's Thesis. There the functional equations of the abelian L-functions are established in a unified manner, but the zeros do not appear at all. It is only recently that progress on this arose, in the work of Connes [20]. We have examined this question anew [13] [14], from the point of view of the study of the interaction between the additive and multiplicative Fourier Transforms [11] [13], which as we saw in the last section is a mechanism underlying the operator theoretic approach to the explicit formula.

It turned out that another ingredient involved (alongside Tate's Thesis and the impulse from

Connes's work) is the theme originating with Nyman [29] and Beurling [5] of the Riemann Hypothesis as a Hilbert Space closure property. This theme, which we will discuss more at our next stop around the triangle, is an outgrowth of the foundational work of Beurling [4] and Lax [27] on the invariant subspaces of Hardy spaces. The Beurling-Lax theory of invariant subspaces is central to the approach to (classical, acoustical) scattering due to Lax and Phillips [28]. It turns out that the scattering framework allows a natural formulation of the Riemann Hypothesis, simultaneous for all L-functions, as a property of *causality* [14].

A key theorem from the global half of Tate's Thesis is the following:

$$\sum_{q\in K}\mathcal{F}(\varphi)(qv)=\frac{1}{|v|}\sum_{q\in K}\varphi(\frac{q}{v})$$

This was called the "Riemann-Roch Theorem" by Tate, but we prefer to call it the *Poisson-Tate formula* (which sounds less definitive, and more to the point). Let us explain the notations: K is a number field, $\varphi(x)$ is a function on the adeles $\mathbb{A}$ of K (satisfying suitable conditions), $q\in K$ is diagonally considered as an adele, $v\in\mathbb{A}^\times$ is an idele and $|v|$ is its module. Finally $\mathcal{F}$ is the adelic additive Fourier Transform (we refer to [31] for the details of the normalizations). We note that it does not matter if we exchange the φ on the right with the $\mathcal{F}(\varphi)$ on the left as $\mathcal{F}(\mathcal{F}(\varphi))(x)=\varphi(-x)$ and $-1\in K^\times$.

Let us write E_0 (later this is dropped in favor of a related E) for the map which to the function $\varphi(x)$ on the *adeles* associates the function $\sum_{q\in K}\varphi(qv)\sqrt{|v|}$ on the *ideles* or even on the *idele class group* $\mathcal{C}_K$ (ideles quotiented by $K^\times$). This map E_0 plays an important rôle in the papers of Connes [19] [20] (where it is used under the additional assumption $\varphi(0)=0=\mathcal{F}(\varphi)(0)$.)

The Poisson-Tate formula tells us that E_0 intertwines the additive Fourier transform with the operator $I: g(v)\mapsto g(1/v)$.

$$(E_0\cdot\mathcal{F})(\varphi)=(I\cdot E_0)(\varphi)$$

Some conditions on φ have to be imposed in order for this to make sense. A suitable class of functions stable under $\mathcal{F}$ for which this works is given by the Bruhat-Schwartz functions: finite linear combinations of infinite product of local factors, almost all of them being the indicator function of the local ring of integers, in the Schwartz class for the infinite places, locally constant with compact support at each finite place. To each such function and unitary character χ on the idele class group Tate associates an L-function $L(\chi,\varphi)(s)=\int_{\text{ideles}}\varphi(v)\chi(v)|v|^s d^*v$, and shows how to choose φ so that this coincides exactly with the complete Hecke L-function with grossencharakter χ.

Let us ([14]) manipulate a little bit the Poisson-Tate formula into:

$$\sqrt{|v|}\sum_{q\in K^\times}\mathcal{F}(\varphi)(qv)-\frac{1}{\sqrt{|v|}}\varphi(0)=\frac{1}{\sqrt{|v|}}\sum_{q\in K^\times}\varphi(\frac{q}{v})-\sqrt{|v|}\int_{\text{adeles}}\varphi(x)\,dx$$

If we write E for the map which to $\varphi(x)$ associates $u\mapsto\sqrt{|v|}\sum_{q\in K^\times}\varphi(qv)-\int\varphi(x)dx/\sqrt{|u|}$ on the idele class group (v in the class u), we still have the intertwining property

$$(E\cdot\mathcal{F})(\varphi)=(I\cdot E)(\varphi)$$

3.1. Note. Let $\phi(x)$ be an even Schwartz function on $\mathbb{R}$. Let $F(x) = \sum_{n\geq 1} \phi(nx)$. We have $\lim_{x\to 0} xF(x) = \int_0^\infty \phi(x)\,dx$. For $\mathrm{Re}(s) > 1$ we may intervert the integral with the summation and this gives $\int_0^\infty F(x)x^{s-1}\,dx = \zeta(s)\int_0^\infty \phi(x)x^{s-1}\,dx$. The analytic continuation of this formula to the critical strip [14] requires a modification of the left hand side (for $0 < \mathrm{Re}(s) < 1$):

$$\int_0^\infty \left(F(x) - \frac{\int_0^\infty \phi(y)\,dy}{x}\right) x^{s-1}\,dx = \zeta(s)\int_0^\infty \phi(x)x^{s-1}\,dx$$

So it is in truth not the original Poisson summation but the modified Poisson (where one takes out $\phi(0)$ and replaces it with $-(\int_{\mathbb{R}} \phi(y)\,dy)/|x|)$) which corresponds to $\zeta(s)$ as multiplier. Once we are in the critical strip we are allowed to exchange s with $1-s$: more on this later. What matters now is that the modified Poisson contrarily to the original Poisson keeps us in a Hilbert space setting.

3.2. Theorem ([14]). *For φ a Bruhat-Schwartz function $E(\varphi)$ is square-integrable on the idele class group $\mathcal{C}_K$ (for the multiplicative Haar measure d^*u), and its unitary multiplicative Fourier transform, as a function of the unitary characters, coincides up to an overall constant with the Tate L-function on the critical line* $\mathrm{Re}(s) = 1/2$. *The functions $E(\varphi)$ are dense in $L^2(\mathcal{C}_K, d^*u)$ and $E(\mathcal{F}(\varphi)) = I(E(\varphi))$.*

Connes [19] [20] had already considered the functions $\sqrt{|v|}\sum_{q\in K^\times} \varphi(qv)$ with $\varphi(0) = 0 = \mathcal{F}(\varphi)(0)$ and he had shown that they are dense in $L^2(\mathcal{C}_K, d^*u)$.

Let $\mathcal{S}_1$ be the set of Bruhat-Schwartz functions $\varphi(x)$ which are supported in a parallelepiped $P(v) = \{\forall\nu\ |x|_\nu \leq |v|_\nu\}$ with $|v| \leq 1$. Let $\mathcal{D}_+ = E(\mathcal{S}_1)^\perp$ and let $\mathcal{D}_- = E(\mathcal{F}(\mathcal{S}_1))^\perp$. The following holds:

3.3. Theorem ([14]). *The subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ of the Hilbert space of square-integrable functions on the idele class group are outgoing and incoming subspaces for a Lax-Phillips scattering system, where the idele class group plays the rôle of time. The Riemann Hypothesis for all abelian L-functions of K holds if and only if the causality axiom $\mathcal{D}_+ \perp \mathcal{D}_-$ is satisfied.*

In the purely local case *at a non-archimedean place* a related but simpler scattering set-up for the multiplicative analysis of the additive Fourier transform may be developed, and the causality holds [13]. The situation is more subtle at an *archimedean place* and the next stop on the triangle from the Introduction will provide keys for this. In fact it is to be expected that a renewed consideration of the material of this section will come from the lessons we are currently learning from this next stop.

4 On Poisson, Tate, and co-Poisson

This section is devoted to some new material, on Adeles and Ideles, and also on $\mathbb{R}$. We have already mentioned the Tate L-functions:

$$L(\chi,\varphi)(s) = \int_{\text{ideles}} \varphi(v)\chi(v)|v|^s d^*v$$

The integral is (generally speaking) absolutely convergent only for $\mathrm{Re}(s) > 1$. Using the Poisson-Tate summation formula, Tate established the analytic continuation and the functional equations:

$$L(\chi,\mathcal{F}(\varphi))(s) = L(\chi^{-1},\varphi)(1-s)$$

This follows from an integral representation

$$L(\chi,\varphi)(s) = C\delta_\chi\left(\frac{\mathcal{F}(\varphi)(0)}{s-1-i\tau} - \frac{\varphi(0)}{s-i\tau}\right) + \int_{|v|\geq 1}\left(\varphi(v)\chi(v)|v|^s + \mathcal{F}(\varphi)(v)\chi(v)^{-1}|v|^{1-s}\right) d^*v$$

where C is a certain constant associated to the number field K (and relating the Haar measures d^*v on $\mathbb{A}^\times$ and d^*u on $\mathcal{C}_K$), and the Kronecker symbol δ_χ is 1 or 0 according to whether $\chi(v) = |v|^{-i\tau}$ for a certain $\tau \in \mathbb{R}$ (principal unitary character) or not (ramified unitary character). The integral over the ideles (this is not an integral over the idele *classes*) with $|v| \geq 1$ is absolutely convergent for *all* $s \in \mathbb{C}$.

In terms of the distributions on the adeles: $\Delta(\chi,s) : \varphi \mapsto L(\chi,\varphi)(s)$, the functional equations are:

$$\mathcal{F}(\Delta(\chi,s)) = \Delta(\chi^{-1},1-s)$$

This is (using explicitely the language of distributions) Tate's version of the functional equations for the abelian L-functions of a number field.

As a distribution (on the adeles) valued function of s, $\Delta(\chi,\cdot)$ has exactly the same zeros and poles as the completed Hecke L-function of the grossencharakter χ. We attempt to formulate things as much as possible in an Hilbert space setting and now discuss some new developments related to this. Let $\varphi(x)$ be a Bruhat-Schwartz function on the adeles. We recall that we associated to it the square-integrable function on the idele class group $\mathcal{C}_K$ (with $u \in \mathcal{C}_K$ the class of $v \in \mathbb{A}^\times$):

$$E(\varphi)(u) = \sqrt{|u|}\sum_{q\in K^\times}\varphi(qv) - \frac{\int_{\text{adeles}}\varphi(x)\,dx}{\sqrt{|u|}}$$

The precise relation [14] to the Tate L-functions is:

$$L(\chi,\varphi)(s) = C\int_{\mathcal{C}_K} E(\varphi)(u)\chi(u)|u|^{s-1/2}\,d^*u$$

This integral representation is absolutely convergent for $0 < \mathrm{Re}(s) < 1$ and we read the functional equations directly from it and from the intertwining property $E\cdot\mathcal{F} = I\cdot E$.

If we picture $\alpha = (\chi, s)$ as corresponding to the quasi-character $\chi(u)^{-1}|u|^{-(s-1/2)}$ on the idele class group, unitary for $\mathrm{Re}(s) = 1/2$, then we may formally Fourier transform $\Delta(\chi, s)$ from (χ, s) to $u \in \mathcal{C}_K$ and obtain a function of u (which takes its values in the distributions on the adeles):

$$\Delta(u) = \int \Delta(\chi, 1/2 + i\tau)\chi(u)^{-1}|u|^{-i\tau} d\alpha$$

We note the formal functional equation $\mathcal{F}(\Delta(u)) = \Delta(1/u)$. It is not obvious at first from the expression for $\Delta(u)$ that it allows to make sense of it pointwise, but at least when applied to a test-function φ on the adeles this should define $\Delta(u)(\varphi)$ as a distribution in u, characterized by its Mellin Transform:

$$\Delta(\chi, 1/2 + i\tau)(\varphi) = \int_{\mathcal{C}_K} \Delta(u)(\varphi)\chi(u)|u|^{i\tau} \, d^*u$$

From the formula above we get

$$\Delta(u)(\varphi) = C \cdot E(\varphi)(u)$$

so that $\Delta(u)$ is in fact the *bona fide* function of u with values in the distribution on the adeles:

$$\Delta(u)(\varphi) = C \cdot \left(\sqrt{|u|} \sum_{q \in K^\times} \varphi(qv) - \frac{\int_{\text{adeles}} \varphi(x)\, dx}{\sqrt{|u|}} \right) = C \cdot E(\varphi)(u)$$

At this stage a complete identification of the modified Poisson-Tate summation with the Hecke-Tate L-functions has been obtained: viewed as functions on the idele class group (resp. the group of unitary Hecke characters) with values in the distributions on the adeles, they are a pair of "Fourier-Mellin" transforms.

From the analysis of our recent works [17] [18] it has emerged that it may be relevant to look at $\Delta(u)$ not as a *function* in u (which it is from the formula above) but as a *distribution* in u (so that Δ is a distribution with values in distributions...) It will take us quite some time to explain that it is not a tautological thing. Basically we shift the emphasis from the Poisson-Tate summation [31] [20] [14] which goes from adeles to ideles, to the *co-Poisson summation* which goes from ideles to adeles. It is only with our discussion of the Nyman-Beurling criterion and of the so-called Hilbert-Pólya idea in the next section that the relevance of this shift will be really apparent. The Poisson-Tate summation is a function with values in distributions, whereas the co-Poisson-Tate summation is a distribution whose values we try to represent as L^2-functions.

Let $g(v)$ be a compact Bruhat-Schwartz function on the idele group $\mathbb{A}^\times$. This is a finite linear combination of infinite products $v \mapsto \prod_\nu g_\nu(v_\nu)$, where almost each component is the indicator function of the ν-adic units, the component $g_\nu(v_\nu)$ at an infinite place is a smooth compactly supported function on $K_\nu^\times$, and the components at finite places are locally constant compactly supported.

4.1. Definition. *The* co-Poisson summation *is the map E' which assigns to each compact Bruhat-Schwartz function $g(v)$ the distribution on the* adeles *given by:*

$$E'(g)(\varphi) = \int_{\mathbb{A}^\times} \varphi(v) \sum_{q \in K^\times} g(qv)\sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} g(v)|v|^{-1/2} d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

4.2. Note. Clearly $E'(g)$ depends on $g(v)$ only through the function $R(g)$ on $\mathcal{C}_K$ given by $R(g)(u) = \sum_{q\in K^\times} g(qv)$, with $u \in \mathcal{C}_K$ the class of $v \in \mathbb{A}^\times$. However, for various reasons (among them avoiding the annoying constant C in all our formulae), it is better to keep the flexibility provided by g. The function $R(g)$ has compact support. To illustrate this with an example, and explain why the integral above makes sense, we take $K = \mathbb{Q}$, $g(v) = \prod_p \mathbf{1}_{|v_p|_p=1}(v_p) \cdot g_\infty(v_\infty)$. Then $\sum_{q\in\mathbb{Q}^\times} g(qv) = g_\infty(|v|) + g_\infty(-|v|)$. We may bound this from above by a multiple of $|v|$ (as g_∞ has compact support in $\mathbb{R}^\times$), and the integral $\int_{\mathbb{A}^\times} \varphi(v)|v|^{3/2}\,d^*v$ converges absolutely as $1 < 3/2$ (we may take $\varphi(x)$ itself to be an infinite product here.)

4.3. Theorem. *The co-Poisson summation intertwines the operator $I : g(v) \mapsto g(1/v)$ with the additive adelic Fourier Transform $\mathcal{F}$:*

$$\mathcal{F}(E'(g)) = E'(I(g))$$

Furthermore it intertwines between the multiplicative translations $g(v) \mapsto g(v/w)$ on ideles and the multiplicative translations on adelic distributions $D(x) \mapsto D(x/w)/\sqrt{|w|}$. And the distribution $E'(g)$ is invariant under the action of the multiplicative group $K^\times$ on the adeles.

Proof. We have:

$$\begin{aligned}
E'(g)(\varphi) &= C\int_{\mathcal{C}_K} \sum_{q\in K^\times} \varphi(qv)R(g)(u)\sqrt{|u|}\,d^*u - \int_{\mathbb{A}^\times} g(v)|v|^{-1/2}d^*v \int_{\mathbb{A}} \varphi(x)\,dx \\
&= C\int_{\mathcal{C}_K} \left(\frac{E(\varphi)(u)}{\sqrt{|u|}} + \frac{\int\varphi(x)\,dx}{|u|}\right) R(g)(u)\sqrt{|u|}\,d^*u - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}}\,d^*v \int_{\mathbb{A}} \varphi(x)\,dx \\
&= C\int_{\mathcal{C}_K} E(\varphi)(u)R(g)(u)\,d^*u + \left(C\int_{\mathcal{C}_K} \frac{R(g)(u)}{\sqrt{|u|}}\,d^*u - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v\right)\int \varphi(x)\,dx \\
&= C\int_{\mathcal{C}_K} E(\varphi)(u)R(g)(u)\,d^*u
\end{aligned}$$

Using the intertwinings $E \cdot \mathcal{F} = I \cdot E$ and $R(I(g))(u) = R(g)(1/u)$ we get:

$$E'(g)(\mathcal{F}(\varphi)) = C\int_{\mathcal{C}_K} E(\varphi)(u)R(g)(\frac{1}{u})\,d^*u = E'(I(g))(\varphi)$$

which completes the proof of $\mathcal{F}(E'(g)) = E'(I(g))$. The compatibility with multiplicative translations is easy, and the invariance under the multiplication $x \mapsto qx$ follows. ∎

4.4. Note. The way the ideles have to act on the distributions on adeles for the intertwining suggests some Hilbert space properties of the distribution $E'(g)$ (more on this later).

4.5. Note. The invariance under $K^\times$ suggests that it could perhaps be profitable to discuss $E'(g)$ from the point of view of the Connes space $\mathbb{A}/K^\times$ [19] [20].

4.6. Theorem. *The following Riemann-Tate formula holds:*

$$E'(g)(\varphi) = \int_{|v|\geq 1} \mathcal{F}(\varphi)(v) \sum_{q\in K^\times} g(q/v)\sqrt{|v|}d^*v + \int_{|v|\geq 1} \varphi(v) \sum_{q\in K^\times} g(qv)\sqrt{|v|}d^*v$$

$$-\varphi(0)\int_{|v|\geq 1} g(1/v)|v|^{-1/2}d^*v - \int_{\mathbb{A}} \varphi(x)\,dx \cdot \int_{|v|\geq 1} g(v)|v|^{-1/2}d^*v$$

Proof. From $E'(g)(\varphi) = C\int_{\mathcal{C}_K} E(\varphi)(u)R(g)(u)\,d^*u$ we get

$$\begin{aligned}
E'(g)(\varphi) &= C\int_{|u|\leq 1} E(\varphi)(u)R(g)(u)\,d^*u + C\int_{|u|\geq 1} E(\varphi)(u)R(g)(u)\,d^*u \\
&= C\int_{|u|\geq 1} E(\mathcal{F}(\varphi))(u)R(g)(1/u)\,d^*u + C\int_{|u|\geq 1} E(\varphi)(u)R(g)(u)\,d^*u \\
&= C\int_{|u|\geq 1} \Big(E(\mathcal{F}(\varphi))(u) + \frac{\varphi(0)}{\sqrt{|u|}}\Big)R(g)(1/u)\,d^*u - \varphi(0)\int_{|v|\geq 1} g(1/v)|v|^{-1/2}d^*v \\
&+ C\int_{|u|\geq 1} \Big(E(\varphi)(u) + \frac{\int_{\mathbb{A}}\varphi(x)\,dx}{\sqrt{|u|}}\Big)R(g)(u)\,d^*u - (\int \varphi)\int_{|v|\geq 1} g(v)|v|^{-1/2}d^*v \\
&= \int_{|v|\geq 1} \mathcal{F}(\varphi)(v) \sum_{q\in K^\times} g(q/v)\sqrt{|v|}d^*v + \int_{|v|\geq 1} \varphi(v) \sum_{q\in K^\times} g(qv)\sqrt{|v|}d^*v \\
&- \varphi(0)\int_{|v|\geq 1} g(1/v)|v|^{-1/2}d^*v - (\int_{\mathbb{A}} \varphi(x)\,dx)\int_{|v|\geq 1} g(v)|v|^{-1/2}d^*v
\end{aligned}$$

which completes the proof. □

We note that if we replace formally $\sum_{q\in K^\times} g(qv)\sqrt{|v|}$ with $\chi(v)|v|^s$ we obtain exactly the Tate formula for $L(\chi,\varphi)(s)$. But some new flexibility arises with a "compact" $g(v)$:

4.7. Theorem. *The following formula holds:*

$$E'(g)(\varphi) = \int_{|v|\leq 1} \mathcal{F}(\varphi)(v) \sum_{q\in K^\times} g(q/v)\sqrt{|v|}d^*v + \int_{|v|\leq 1} \varphi(v) \sum_{q\in K^\times} g(qv)\sqrt{|v|}d^*v -$$

$$-\varphi(0)\int_{|v|\leq 1} g(1/v)|v|^{-1/2}d^*v - \int_{\mathbb{A}} \varphi(x)\,dx \cdot \int_{|v|\leq 1} g(v)|v|^{-1/2}d^*v$$

Proof. Exactly the same as above exchanging everywhere $|u|\geq 1$ with $|u|\leq 1$ (we recall that $R(g)$ has compact support and also that as we are dealing with a number field $|v|=1$ has zero measure). This is not possible with a quasicharacter in the place of $R(g)(u)$. Alternatively one adds to the previous formula and checks that one obtains $2E'(g)(\varphi)$ (using $E'(I(g))(\mathcal{F}(\varphi)) = E'(g)(\varphi)$). □

To illustrate some Hilbert Space properties of the co-Poisson summation, we will assume $K = \mathbb{Q}$. The components (a_ν) of an adele a are written a_p at finite places and a_r at the real place. We have an embedding of the Schwartz space of test-functions on $\mathbb{R}$ into the Bruhat-Schwartz space on $\mathbb{A}$ which sends $\psi(x)$ to $\varphi(a) = \prod_p \mathbf{1}_{|a_p|_p \le 1}(a_p) \cdot \psi(a_r)$, and we write $E'_{\mathbb{R}}(g)$ for the distribution on $\mathbb{R}$ thus obtained from $E'(g)$ on $\mathbb{A}$.

4.8. Theorem. *Let g be a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The co-Poisson summation $E'_{\mathbb{R}}(g)$ is a square-integrable function (with respect to the Lebesgue measure). The $L^2(\mathbb{R})$ function $E'_{\mathbb{R}}(g)$ is equal to the constant $-\int_{\mathbb{A}^\times} g(v)|v|^{-1/2} d^*v$ in a neighborhood of the origin.*

Proof. We may first, without changing anything to $E'_{\mathbb{R}}(g)$, replace g with its average under the action of the finite unit ideles, so that it may be assumed invariant. Any such compact invariant g is a finite linear combination of suitable multiplicative translates of functions of the type $g(v) = \prod_p \mathbf{1}_{|v_p|_p = 1}(v_p) \cdot f(v_r)$ with $f(t)$ a smooth compactly supported function on $\mathbb{R}^\times$, so that we may assume that g has this form. We claim that:

$$\int_{\mathbb{A}^\times} |\varphi(v)| \sum_{q \in \mathbb{Q}^\times} |g(qv)| \sqrt{|v|}\, d^*v < \infty$$

Indeed $\sum_{q \in \mathbb{Q}^\times} |g(qv)| = |f(|v|)| + |f(-|v|)|$ is bounded above by a multiple of $|v|$. And $\int_{\mathbb{A}^\times} |\varphi(v)||v|^{3/2}\, d^*v < \infty$ for each Bruhat-Schwartz function on the adeles (basically, from $\prod_p (1 - p^{-3/2})^{-1} < \infty$). So

$$E'(g)(\varphi) = \sum_{q \in \mathbb{Q}^\times} \int_{\mathbb{A}^\times} \varphi(v) g(qv) \sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

$$E'(g)(\varphi) = \sum_{q \in \mathbb{Q}^\times} \int_{\mathbb{A}^\times} \varphi(v/q) g(v) \sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

Let us now specialize to $\varphi(a) = \prod_p \mathbf{1}_{|a_p|_p \le 1}(a_p) \cdot \psi(a_r)$. Each integral can be evaluated as an infinite product. The finite places contribute 0 or 1 according to whether $q \in \mathbb{Q}^\times$ satisfies $|q|_p < 1$ or not. So only the inverse integers $q = 1/n$, $n \in \mathbb{Z}$, contribute:

$$E'_{\mathbb{R}}(g)(\psi) = \sum_{n \in \mathbb{Z}^\times} \int_{\mathbb{R}^\times} \psi(nt) f(t) \sqrt{|t|} \frac{dt}{2|t|} - \int_{\mathbb{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbb{R}} \psi(x)\, dx$$

We can now revert the steps, but this time on $\mathbb{R}^\times$ and we get:

$$E'_{\mathbb{R}}(g)(\psi) = \int_{\mathbb{R}^\times} \psi(t) \sum_{n \in \mathbb{Z}^\times} \frac{f(t/n)}{\sqrt{|n|}} \frac{dt}{2\sqrt{|t|}} - \int_{\mathbb{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbb{R}} \psi(x)\, dx$$

Let us express this in terms of $\alpha(y) = (f(y) + f(-y))/2\sqrt{|y|}$:

$$E'_{\mathbb{R}}(g)(\psi) = \int_{\mathbb{R}} \psi(y) \sum_{n \ge 1} \frac{\alpha(y/n)}{n} dy - \int_0^\infty \frac{\alpha(y)}{y} dy \int_{\mathbb{R}} \psi(x)\, dx$$

So the distribution $E'_{\mathbb{R}}(g)$ is in fact the even smooth function

$$E'_{\mathbb{R}}(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

As $\alpha(y)$ has compact support in $\mathbb{R} \setminus \{0\}$, the summation over $n \geq 1$ contains only vanishing terms for $|y|$ small enough. So $E'_{\mathbb{R}}(g)$ is equal to the constant $-\int_0^\infty \frac{\alpha(y)}{y} dy = -\int_{\mathbb{R}^\times} \frac{f(y)}{\sqrt{|y|}} \frac{dy}{2|y|} = -\int_{\mathbb{A}^\times} g(t)/\sqrt{|t|}\, d^*t$ in a neighborhood of 0. To prove that it is L^2, let $\beta(y)$ be the smooth compactly supported function $\alpha(1/y)/2|y|$ of $y \in \mathbb{R}$ ($\beta(0) = 0$). Then ($y \neq 0$):

$$E'_{\mathbb{R}}(g)(y) = \sum_{n\in\mathbb{Z}} \frac{1}{|y|} \beta(\frac{n}{y}) - \int_{\mathbb{R}} \beta(y)\, dy$$

From the usual Poisson summation formula, this is also:

$$\sum_{n\in\mathbb{Z}} \gamma(ny) - \int_{\mathbb{R}} \beta(y)\, dy = \sum_{n\neq 0} \gamma(ny)$$

where $\gamma(y) = \int_{\mathbb{R}} \exp(i\, 2\pi y w)\beta(w)\, dw$ is a Schwartz rapidly decreasing function. From this formula we deduce easily that $E'_{\mathbb{R}}(g)(y)$ is itself in the Schwartz class of rapidly decreasing functions, and in particular it is is square-integrable. □

It is useful to recapitulate some of the results arising in this proof:

4.9. Theorem. *Let g be a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The co-Poisson summation $E'_{\mathbb{R}}(g)$ is an even function on $\mathbb{R}$ in the Schwartz class of rapidly decreasing functions. It is constant, as well as its Fourier Transform, in a neighborhood of the origin. It may be written as*

$$E'_{\mathbb{R}}(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

with a function $\alpha(y)$ smooth with compact support away from the origin, and conversely each such formula corresponds to the co-Poisson summation $E'_{\mathbb{R}}(g)$ of a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The Fourier transform $\int_{\mathbb{R}} E'_{\mathbb{R}}(g)(y) \exp(i2\pi wy)\, dy$ corresponds in the formula above to the replacement $\alpha(y) \mapsto \alpha(1/y)/|y|$.

Everything has been obtained previously. The intertwining property was proven as a result on the adeles and ideles, but obviously the proof can be written directly on $\mathbb{R}$. It will look like this, with $\varphi(y)$ an even Schwartz function (and $\alpha(y)$ as above):

Proof. From $\int_{\mathbb{R}} \sum_{n\geq 1} |\varphi(ny)||\alpha(y)|\, dy < \infty$:

$$\begin{aligned} \int_{\mathbb{R}} \sum_{n\geq 1} \varphi(ny)\alpha(y)\, dy &= \sum_{n\geq 1} \int_{\mathbb{R}} \varphi(ny)\alpha(y)\, dy \\ = \sum_{n\geq 1} \int_{\mathbb{R}} \varphi(y) \frac{\alpha(y/n)}{n}\, dy &= \int_{\mathbb{R}} \varphi(y) \sum_{n\geq 1} \frac{\alpha(y/n)}{n}\, dy \end{aligned}$$

On the other hand applying the usual Poisson summation formula:

$$\begin{aligned}
&\int_{\mathbb{R}} \sum_{n\geq 1} \varphi(ny)\alpha(y)\,dy \\
= &\int_{\mathbb{R}} \left(\sum_{n\geq 1} \frac{\mathcal{F}(\varphi)(n/y)}{|y|} - \frac{\varphi(0)}{2} + \frac{\mathcal{F}(\varphi)(0)}{2|y|} \right) \alpha(y)\,dy \\
= &\int_{\mathbb{R}} \Big(\sum_{n\geq 1} \mathcal{F}(\varphi)(ny) \Big) \frac{\alpha(1/y)}{|y|}\,dy - \varphi(0) \int_0^\infty \alpha(y)\,dy + \mathcal{F}(\varphi)(0) \int_0^\infty \frac{\alpha(y)}{|y|}\,dy \\
= &\int_{\mathbb{R}} \mathcal{F}(\varphi)(y) \sum_{n\geq 1} \frac{\alpha(n/y)}{|y|}\,dy - \varphi(0) \int_0^\infty \alpha(y)\,dy + \mathcal{F}(\varphi)(0) \int_0^\infty \frac{\alpha(y)}{|y|}\,dy
\end{aligned}$$

The conclusion being:

$$\int_{\mathbb{R}} \varphi(y) \sum_{n\geq 1} \frac{\alpha(y/n)}{n}\,dy - \mathcal{F}(\varphi)(0) \int_0^\infty \frac{\alpha(1/y)}{y}\,dy = \int_{\mathbb{R}} \mathcal{F}(\varphi)(y) \sum_{n\geq 1} \frac{\alpha(n/y)}{|y|}\,dy - \varphi(0) \int_0^\infty \alpha(y)\,dy$$

which, after exchanging φ with $\mathcal{F}(\varphi)$, is a distribution theoretic formulation of the intertwining property:

$$\mathcal{F}\left(\sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy \right) = \sum_{n\geq 1} \frac{\alpha(n/y)}{|y|} - \int_0^\infty \alpha(y) dy$$

□

4.10. Note. In this familiar $\mathbb{R}$ setting our first modification of the Poisson formula was very cosmetic: the Poisson formula told us the equality of two functions and we exchanged a term on the left with a term on the right. This was to stay in a Hilbert space, but it remained a statement about the equality of two functions (and in the adelic setting, the equality of two functions with values in the distribution on the adeles). With the co-Poisson if we were to similarly exchange the integral on the left with the integral on the right, we would have to use Dirac distributions, and the nature of the identity would change. So the co-Poisson is a more demanding mistress than the (modified) Poisson. Going from Poisson to co-Poisson can be done in many ways: conjugation with I, or conjugation with $\mathcal{F}$, or Hilbert adjoint, or more striking still and at the same time imposed upon us from adeles and ideles, the switch from viewing a certain quantity as a function (Poisson) to viewing it as a distribution (co-Poisson). The co-Poisson is a distribution whose values we try to understand as L^2-functions, whereas the (modified) Poisson is a function with values in distributions (bad for Hilbert space).

For additional perspective and emphasis we state one more time the important intertwining property as a theorem, with an alternative proof:

4.11. Theorem. *Let $\alpha(y)$ be a smooth even function on $\mathbb{R}$ with compact support away from the origin. Let $P'(\alpha)$ be its co-Poisson summation:*

$$P'(\alpha)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y}\,dy$$

Then the additive Fourier Transform of $P'(\alpha)$ is $P'(I(\alpha))$ with $I(\alpha)(y) = \alpha(1/y)/|y|$.

Proof. Let P be the (modified) Poisson summation on even functions:

$$P(\alpha)(y) = \sum_{n\geq 1} \alpha(ny) - \frac{\int_0^\infty \alpha(y)\,dy}{|y|}$$

Obviously $P' = I \cdot P \cdot I$. And we want to prove $\mathcal{F}P' = P'I$. Let us give a formal operator proof:

$$\mathcal{F}P' = \mathcal{F}IPI = P\mathcal{F}II = P\mathcal{F} = IP = P'I$$

Apart from the usual Poisson summation formula $P\mathcal{F} = IP$ the crucial step was the commutativity of $\mathcal{F}I$ and P. This follows from the fact that both operators commute with the multiplicative action of $\mathbb{R}^\times$, so they are simultaneously diagonalized by multiplicative characters, hence they have to commute.

To elucidate this in a simple manner we extend our operators I, $\mathcal{F}$, P and P' to a larger class of functions, a class stable under all four operators. It is not difficult [14] to show that for each Schwartz function β (in particular for $\beta = I(\alpha)$) the Mellin Transform of $P(\beta)(y)$ is:

$$\int_0^\infty P(\beta)(y) y^{s-1}\,dy = \zeta(s) \int_0^\infty \beta(y) y^{s-1}\,dy$$

initially at least for $0 < \mathrm{Re}(s) < 1$. Let us consider the class of functions $k(1/2 + i\tau)$ on the critical line which decrease faster than any inverse power of τ when $|\tau| \to \infty$. On this class of functions we define I as $k(s) \mapsto k(1-s)$, P as $k(s) \mapsto \zeta(s)k(s)$, P' as $k(s) \mapsto \zeta(1-s)k(s)$, and $\mathcal{F}I$ as $k(s) \mapsto \gamma_+(s)k(s)$ with $\gamma_+(s) = \pi^{-(s-1/2)}\Gamma(s/2)/\Gamma((1-s)/2)$. The very crude bound (on $\mathrm{Re}(s) = 1/2$) $|\zeta(s)| = O(|s|)$ (for example, from $\zeta(s)/s = 1/(s-1) - \int_0^1 \{1/t\} t^{s-1}\,dt$) shows that it is a multiplier of this class (it is also a multiplier of the Schwartz class from the similar crude bounds on its derivatives one obtains from the just given formula). And $|\gamma_+(s)| = 1$, so this works for it too (and also for the Schwartz class, see [11]). The above formal operator proof is now not formal anymore (using, obviously, that the Mellin transform is one-to-one on our α's). The intertwining property for P' is equivalent to the intertwining property for P, because both are equivalent, but in different ways, to the functional equation for the zeta function. ∎

4.12. Note. As was stated in the previous proof a function space which is stable under all four operators I, $\mathcal{F}$, P and P' is the space of inverse Mellin transforms of Schwartz functions on the critical line: these are exactly the even functions on $\mathbb{R}$ with the form $k(\log|y|)/\sqrt{|y|}$ where $k(a)$ is a Schwartz function of $a \in \mathbb{R}$. We pointed out the stability under Fourier Transform in [11]. We note that although P and P' make sense when applied to $k(\log|y|)/\sqrt{|y|}$ and that they give a new function of this type, this can not always be expressed as in their original definitions, for example because the integrals involved have no reason to be convergent (morally they correspond to evaluations away from the critical line at 0 and at 1.) Of course we can also study I, $\mathcal{F}$, P and P' on L^2, but then care as to be taken because P and its adjoint P' are not bounded. Nevertheless they are closed operators and they commute with the Abelian von Neumann algebra of bounded operators commuting with $\mathbb{R}^\times$ (see [15]).

We believe that the problem of understanding the spaces of L^2-functions which are constant (or vanishing), together with their Fourier Transform in a neighborhood of the origin, is important simultaneously for Analysis and Arithmetic. It is a remarkable old discovery of de Branges [6] [7] that such spaces have the extremely rich structure which he has developpped in his Theory of Hilbert Spaces of entire functions, and he called them "Sonine spaces".

The co-Poisson summations provide examples of such functions and we show in the next sections that the zeros of the Riemann zeta function provide obstructions for them to fill up the full spaces.

5 On the Nyman-Beurling criterion, the so-called Hilbert-Pólya idea, and the Báez-Duarte, Balazard, Landreau and Saias theorem

How could it be important to replace $\zeta(s)$ with $\zeta(1-s)$? Clearly only if we leave the critical line and start paying attention to the difference between the right half-plane $\mathrm{Re}(s) > 1/2$ and the left half-plane $\mathrm{Re}(s) < 1/2$. Equivalently if we switch from the full group of contractions-dilations C_λ, $0 < \lambda < \infty$, which acts as $\phi(x) \mapsto \phi(x/\lambda)/\sqrt{|\lambda|}$ on even functions on $\mathbb{R}$, or as $Z(s) \mapsto \lambda^{s-1/2} Z(s)$ on their Mellin Transforms, to its sub-semi-group of contractions ($0 < \lambda \leq 1$). The contractions act as isometries on the Hardy space $\mathbb{H}^2(\mathrm{Re}(s) > 1/2)$ or equivalently on its inverse Mellin transform the space $L^2(]0,1[, dt)$.

5.1. Note. So far we have kept letters like t, u, v, for elements of multiplicative groups and x, y, for the additive groups. It is hard to be completely consistent with this, and in this section the functions of t are elements in $L^2(\mathbb{R}, dt)$, or rather in $L^2((0,\infty), dt)$, not in $L^2((0,\infty), dt/t)$.

It is a theme contemporaneous to Tate's Thesis and Weil's first paper on the explicit formula that it is possible to formulate the Riemann Hypothesis in such a semi-group set-up: this is due to Nyman [29] and Beurling [5] and builds on the Beurling [4] (and later for the half-plane) Lax [27] theory of invariant subspaces of the Hardy spaces. The criterion of Nyman reads as follows: the linear combinations of functions $t \mapsto \{1/t\} - \{a/t\}/a$, for $0 < a < 1$, are dense in $L^2(]0,1[, dt)$ if and only if the Riemann Hypothesis holds. It is easily seen that the smallest closed subspace containing these functions is stable under contractions, so to test the closure property it is only necessary to decide whether the constant function $\mathbf{1}$ on $]0,1[$ may be approximated. The connection with the zeta function is established through ($0 < \mathrm{Re}(s) < 1$):

$$\int_0^\infty \{\frac{1}{t}\} t^{s-1} dt = -\frac{\zeta(s)}{s}$$

This gives for our functions, with $0 < a < 1$ and $0 < \mathrm{Re}(s)$:

$$\int_0^1 (\{\frac{1}{t}\} - \frac{1}{a}\{\frac{a}{t}\}) t^{s-1} dt = (a^{s-1} - 1)\frac{\zeta(s)}{s}$$

The question is whether the invariant (under contractions) subspace of $\mathbb{H}^2(\mathrm{Re}(s) > 1/2)$ of linear combinations of these Mellin transforms is dense or not. Each zero ρ of $\zeta(s)$ in $\mathrm{Re}(s) > 1/2$ is an obvious obstruction as (the complex conjugate of) $t^{\rho-1}$ belongs to $L^2(]0,1[, dt)$. The Beurling-Lax theory describing the structure of invariant subspaces allows the conclusion in that case that there are no other obstructions.

We showed [16] as an addendum that the norm of the orthogonal projection of $\mathbf{1}$ to the Nyman space is $\prod_{\mathrm{Re}(\rho)>1/2} |(1-\rho)/\rho|$, where the zeros are counted with their multiplicities.

From this short description (see [3] for a more detailed exposition and additional references) we could think that the zeros on the critical line are completely out of the scope of the Nyman criterion. So it has been a very novel thing when Báez-Duarte, Balazard, Landreau and Saias asked the right question and provided a far from obvious answer [2]. First a minor variation is to replace the Nyman criterion with the question whether the function $\mathbf{1}_{0<t<1}$ can be approximated in $L^2(0,\infty)$ with linear combinations of the contractions of $\{1/t\}$. Let $D(\lambda)$ be the Hilbert space distance between $\mathbf{1}_{0<t<1}$ and linear combinations of contractions $C_\theta(\{1/t\})$, $\lambda \leq \theta \leq 1$. The Riemann Hypothesis holds if and only if $\lim D(\lambda) = 0$.

Theorem of Báez-Duarte, Balazard, Landreau and Saias [2]: *One has the lower bound:*

$$\liminf \ |\log(\lambda)| \ D(\lambda)^2 \geq \sum_\rho \frac{1}{|\rho|^2}$$

where the sum is over all non-trivial zeros of the zeta function, counted only once independently of their multiplicity.

The authors of [2] conjecture that equality holds (also with lim in place of lim inf) when one counts the zeros with their multiplicities: our next result shows that their conjecture not only implies the Riemann Hypothesis but it also implies the simplicity of all the zeros:

5.2. Theorem ([17]). *The following lower bound holds:*

$$\liminf \ |\log(\lambda)| \ D(\lambda)^2 \geq \sum_\rho \frac{m_\rho^2}{|\rho|^2}$$

Our proof relies on the link we have established between the study originated by Báez-Duarte, Balazard, Landreau and Saias of the distance function $D(\lambda)$ and the so-called Hilbert-Pólya idea. This idea will be taken here in the somewhat vague acceptation that the zeros of L-functions may have a natural interpretation as Hilbert space vectors, eigenvectors for a certain self-adjoint operator. If we had such vectors in $L^2((0,\infty), dt)$, perpendicular to $\{1/t\}$ and to its contractions $C_\theta(\{1/t\})$, $\lambda \leq \theta \leq 1$ then we would be in position to obtain a lower bound for $D(\lambda)$ from the orthogonal projection of $\mathbf{1}_{0<t<1}$ to the space spanned by the vectors. This lower bound would be presumably easily expressed as a sum indexed by the zeros from the fact that eigenspaces of a self-adjoint operator are mutually perpendicular. The first candidates are $t \mapsto t^{-\rho}$: they satisfy formally the perpendicularity condition to $\{1/t\}$ and its contractions, but they do not belong to L^2.

Nevertheless we could be in a position to approximately implement the idea if we used instead the square integrable vectors $t \mapsto t^{-(\rho-\epsilon)}\mathbf{1}_{0<t<1}$, $\epsilon > 0$, $\mathrm{Re}(\rho) = 1/2$. The authors of [2] followed more or less this strategy, but as they did not benefit from exact perpendicularity, they had to provide not so easily obtained estimates. It appears that the $\epsilon > 0$ does not seem to allow to take easily into account the multiplicities of the zeros. For their technical estimates the authors of [2] used to great advantage a certain scale invariant operator U, which had been introduced by Báez-Duarte in an earlier paper [1] discussing the Nyman-Beurling problem.

How is it possible that the use of this Báez-Duarte operator U (whose definition only relies on some ideas of harmonic analysis, and some useful integral formulae, with at first sight no arithmetic involved) allows Báez-Duarte, Balazard, Landreau and Saias to make progress on the Nyman-Beurling criterion? We related the mechanism underlying the insightful Báez-Duarte construction [1] of the operator U to a construction quite natural in scattering theory and it appeared then that it was possible to use it (or a variant V) to construct true Hilbert space vectors $Y^\lambda_{s,k}$ indexed by s on the critical line, and $k \in \mathbb{N}$, having the property of expressing the values of the Riemann zeta function and its derivatives on the critical line as Hilbert space scalar products:

$$(A, Y^\lambda_{s,k}) = (-\frac{d}{ds})^k \frac{s-1}{s}\frac{\zeta(s)}{s}$$

The additional factors are such that $\frac{s-1}{s}\frac{\zeta(s)}{s}$ belongs to the Hardy space of the right half-plane and $A(t)$ is its inverse Mellin transform, an element of $L^2((0,1), dt)$. What is more, we can replace A with its contractions $C_\theta(A)$ as long as $\lambda \leq \theta \leq 1$:

$$\lambda \leq \theta \leq 1 \Rightarrow (C_\theta(A), Y^\lambda_{s,k}) = (-\frac{d}{ds})^k\ \theta^{s-1/2}\ \frac{s-1}{s}\frac{\zeta(s)}{s}$$

So the only ones among the $Y^\lambda_{s,k}$ which are (exactly, not approximately) perpendicular to A and its contractions up to λ are the $Y^\lambda_{\rho,k}$, $\zeta(\rho) = 0$, $k < m_\rho$ ($\lambda < 1$). This connects the Nyman-Beurling criterion with the so-called Hilbert-Pólya idea. From the explicit integral formulae of Báez-Duarte for his operator U one can write formulae for the vectors $Y^\lambda_{s,k}$ from which the following asymptotic behavior emerges:

$$\begin{aligned} \lim_{\lambda\to 0} |\log(\lambda)|^{-1-k-l} \cdot (Y^\lambda_{s_1,k}, Y^\lambda_{s_2,l}) &= 0 \qquad (s_1 \neq s_2) \\ \lim_{\lambda\to 0} |\log(\lambda)|^{-1-k-l} \cdot (Y^\lambda_{s,k}, Y^\lambda_{s,l}) &= \frac{1}{k+l+1} \end{aligned}$$

In particular the rescaled vectors $X^\lambda_{s,0} = Y^\lambda_{s,0}/\sqrt{|\log(\lambda)|}$ become orthonormal in the limit when $\lambda \to 0$.

5.3. Note. Of course the limit can not work inside $L^2((0,\infty))$ because this is a separable space! In fact one shows that the vectors $X^\lambda_{s,k}$ weakly converge to 0 as $\lambda \to 0$.

Our theorem [17]

$$\liminf\ |\log(\lambda)|\ D(\lambda)^2 \geq \sum_\rho \frac{m_\rho^2}{|\rho|^2}$$

is easily deduced from the above estimates.

We have not explained yet what is U and how the $Y_{s,k}^{\lambda}$ are constructed with the help of it. The Báez-Duarte operator is the unique scale invariant operator with sends $\{1/t\}$ to its image under the *time reversal* $J : \varphi(t) \mapsto \overline{\varphi(1/t)/t}$, which is here the function $\{t\}/t$. Equivalently at the level of the Mellin Transforms, U acts as a multiplicator with multiplier on the critical line

$$U(s) = \frac{\overline{\zeta(s)/s}}{\zeta(s)/s} = \frac{\zeta(1-s)}{\zeta(s)} \frac{s}{1-s}$$

So this construction is extremely general: as soon as the function $A \in L^2((0,\infty), dt)$ has an almost everywhere non-vanishing Mellin transform $Z(s)$ on the critical line (which by a theorem of Wiener is equivalent to the fact that the multiplicative translates $C_\theta(A)$, $0 < \theta < \infty$, span L^2) then we may associate to it the scale invariant operator V with acts as $\overline{Z(s)}/Z(s)$, and sends A to its time reversal $J(A)$. We see that V is necessarily unitary. Let us in particular suppose that A is in $L^2((0,1), dt)$: then $J(A)$ has support in $[1,\infty)$ and the images under V of the contractions $C_\theta(A)$, $\lambda \leq \theta \leq 1$, being contractions of $J(A)$, will be in $L^2((\lambda,\infty), dt)$. In the case at hand we have:

$$Z(s) = \frac{s-1}{s}\frac{\zeta(s)}{s} \qquad V(s) = \frac{\zeta(1-s)}{\zeta(s)}\left(\frac{s}{1-s}\right)^3$$

The Báez-Duarte operator U and its cousin V depend on $\zeta(s)$ only through its functional equation, which means that they are associated with the (even) Fourier transform $\mathcal{F}_+$ (the cosine transform). We can use (almost) the same operators for Dirichlet L-series with an even character (with due attention paid to the conductors $q > 1$), and there are other operators we would use for odd characters, associated with the sine transform $\mathcal{F}_-$.

The vectors $Y_{s,k}^{\lambda}$ are obtained as follows: we start from $|\log(t)|^k\, t^{-(s-\epsilon)}\mathbf{1}_{0<t<1}$, apply V, restrict to $[\lambda, +\infty[$, take the limit which now exists in L^2 as $\epsilon \to 0$, and apply V^{-1}. What happens is that $V(t^{-1/2-i\tau}\mathbf{1}_{0<t<1})$ does not belong to L^2 but this is entirely due to its singularity at 0, which, it turns out, is $V(1/2+i\tau)t^{-1/2-i\tau}$. We would not expect it to be possible that a localized singularity would remain localized after the action of $\mathcal{F}_+$ but the point is that the operator with multiplier $\zeta(1-s)/\zeta(s)$ is the composite $\mathcal{F}_+ \cdot I$ with $I(\phi)(x) = \phi(1/x)/|x|$. So the singularity is first sent to infinity and $\mathcal{F}_+$ puts it back at the origin.

5.4. Note. Let us denote by L the scale invariant unitary operator with spectral function $L(s) = s/(s-1)$. One has $L = 1 - M$ where M is the Hardy averaging $\phi(t) \mapsto (\int_0^t \phi(u)\,du)/t$. The operator V is $(-L)^3\mathcal{F}_+I = \mathcal{F}_+I(-L)^3$. One has $ILI = L^{-1}$ and $LL^* = 1$. The operator L is "real", meaning that it commutes with the anti-unitary complex conjugation $\phi(t) \mapsto \overline{\phi(t)}$.

5.5. Note. One has $IV = I\mathcal{F}_+I(-L)^3$. We will write $\mathcal{G}_+ = I\mathcal{F}_+I$, so that $IV = \mathcal{G}_+(-L)^3 = (-L)^{-3}\mathcal{G}_+$. The operator $\mathcal{G}_+$ is unitary and satisfies $\mathcal{G}_+^2 = 1$. The operator $\mathcal{G}_+$ is real. The U operator of Báez-Duarte is $\mathcal{F}_+I(-L)$, so $V = UL^2 = L^2U$. The operators I, $\mathcal{F}_+$, $\mathcal{G}_+$, U and V are real.

5.6. Note. In our earlier papers we called I the "inversion". But if we think of the traditional meaning of inversion in a circle or in a line (perpendicular reflexion accross the line) we see that inversion is more alike the anti-unitary "time-reversal" J. So in our later papers we stopped giving a name to I. In the papers of Báez-Duarte, Balazard, Landreau and Saias, the letter S is used instead of I. We are tempted to switch to S too, but it is a little late now.

We now motivate the link leading to our Note [18], where a better approximation to the so-called Hilbert-Pólya idea is given, with a more detailed study of the vectors $Y^{\lambda}_{s,k}$, and of their use to express values of Mellin transforms and their derivatives on and off the critical line as Hilbert space scalar products.

We established

$$(B, Y^{\lambda}_{s,k}) = (-\frac{d}{ds})^{k} \widehat{B}(s)$$

for the Hardy functions $C_\theta(A)$, $\lambda \leq \theta \leq 1$. Their Mellin Transforms $\int_0^\infty C_\theta(A)(t)t^{s-1}\,dt = \theta^{s-1/2}\,\frac{s-1}{s}\frac{\zeta(s)}{s}$ are analytic in the entire complex plane except for a double pole at $s = 0$. The vectors $Y^{\lambda}_{s,k}$ are the analytic continuation to $\mathrm{Re}(s) = 1/2$ of vectors with the same definition $Y^{\lambda}_{w,k}$, $\mathrm{Re}(w) < 1/2$. So the above equation has the annoying aspect that its right hand side is analytic in s but the left-hand side sounds anti-analytic, as our scalar product is linear in its first factor and conjugate linear in its second factor. So we will use rather the *euclidean scalar product* $[B, C] = \int_0^\infty B(t)C(t)\,dt$. The spaces we consider are stable under complex conjugation $B(t) \mapsto \overline{B(t)}$, and the operators we use are real, so statements of perpendicularity may equivalently be stated using either $[\cdot,\cdot]$ or $(\cdot,\cdot)$. The identity can then be restated for all finite linear combinations B of our $C_\theta(A)$'s, $\lambda \leq \theta \leq 1$, as

$$[B, Y^{\lambda}_{w,k}] = (+\frac{d}{dw})^{k} \int_0^1 t^{-w} B(t)\,dt = (\frac{d}{dw})^{k} \left(\widehat{B}(1-w)\right)$$

for $\mathrm{Re}(w) \leq 1/2$.

If we look at the proof of the main theorem in [17] we see that the only thing that matters about B is that it should be supported in $[0, 1]$ and that $V(B)$ should be supported in $[\lambda, \infty)$, equivalently that $(IV)(B)(t)$ has support in $[0, \Lambda]$, $\Lambda = 1/\lambda$. Let us note the following:

5.7. Theorem. *The real unitary operator IV satisfies $(IV)^2 = 1$.*

Proof. This is clear from the spectral representation

$$V(s) = \frac{\zeta(1-s)}{\zeta(s)} \left(\frac{s}{1-s}\right)^3$$

which shows that $IVI = V^*$. □

5.8. Note. In particular IV is what Báez-Duarte calls "a skew-root" [1].

We let $\Lambda = 1/\lambda$ and $\mathcal{G}_\Lambda = IVC_\Lambda = C_\lambda IV$. We note that $(\mathcal{G}_\Lambda)^2 = 1$. We also note $V\mathcal{G}_\Lambda = C_\lambda VIV = C_\lambda I$.

Let $M_\Lambda = \mathbb{H}^2 \cap \mathcal{G}_\Lambda(\mathbb{H}^2)$, where we use the notation $\mathbb{H}^2 = L^2((0,1), dt)$. Obviously $\mathcal{G}_\Lambda(M_\Lambda) = M_\Lambda$. The function A as well as its contractions $C_\theta(A)$, $\lambda \leq \theta \leq 1$ belong to M_Λ. Indeed $\mathcal{G}_\Lambda(A) = C_\lambda(A)$. We note that the Mellin transform $\int_0^1 B(t)t^{w-1}\,dt$ of $B \in M_\Lambda$ is analytic at least in $\mathrm{Re}(w) > 1/2$. Let Q_λ be the orthogonal projection to $L^2(\lambda, \infty)$.

5.9. Theorem. *The vectors $Y^\lambda_{w,k}$, originally defined for $\mathrm{Re}(w) < 1/2$ as*

$$V^{-1}Q_\lambda V(|\log(t)|^k t^{-w}\mathbf{1}_{0<t<1})$$

have (inside L^2) an analytic continuation in w to the entire complex plane $\mathbb{C}$ except at $w = 1$.

The Mellin Transform of $B \in M_\Lambda$ has an analytic continuation to $\mathbb{C} \setminus \{0\}$, with at most a pole of order 2 at $w = 0$. One has for $w \neq 1$ and $k \in \mathbb{N}$:

$$[B, Y^\lambda_{w,k}] = (\frac{d}{dw})^k \left(\widehat{B}(1-w)\right)$$

The following functional equation holds:

$$\widehat{\mathcal{G}_\Lambda(B)}(w) = \lambda^{w-1/2}V(1-w)\widehat{B}(1-w)$$

One has

$$\forall B \in M_\Lambda \quad \widehat{B}(-2) = \widehat{B}(-4) = \cdots = 0$$

Proof. We leave the details of the case $k > 0$ to the reader. Let first $\mathrm{Re}(w) > 1/2$. We have for $B \in M_\Lambda$:

$$\widehat{B}(w) = \int_0^1 B(t)t^{w-1}\,dt = [B, t^{w-1}\mathbf{1}_{0<t<1}] = [V(B), V(t^{w-1}\mathbf{1}_{0<t<1})]$$

Writing $B = \mathcal{G}_\Lambda(C)$, with $C \in \mathbb{H}^2$, we get $V(B) = C_\lambda I(C)$. So $V(B)$ has its support in $[\lambda, \infty)$ and:

$$\widehat{B}(w) = \int_\lambda^\infty V(B)(u)V(t^{w-1}\mathbf{1}_{0<t<1})(u)\,du$$

We will show that $V(t^{w-1}\mathbf{1}_{0<t<1})(u)$ is analytic, for fixed u, in $w \in \mathbb{C} \setminus \{0\}$ and that it is $O((1 + |\log(u)|)/u)$ on $[\lambda, \infty)$, uniformly when w is in a compact subset of $\mathbb{C} \setminus \{0\}$ (this is one logarithm better than the estimate in [17] for $\mathrm{Re}(1-w) < 1$). We will thus have obtained the analytic continuation of the vectors $Y^\lambda_{1-w,0}$ from $\mathrm{Re}(w) > 1/2$ and at the same time the analytic continuation of $\widehat{B}(w)$ as well as the formula:

$$w \neq 0 \;\Rightarrow\; [B, Y^\lambda_{1-w,0}] = \widehat{B}(w)$$

So the problem is to study the analytic continuation of $V(t^{-z}\mathbf{1}_{0<t<1})(u)$ from $\mathrm{Re}(z) < 1/2$. If we followed the method of [17], we would write $V = (1-M)^2U$, compute some explicit formula for $U(t^{-z}\mathbf{1}_{0<t<1})(u)$ and work with it. This works fine for the continuation to $\mathrm{Re}(z) < 1$, but for $\mathrm{Re}(z) \geq 1$ there is a problem with applying M (which we must do before Q_λ) as the singularity at 0 is of the kind u^{-z} and is not integrable anymore. So we apply first $L^2 = (1-M)^2$ and only later U.

We compute:

$$\begin{aligned}
M(t^{-z}\mathbf{1}_{0<t<1})(u) &= \frac{\int_0^{\min(1,u)} t^{-z}\,dt}{u} = \frac{u^{-z}\mathbf{1}_{u\leq 1}(u)}{1-z} + \frac{1}{1-z}\frac{\mathbf{1}_{u>1}(u)}{u} \\
M^2(t^{-z}\mathbf{1}_{0<t<1})(u) &= \frac{u^{-z}\mathbf{1}_{u\leq 1}(u)}{(1-z)^2} + \frac{1}{(1-z)^2}\frac{\mathbf{1}_{u>1}(u)}{u} + \frac{1}{1-z}\frac{\log(u)\mathbf{1}_{u>1}(u)}{u} \\
L^2(t^{-z}\mathbf{1}_{0<t<1})(u) &= \frac{z^2}{(z-1)^2}\,u^{-z}\mathbf{1}_{u\leq 1} + (\frac{z^2}{(z-1)^2} - 1)\frac{\mathbf{1}_{u>1}(u)}{u} + \frac{1}{1-z}\frac{\log(u)\mathbf{1}_{u>1}(u)}{u}
\end{aligned}$$

We note that $U(\mathbf{1}_{u>1}/u) = UI(\mathbf{1}_{u<1}) = (M-1)\mathcal{F}_+(\mathbf{1}_{u<1}) = (M-1)(\sin(2\pi u)/(\pi u))$ is $O(1/u)$ (from the existence of the Dirichlet integral) for $u > \lambda$ and then that $U(\log(u)\mathbf{1}_{u>1}/u) = UMI(\mathbf{1}_{u<1}) = MUI(\mathbf{1}_{u<1}) = M(M-1)(\sin(2\pi u)/(\pi u)$ is $O((1+|\log(u)|)/u)$. Clearly this reduces the problem of $V(t^{-z}\mathbf{1}_{0<t<1})(u)$ to the problem of the analytic continuation and estimation of $U(t^{-z}\mathbf{1}_{0<t<1})(u)$. From [2], proof of Lemme 6, one has

$$U(t^{-z}\mathbf{1}_{0<t<1})(u) = \frac{\sin(2\pi u)}{\pi u} + \frac{z}{\pi u}\int_1^\infty t^{z-1}\sin(2\pi ut)\,\frac{dt}{t}$$

and (for example) from [18] we know that the integral is an entire function of z which is $O(1/u)$ on $[\lambda,\infty)$, uniformly in z when $|z|$ is bounded. We also see from this and from the integral representation of $\widehat{B}(w)$ that it has at most a pole of order 2 at $w=0$ (which is $z=1$).

The functional equation holds on the critical line from the spectral representation of $\mathcal{G}_\Lambda = C_\lambda IV$, hence it holds on $\mathbb{C}$ by analytic continuation. As $V(1-w)$ has poles at $1-w = -2,-4,\ldots$, and the left hand side is regular at these values of w it follows that $\widehat{B}(1-w)$ has to vanish for $1-w = -2,-4,\ldots$. □

The distance function $D(\lambda)^2$ has two components: one corresponding to the distance to the subspace M_Λ in $\mathbb{H}^2$ and then another one corresponding to the additional distance inside this space to the translates $C_\theta(A)$, $\lambda \leq \theta \leq 1$. The first step has absolutely no arithmetic, it is a problem of analysis. In the second step the orthogonal projections to M_Λ of the vectors $Y^\lambda_{\rho,k}$, $\zeta(\rho) = 0$, $k < m_\rho$ are obstructions, perhaps the only ones. When $\lambda \to 0$ ($\Lambda \to \infty$) the first contribution is presumably much smaller than the second, and the original vectors $Y^\lambda_{\rho,k}$ will not themselves differ much from their orthogonal projections to M_Λ. If all goes well this should say that the estimate $(\sum_\rho m_\rho^2/|\rho|^2)/|\log(\lambda)|$ gives the exact asymptotic decrease of $D(\lambda)^2$ (when the Riemann Hypothesis holds). We leave here this challenging problem to the interested reader and with it the operators U and V and proceed to concentrate on $\mathcal{G}_+ = I\mathcal{F}_+I$ which allows a simpler approximation to the so-called Hilbert-Pólya idea.

6 On de Branges Sonine spaces, the spaces HP_Λ, and the vectors $Z^\Lambda_{\rho,k}$

Let $K = L^2((0,\infty), dt)$ (complex-valued functions, $(f,g) = \int_0^\infty f(t)\overline{g(t)}\,dt$, and we will also use $[f,g] = \int_0^\infty f(t)g(t)\,dt$). The cosine transform $\mathcal{F}_+$ acts (in the L^2 sense) as $\mathcal{F}_+(f)(t) = 2\int_0^\infty \cos(2\pi tu)f(u)du$. It is a real operator. The operator I is $f(t) \mapsto f(1/t)/t$. The composite $\Gamma_+ = \mathcal{F}_+ I$ is scale invariant. The associated Tate Gamma function is:

$$\gamma_+(s) = \pi^{-(s-1/2)}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)} = \frac{\zeta(1-s)}{\zeta(s)} = 2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)$$

One has formally

$$2\int_0^\infty \cos(2\pi tu)|u|^{s-1}du = \gamma_+(s)\,|t|^{-s}$$

and this is litterally true as an identity of tempered distributions on $\mathbb{R}$ for $0 < \mathrm{Re}(s) < 1$. A useful related integral formula is ($0 < \mathrm{Re}(s) < 1$):

$$\gamma_+(s) = (1-s)\int_0^\infty u^{s-1}\frac{\sin(2\pi u)}{\pi u}\,du$$

One has $\mathcal{F}_+^2 = 1$, so K is the orthogonal sum of the subspaces of invariant functions under $\mathcal{F}_+$ and the subspaces of anti-invariant functions. As the spaces and operators we consider commute with the complex conjugation, we may also use the Euclidean scalar product $[f,g] = \int_0^\infty f(t)g(t)\,dt$ for statements of perpendicularity (of course this is a scalar product only for real-valued functions). It is well known that if $f(t)$ is compactly supported then $\mathcal{F}_+(f)$ is an entire function. In particular it has only countably many zeros (or it vanishes identically). So, as is well known, it is impossible for a non-zero function $f(t)$ with compact support to be such that $\mathcal{F}_+(f)$ again has compact support.

We have shown the intertwining property for co-Poisson summations:

$$\mathcal{F}\left(\sum_{n\geq 1}\frac{\alpha(t/n)}{n} - \int_0^\infty \frac{\alpha(1/t)}{t}\,dt\right) = \sum_{n\geq 1}\frac{\alpha(n/t)}{t} - \int_0^\infty \alpha(t)\,dt$$

which shows how to give example of functions $f(t)$ (which actually belong to the Schwartz class) which vanish identically in a neighborhood $(0,\lambda)$ of the origin and such that their Fourier cosine transform has the same property. For this we take $\alpha(t)$, smooth with support in $[\lambda,\Lambda]$ ($\Lambda = 1/\lambda$) and such that $\int_0^\infty \alpha(t)\,dt = 0 = \int_0^\infty (\alpha(1/t)/t)\,dt$. Obviously a non zero function can be obtained this way only for $\lambda < 1$ (equivalently we need $\Lambda > 1$).

Nevertheless there exists for arbitrary $\lambda > 0$ non-zero square integrable functions $f(t)$ vanishing in $(0,\lambda)$, and such that $\mathcal{F}_+(f)(t)$ also vanishes in $(0,\lambda)$. This was (to the best of the author's knowledge) first observed by de Branges [6] [7]. He called the corresponding spaces "Sonine spaces", as Sonine had given in the nineteenth century examples of such functions with this property where the Fourier Transform is replaced with the general Hankel transform of parameter ν. For $\nu = -1/2$ one has the cosine transform and for $\nu = 1/2$ one has the sine transform. To the best of the author's knowledge the examples given by Sonine are not square-integrable when $\nu = -1/2$ (cosine transform). Nevertheless de Branges showed that for arbitrary $\nu > -1$ the corresponding spaces of square integrable functions are non-trivial, and that the spaces of Mellin transforms of these functions (with an additional suitable Gamma factor) are Hilbert spaces of entire functions, in the sense of his general theory [7] (the Hilbert structure is the one inherited from K).

6.1. Note. In the Theory of de Branges the horizontal axis is the axis of symmetry; in the theory of L-functions it is obviously the critical line $\mathrm{Re}(s) = 1/2$. So comparison with de Branges investigations requires a trivial change of variable.

It is a general feature of a de Branges space that under very general assumptions it can be uniquely considered as a member of a family indexed by a parameter λ, $0 < \lambda < \infty$. This fascinating story is the topic of his book. For the Sonine spaces associated to Hankel transforms of integer orders this general structure has been elucidated by de Branges [6] and Rovnyak-Rovnyak [30]. But the Sonine spaces associated with the cosine and sine transforms remain more out of reach. De Branges has provided some additional information in his paper [9]. In his later paper [10] he also considered "double-Sonine spaces".

6.2. Note. As de Branges provides this additional information on Sonine and double-Sonine spaces in papers whose topic is the Riemann Hypothesis it should be explicitely pointed out by us, that to the best of our current knowledge, there is no intersection with our constructions as described in this section. We will not in fact really put to use the extremely powerful de Branges theory, but obviously further developments could considerably benefit from progress on this underlying structure.

Let $\mathcal{G}_+ = I\mathcal{F}_+I$. Let $H_\Lambda = L^2((0,\Lambda),dt) \cap \mathcal{G}_+(L^2((0,\Lambda),dt))$.

6.3. Theorem. *The spaces H_Λ are all non-reduced to $\{0\}$. The Mellin transforms $\widehat{f}(s) = \int_0^\infty f(t)t^{s-1}\,dt$ are entire functions with trivial zeros at $s = -2n$, $n \in \mathbb{N}$. The entire functions $M(f)(s) = \pi^{-s/2}\Gamma(s/2)\widehat{f}(s)$ satisfy the functional equations*

$$M(\mathcal{G}_+(f))(s) = M(f)(1-s)$$

For each $w \in \mathbb{C}$, each $k \in \mathbb{N}$, the linear forms $f \mapsto M(f)^{(k)}(w)$ are continuous and correspond to (unique) vectors $Z^\Lambda_{w,k} \in H_\Lambda$: $\forall f \in H_\Lambda$ $[f, Z^\Lambda_{w,k}] = M(f)^{(k)}(w)$.

Proof. As we said this is a special instance of de Branges theory of Sonine spaces. The L^2-boundedness of the evaluations of the derivatives ($k \geq 1$) follow by the Banach-Steinhaus theorem from the case $k = 0$. We provide elementary proofs of all statements in [18]. ∎

If we take an arbitrary sequence of (distinct) complex numbers with an accumulation point the corresponding evaluators $Z^{\Lambda}_{w,0}$ will span H_Λ. But any finite system is linearly independent:

6.4. Theorem ([18]). *Any finite collection of vectors $Z^{\Lambda}_{w,k}$ is a linearly independent system. In particular the vectors $Z^{\Lambda}_{w,k}$ are all non-vanishing.*

We saw that the co-Poisson summations gave elements of $I(H_\Lambda)$, so the Poisson summations give elements of H_Λ. In fact we work with H_Λ and not with $I(H_\Lambda)$ because we want the Mellin transforms to have properties analogous to the Riemann zeta function $\zeta(s)$ and not analogous to $\zeta(1-s)$. An alternative would be to change s to $1-s$ in the Mellin transform and work inside K with $I(H_\Lambda)$. Here we stick with H_Λ and rather than think that Poisson summation gives elements of H_Λ we will keep in mind that it is the *(inverted) co-Poisson summation.*

So let W_Λ be the closure in K of the functions $\sum_{n\geq 1}\phi(nu)$, with $\phi(u)$ smooth with support in $[\lambda,\Lambda]$, and such that $\widehat{\phi}(0)=0=\widehat{\phi}(1)$.

6.5. Theorem ([18]). *Let $\Lambda>1$. One has $W_\Lambda\subset H_\Lambda$. A vector $Z^{\Lambda}_{w,k}$ is perpendicular to W_Λ if and only if w is a non-trivial zero ρ of the Riemann zeta function and $k<m_\rho$.*

Let $HP_\Lambda = H_\Lambda\setminus W_\Lambda$, and let Z_Λ be the span in H_Λ of the vectors $Z^{\Lambda}_{\rho,k}$, $k<m_\rho$. We note that Z_Λ makes sense for arbitrary $\Lambda>0$. We will write $W'_\Lambda = H_\Lambda\setminus Z_\Lambda$ so that $W'_\Lambda\supset W_\Lambda$ (with W_Λ defined to be $\{0\}$ for $\Lambda\leq 1$.)

We have $Z_\Lambda\subset HP_\Lambda$ for $\Lambda>1$ but we have not decided so far whether it is possible for the inclusion to be strict. We explain later how one can translate this into a question about the Krein spaces associated with the measure $|\zeta(s)|^2 d\tau/2\pi$ on the critical line. For this discussion it is useful to have at our disposal slightly enlarged spaces $G_\Lambda\supset H_\Lambda$, whose elements' (Gamma-completed) Mellin Transforms may have poles at 0 and 1.

Let for each $\Lambda>0$: $G_\Lambda = L\cdot L^2((0,\Lambda),dt)\cap \mathcal{G}_+\cdot L\cdot L^2((0,\Lambda),dt))$. We recall that L is the unitary invariant operator with spectral function $s/(s-1)$.

6.6. Theorem. *Let $\Lambda>0$. Let $f\in G_\Lambda$. The Mellin transform $\widehat{f}(s)=\int_0^\infty f(t)t^{s-1}\,dt$ is an analytic function in $\mathbb{C}\setminus\{1\}$ with at most a pole of order 1 at $s=1$. It has trivial zeros at $s=-2n$, $n\geq 1$. The function $M(f)(s)=\pi^{-s/2}\Gamma(s/2)\widehat{f}(s)$, analytic in $\mathbb{C}\setminus\{0,1\}$, satisfies the functional equation*

$$M(\mathcal{G}_+(f))(s) = M(f)(1-s)$$

For each $w\in\mathbb{C}\setminus\{0,1\}$, each $k\in\mathbb{N}$, the linear forms $f\mapsto M(f)^{(k)}(w)$ are continuous.

Proof. The square integrable function $f(t)$ is equal to $\alpha(f)/t$ for $t>\Lambda$, with a certain constant $\alpha(f)$ (which is a continuous linear form in f). So $\int_0^\infty f(t)t^{s-1}\,dt$ is absolutely convergent and analytic at least for $1/2<\mathrm{Re}(s)<1$. In this strip we can write it as:

$$\int_0^\infty f(t)t^{s-1}\,dt = \int_0^\Lambda f(t)t^{s-1}\,dt + \frac{\alpha(f)\Lambda^{s-1}}{1-s}$$

which gives its analytic continuation to the right half-plane $\mathrm{Re}(s) > 1/2$ with at most a pole at $s = 1$. Let us also note that the evaluation at these points are clearly continuous for the Hilbert structure (as an intersection G_Λ is obviously a closed subspace of $L^2((0,\infty), dt)$). Let $g(t) = f(1/t)/t$. We have:

$$\int_0^\Lambda f(t)t^{s-1}\,dt = \int_\lambda^\infty g(t)t^{-s}\,dt = \int_0^\infty \mathcal{F}_+(g)(u)\mathcal{F}_+(\mathbf{1}_{t>\lambda}t^{-s})(u)\,du$$

We known from [18, Lemme 1.3.] that the function $\mathcal{F}_+(\mathbf{1}_{t>\lambda}t^{-s})(u)$ is an entire function of s, that it is (uniformly when $|s|$ is bounded) $O(1/u)$ on $[\lambda, \infty[$, and also that it is $\gamma_+(1-s)u^{s-1} + O(1)$ on $(0, \lambda)$, when $\mathrm{Re}(1-s) > 0$. Moreover $\mathcal{F}_+(g)(u)$ is a constant in the interval $(0, \lambda)$ (from $f \in G_\Lambda$). Combining these informations we get that the right-most term of the above displayed equation has an analytic continuation to the critical strip $0 < \mathrm{Re}(s) < 1$. In this critical strip we have the functional equation:

$$\widehat{f}(s) = \gamma_+(1-s)\widehat{\mathcal{G}_+(f)}(1-s)$$

as it holds on the critical line. From this we get the analytic continuation of $\widehat{f}(s)$ to $\mathrm{Re}(s) < 1$. We note that $\gamma_+(1-s)$ vanishes at $s = 0$ and that this counterbalances the (possible) pole of $\widehat{\mathcal{G}_+(f)}(1-s)$. Also this functional equation shows that $\widehat{f}(s)$ vanishes at $s = -2n$, $n \geq 1$.

The evaluations will be continuous linear forms off the critical strip, hence everywhere (except at $s = 1$ of course) from the Banach-Steinhaus theorem. □

We now state two theorems about W_Λ and W'_Λ and will complete their proofs after a number of intermediate steps.

6.7. Theorem. *A function $\alpha(s)$ on $\mathrm{Re}(s) = 1/2$ is the Mellin transform of an element of W'_Λ if and only if:*
1. It is square integrable on the critical line for $d\tau/2\pi$.
2. One has $\alpha(s) = \zeta(s)s(s-1)\beta(s)$ with $\beta(s)$ an entire function of finite exponential type at most $\log(\Lambda)$.

6.8. Theorem. *When $\Lambda \leq 1$ one has $W'_\Lambda = \{0\}$: the vectors $Z^\Lambda_{\rho,k}$, $k < m_\rho$, span H_Λ if and only if $\Lambda \leq 1$.*

Proof of 6.7 and 6.8 (first part). Let $f \in W'_\Lambda$. By definition its Mellin transform vanishes at the non-trivial zeros of ζ. It also vanishes at the trivial zeros and at 0 so it can be written

$$\widehat{f}(s) = s(s-1)\zeta(s)\theta(s)$$

with an entire function $\theta(s)$. From the functional equation one has

$$\widehat{\mathcal{G}_+(f)}(s) = s(s-1)\zeta(s)\theta(1-s)$$

In the right half-plane $\theta(s)$ is in the Nevanlinna class (of quotients of bounded analytic functions) because both $\widehat{f}(s)$ and $s(s-1)\zeta(s)$ are meromorphic in this class. And the same holds in the left half-plane. We now use a fundamental theorem of Krein [26] and conclude that the entire function $\theta(s)$ has finite exponential type which is given as

$$\max(\limsup_{\sigma\to\infty}\frac{\log|\theta(\sigma)|}{\sigma},\limsup_{\sigma\to\infty}\frac{\log|\theta(-\sigma)|}{\sigma})$$

This formula (elements of $\mathbb{H}^2$ are bounded in $\mathrm{Re}(s)\geq 1$) shows that the exponential type of $\theta(s)$ is at most $\log(\Lambda)$. This shows $W'_\Lambda=\{0\}$ for $\Lambda<1$.

Let us prove this also for $\Lambda=1$: on the line $\mathrm{Re}(s)=+2$ one has $\widehat{f}(s)=O(1)$ (as it belongs to $\mathbb{H}^2(\mathrm{Re}(s)>1/2)$) hence $\theta(s)$ is $O(1/s(s-1))$. So it is square integrable on this line and by the Paley-Wiener theorem it vanishes identically as it is of minimal exponential type.

The completion of the proof of 6.7 for $\Lambda=1$ and $\Lambda>1$ is given later. $\square$

6.9. Theorem. *A function $\alpha(s)$ on $\mathrm{Re}(s)=1/2$ is the Mellin transform of an element of W_Λ if and only if:*
1. It is square integrable on the critical line for $d\tau/2\pi$.
2. It is in the closure of the square integrable functions $\alpha(s)=\zeta(s)s(s-1)\beta(s)$ with $\beta(s)$ an entire function of finite exponential type strictly less than $\log(\Lambda)$.

We note that the theorem is obviously true for $\Lambda\leq 1$ as W_Λ is then defined to be $\{0\}$.

From these theorems the space W'_Λ (resp. W_Λ) is isometric to the subspace of $L^2(\mathrm{Re}(s)=1/2,|s(s-1)\zeta(s)|^2\,d\tau/2)$ of (resp. spanned by) entire functions of exponential type at most $\log(\Lambda)$ (resp. strictly less than $\log(\Lambda)$.) The following corollary holds:

6.10. Corollary. *One has $W'_\Lambda=\cap_{\mu>\Lambda}W_\mu$.*

Proof. From the two theorems one clearly has $W'_\Lambda\subset W_\mu\subset W'_\mu$ for $\mu>\Lambda$ and also $W'_\Lambda=\cap_{\mu>\Lambda}W'_\mu$. $\square$

This shows that the set of Λ's for which the inclusion $W_\Lambda\subset W'_\Lambda$ is strict is at most countable.

We now obtain the completion of the proofs of both theorems 6.7 and 6.9, as a corollary to a sequence of steps devoted to some general aspects of the Krein type spaces for the measure $|\zeta(s)|^2d\tau/2\pi$ on the critical line. The theory of Krein is explained in the book [22] by Dym and McKean, which also contains an introduction to the de Branges theory.

From now on we let $\Lambda\geq 1$.

Let $Z=L^2(\mathrm{Re}(s)=1/2,|\zeta(s)|^2d\tau/2\pi)$ ($s=1/2+i\tau$). We use the notation Z for easier comparison with [22]. We recall that it is known that $\int|\zeta(1/2+i\tau)|^2\,d\tau=\infty$ and $\int|\zeta(1/2+i\tau)|^2/|s|^2\,d\tau<\infty$. Let I^Λ be the subspace of Z of functions $F(s)$ which are entire functions of exponential type at most $\log(\Lambda)$. Let J^Λ be the subspace of Z of functions $F(s)$ which are entire functions of exponential type strictly less than $\log(\Lambda)$ (for $\Lambda=1$ this means $J^1=\{0\}$).

6.11. Lemma (Step 1). *A function $G \in Z$ is perpendicular to J^Λ if and only if it is perpendicular to all functions $(u^s - 1)/s$ for $\lambda \le u \le \Lambda$. Hence the closure $\overline{J^\Lambda}$ is also the closure of the finite linear combinations $(u^s - 1)/s$ for $\lambda \le u \le \Lambda$.*

Proof. One direction is obvious. Let us now assume that $G \perp (u^s - 1)/s$ for $\lambda \le u \le \Lambda$. Let $F \in J^\Lambda$ and let $\epsilon > 0$ be such that the type of F is $< \log(\Lambda) - \epsilon$. We consider

$$\int F(s) \frac{e^{\epsilon s} - 1}{s} \overline{g(s)} |\zeta(s)|^2 \, d\tau$$

If we take $F(s) = u^s$ with $e^\epsilon \lambda \le u \le e^{-\epsilon}\Lambda$ this integral vanishes. Using the Pollard-de Branges-Pitt "lemma" (*sic*) from [22], we deduce that the integral with the original $F(s)$ vanishes too. Then from $|(e^{\epsilon s} - 1)/\epsilon s| \le 2(e^{\epsilon/2} - 1)/\epsilon$ and dominated convergence we get the desired conclusion. □

6.12. Lemma (Step 2). *Any $F \in I^\Lambda$ is for each $\epsilon > 0$ in the closure of finite sums of functions $(u^s - 1)/s$ for $e^{-\epsilon}\lambda \le u \le e^{+\epsilon}\Lambda$.*

Proof. One has $I^\Lambda \subset J^{\Lambda \exp(\epsilon)}$. □

6.13. Lemma (Step 3). *Let $F(s) \in I^\Lambda$. One has $F(s)\zeta(s) \in L(\Lambda^s \mathbb{H}^2)$ and also $F(1-s)\zeta(s) \in L(\Lambda^s \mathbb{H}^2)$ (we write $\mathbb{H}^2$ for the Hardy space of the right half-plane and we recall that L is the operator of multiplication with $s/(s-1)$.)*

Proof. The product $F(s)\zeta(s)$ belongs to $L^2(\mathrm{Re}(s) = 1/2, d\tau/2\pi)$ and is from step 2 in the closure of finite sums of $(u^s - 1)\zeta(s)/s$ for $e^{-\epsilon}\lambda \le u \le e^{+\epsilon}\Lambda$. It thus belongs to the closed space $L((e^\epsilon\Lambda)^s \mathbb{H}^2)$ as $\zeta(s)/s$ belongs to $L(\mathbb{H}^2)$. We note that this space is the image under L of the Mellin transform of $L^2((0, e^\epsilon\Lambda), dt)$ so we may take the limit $\epsilon \to 0$. We note that $F(s) \to F(1-s)$ is an isometry of I^Λ and the conclusion follows. □

6.14. Theorem (Step 4). *An entire function $F(s)$ belongs to I^Λ (i.e. it is in Z and of exponential type at most $\log(\Lambda)$) if and only if $F(s)\zeta(s)$ is the Mellin transform of an element in G_Λ. The space I^Λ is a closed subspace of Z and is isometric through $F(s) \to \zeta(s)F(s)$ to the subspace of G_Λ of functions whose Mellin transform vanish at the zeros of the zeta function with at least the same multiplicity. For each complex number w the evaluations $F \mapsto F(w)$ are continuous linear forms on I^Λ.*

Proof. From **step 3** we know that the map $F(s) \to \zeta(s)F(s)$ is an isometric embedding into $\widehat{G_\Lambda}$. If an element $G(s)$ from $\widehat{G_\Lambda}$ vanishes at the non-trivial zeros of the zeta function (taking into account the multiplicities) then it factorizes as $G(s) = F(s)\zeta(s)$ with an entire function $F(s)$ (as $G(s)$ also vanishes at the trivial zeros and as at most a pole of order 1 at $s = 1$). From this $F(s)$ is in the right half-plane in the Nevanlinna class (of quotients of bounded analytic functions) because both $F(s)\zeta(s)$ and $\zeta(s)$ are meromorphic functions in this class. And the same holds in the left half-plane, as $\mathcal{G}_+(G)(s) = F(1-s)\zeta(s)$. We now use a fundamental theorem of Krein [26] which tells us that under such circumstances the entire function $F(s)$ has finite exponential type which is given as

$$\max(\limsup_{\sigma\to\infty}\frac{\log|F(\sigma)|}{\sigma},\limsup_{\sigma\to\infty}\frac{\log|F(-\sigma)|}{\sigma})$$

From this formula, and from $F(s)\zeta(s) \in L(\Lambda^s\mathbb{H}^2)$, $F(1-s)\zeta(s) \in L(\Lambda^s\mathbb{H}^2)$, and from the fact that elements of $\mathbb{H}^2$ are bounded in $\mathrm{Re}(s) \geq 1/2+\epsilon > 1/2$, we deduce that the exponential type of $F(s)$ is at most $\log(\Lambda)$. So I^Λ is isometrically identified with the functions in $\widehat{G_\Lambda}$ vanishing at least as $\zeta(s)$ does. This space is closed because the evaluators are continuous linear forms on G_Λ.
From this we see that the evaluators $F \mapsto F(s)$ are continuous linear forms except possibly at the zeros and poles of $\zeta(s)$, and the final statement then follows from this and the Banach-Steinhaus theorem (as I^Λ is a Hilbert space from the preceding). □

We are now in a position to prove 6.7 for $\Lambda \geq 1$:

Completion of the proof of 6.7. Let $F(s) \in I^\Lambda$. If $F(1) = 0$ then $F(s)\zeta(s)$ is analytic at $s = 1$ hence belongs to the codimension 1 subspace $\Lambda^s\mathbb{H}^2$ of $L(\Lambda^s\mathbb{H}^2)$, and of course the converse holds. So the conditions $F(s) \in I^\Lambda$, $F(0) = 0$, $F(1) = 0$ are equivalent to the condition $F(s)\zeta(s) \in \Lambda^s\mathbb{H}^2 \cap \mathcal{G}_+(\Lambda^s\mathbb{H}^2)$ which means exactly that $F(s)\zeta(s)$ is the Mellin transform of an element in H_Λ, vanishing at the zeros of zeta with at least the same multiplicities. □

6.15. Lemma (Step 5). *Any function F in I^Λ is $O(1)$ on the critical line.*

Proof. From the fact that $F(s)\zeta(s)$ is bounded on the line $\mathrm{Re}(s) = 2$ one deduces that $F(s)$ is bounded on $\mathrm{Re}(s) = 2$, hence also on $\mathrm{Re}(s) = -1$. As it has finite exponential type we may apply the Phragmen-Lindelöf theorem to deduce that $F(s)$ is $O(1)$ on the critical line. □

We now assume $\Lambda > 1$.

6.16. Lemma (Step 6). *Let K^Λ be the closure of J^Λ in Z. Let K_0^Λ be the subspace of functions vanishing at 0 and at 1, and similarly let J_0^Λ be the subspace of J^Λ of functions vanishing at 0 and at 1. Then K_0^Λ is the closure of J_0^Λ.*

Proof. Let for $1 < \mu < \Lambda$:

$$A_\mu(s) = \frac{(\mu^{s/2}-1)(\mu^{s/2}-\mu^{1/2})}{\log(\mu)(1-\mu^{1/2})/2}\frac{1}{s}$$

This is an entire function of exponential type $\log(\mu)$, in Z and with $A_\mu(0) = 1$, $A_\mu(1) = 0$. Let also $B_\mu(s) = A_\mu(1-s)$. Let $F \in K_0^\Lambda$ and let us write $F = \lim F_\mu$ with $F_\mu \in J^\mu$, $\mu < \Lambda$. One has $F_\mu(0) \to F(0) = 0$ and $F_\mu(1) \to F(1) = 0$, because evaluations are continuous linear forms on I^Λ. So $F = \lim(F_\mu - F_\mu(0)A_\mu - F_\mu(1)B_\mu)$ (clearly the norms of A_μ and B_μ are bounded as $\mu \to \Lambda$). □

Proof of theorem 6.9. Let $\Lambda > 1$. Let F in K_0^Λ. We want to show that $F(s)\zeta(s)$ is in the (Mellin transform of the) space W_Λ. The converse is obvious, as we can always approximate a smooth function with support in $[\lambda, \Lambda]$ with smooth functions with support strictly included in $[\lambda, \Lambda]$.

We may approximate F with an element of J_0^Λ, so we may assume F itself to be of positive exponential type $\log(\mu) < \log(\Lambda)$. Let θ be a smooth function with support in $[1/e, e]$, with $\widehat{\theta}(1/2) = 1$. Let $\widehat{\theta}_\epsilon(s) = \widehat{\theta}(\epsilon(s - 1/2) + 1/2)$. Let $F_\epsilon = \widehat{\theta}_\epsilon F$. In Z the functions F_ϵ converge to F. As F is $O(1)$ on the critical line the functions F_ϵ are $O(|s|^{-N})$ for any $N \in \mathbb{N}$. From the Paley-Wiener theorem they are the Mellin transforms of L^2 functions $f_\epsilon(t)$ with support in $[e^{-\epsilon}\mu^{-1}, e^\epsilon\mu]$. For ϵ small enough this will be included in $[\lambda, \Lambda]$. From the decrease on the critical line the functions $f_\epsilon(t)$ are smooth. As $\widehat{f_\epsilon}(0) = 0 = \widehat{f_\epsilon}(1)$ this tells us that $F_\epsilon(s)\zeta(s)$ is the Mellin transform of a Poisson summation, and this implies that $F(s)\zeta(s)$ belongs to the (Mellin transform of) W_Λ, as W_Λ is defined as the closure of such Poisson summations (of smooth functions). □

7 On the zeta function, the renormalization group, and duality

We briefly explain how the considerations of the previous section extend to Dirichlet L-series. For an odd character the cosine transform $\mathcal{F}_+$ is replaced with the sine transform $\mathcal{F}_-$, so we will stick with an even (primitive) Dirichlet character: $\chi(-1) = 1$. Let us recall the functional equation of $L(s, \chi) = \sum_{n\geq 1} \chi(n) n^{-s}$:

$$L(1 - s, \overline{\chi}) = w_\chi q^{s-1/2} \gamma_+(s) L(s, \chi)$$

where w_χ is a certain complex number of modulus 1 and q is the conductor (= period) of the primitive character χ. One has $\overline{w_\chi} = w_{\overline{\chi}}$. Tate's Thesis [31] gives a unified manner of deriving all these functional equations as a corollary to the one-and-only Poisson-Tate intertwining formula on adeles and ideles (and additional local computations). A reference for the more classical approach is, for example, [21] (for easier comparison with the classical formula, we have switched from $L(\chi, s)$ to $L(s, \chi)$). The Poisson-Tate formula specializes to twisted Poisson summation formulae on $\mathbb{R}$, or rather on the even functions on $\mathbb{R}$ as we are dealing only with even characters.

Let $\phi(t)$ be an even Schwartz function, and let:

$$P_\chi(\phi)(t) = \sum_{n\geq 1} \chi(n) \phi(nt)$$

We suppose here that χ is not the principal character so there is no term $-(\int_0^\infty \alpha(y)dy)/|y|$ (which was engineered to counterbalance the pole of the Riemann zeta function at $s = 1$). At the level of Mellin transforms P_χ corresponds to multiplication by $L(s, \chi)$.

So the composite $P_\chi \mathcal{F}_+ = P_\chi \mathcal{F}_+ II$ acts as

$$\widehat{\phi}(s) \mapsto L(s, \chi)\gamma_+(s)\widehat{\phi}(1 - s) = w_\chi^{-1} q^{-(s-1/2)} L(1 - s, \overline{\chi})\widehat{\phi}(1 - s)$$

and this gives the χ-Poisson intertwining:

$$P_\chi \mathcal{F}_+ = \overline{w_\chi} C_{1/q} I P_{\overline{\chi}}$$

Let us define the χ-co-Poisson P'_χ on smooth even functions compactly supported away from 0 as:

$$P'_\chi(\phi)(t) = \sum_{n\geq 1} \overline{\chi(n)} \frac{\alpha(t/n)}{n}$$

We have $P'_\chi = I P_{\overline{\chi}} I$, and P'_χ is the scale invariant operator with multiplier $L(1-s,\overline{\chi})$. From the commutativity of P_χ with $\Gamma_+ = \mathcal{F}_+ I$ and the χ-Poisson intertwining we get the χ-co-Poisson intertwining:

$$\mathcal{F}_+ P'_{\overline{\chi}} = \overline{w_\chi} C_{1/q} P'_\chi I$$

or exchanging χ with $\overline{\chi}$:

$$\mathcal{F}_+ P'_\chi = w_\chi C_{1/q} P'_{\overline{\chi}} I$$

Of course we can also get this equation directly from the functional equation of the L-function, or we can deduce it using the language of distributions as we did for the co-Poisson on adeles and ideles. The operator $C_{1/q}$ is the contraction with ratio $q > 1$: its placement on the right-side of the Intertwining equation is very important!

If $\alpha(t)$ (from now on we restrict to $L^2((0,\infty), dt)$) is supported in $[\lambda_1, \lambda_2]$ then its co-Poisson $f(t)$ summation will be supported in $[\lambda_1, \infty)$ and the Fourier cosine transform of $f(t)$ will be supported in $[1/(q\lambda_2), \infty)$. The product of the lower ends of these two intervals is strictly less than $1/q$ (if α is not identically zero). So this means that we obtain (non-zero) functions which together with their cosine transform are supported in $[\lambda, \infty[$ only for $\lambda < 1/\sqrt{q}$.

Equivalently, applying I, we get non-trivial subspaces $W_\Lambda^\chi \subset H_\Lambda$ from $\Lambda > \sqrt{q}$ on. The Mellin transforms of the functions in W_Λ^χ are the functions $L(s,\overline{\chi})\widehat{\alpha}(1-s)$ where $\alpha(t)$ is a smooth function compactly supported in $[\lambda, \Lambda/q]$ ($\Lambda = 1/\lambda$, $\Lambda > \sqrt{q}$). Let us note that this subspace is real only if χ is a real character. A vector $Z^\Lambda_{\rho,k}$ is (Hilbert-)perpendicular to W_Λ^χ if and only if $Z^\Lambda_{\overline{\rho},k}$ is Euclide-perpendicular to W_Λ^χ if and only if $\overline{\rho}$ is a (non-trivial) zero of $L(s,\overline{\chi})$ of multiplicity strictly greater than k, if and only ρ is a (non-trivial) zero of $L(s,\chi)$ of multiplicity strictly greater than k. So:

7.1. Theorem. *Let $\Lambda > \sqrt{q}$. A vector $Z^\Lambda_{\rho,k} \in H_\Lambda$ is perpendicular to W_Λ^χ if and only if ρ is a non-trivial zero ρ of the Dirichlet L-function $L(s,\chi)$ of multiplicity $m_\rho > k$.*

We complement this with a statement whose analog Theorem 6.8 we have already stated and proven for the Riemann zeta function. The proof is almost exactly identical, but as the statement is so important we retrace the steps here.

7.2. Theorem. *The vectors $Z^\Lambda_{\rho,k} \in H_\Lambda$, $L(\rho,\chi) = 0$, $k < m_\rho$, associated with the non-trivial zeros of the Dirichlet L-function (and with their multiplicities) span H_Λ if and only if $\Lambda \leq \sqrt{q}$.*

Proof. They can not span if $\Lambda > \sqrt{q}$ from the existence of W_Λ^χ. Let us suppose $\Lambda \leq \sqrt{q}$. Let $\widehat{f}(s)$ be the Mellin Transform of an element of H_Λ which is (Hilbert)-perpendicular to all $Z_{\rho,k}^\Lambda$, $L(\rho,\chi) = 0$ (non-trivial), $k < m_\rho$. This says that $f(s)$ vanishes at the $\overline{\rho}$'s. We know already that $f(s)$ vanishes at the trivial zeros. So one has:

$$\widehat{f}(s) = L(s, \overline{\chi})\theta_1(s)$$

with an entire function $\theta_1(s)$. The image of f under the unitary $\mathcal{G}_+$ will be perpendicular to $\mathcal{G}_+(Z_{\rho,k}^\Lambda)$. We forgot to state that $\mathcal{G}_+(Z_{\rho,k}^\Lambda) = (-1)^k Z_{1-\rho,k}^\Lambda$ and so $\mathcal{G}_+(f)$ is Euclide-perpendicular to the $\overline{1-\rho}$, which are the zeros of $L(s,\chi)$, hence:

$$\widehat{\mathcal{G}_+(f)}(s) = L(s,\chi)\theta_2(s)$$

with an entire function $\theta_2(s)$. From the functional equation:

$$\widehat{\mathcal{G}_+(f)}(s) = \gamma_+(1-s)\widehat{f}(1-s)$$

we get

$$\gamma_+(1-s)L(1-s,\overline{\chi})\theta_1(1-s) = L(s,\chi)\theta_2(s)$$

and combining with

$$L(1-s,\overline{\chi}) = w_\chi q^{s-1/2}\gamma_+(s)L(s,\chi)$$

this gives:

$$w_\chi q^{s-1/2}\theta_1(1-s) = \theta_2(s)$$

Using the fundamental theorem of Krein [26] we deduce that $F(s) = q^{-(s-1/2)/2}\theta_1(s)$ has finite exponential type which is equal to

$$\max(\limsup_{\sigma\to\infty} \frac{\log|F(\sigma)|}{\sigma}, \limsup_{\sigma\to\infty} \frac{\log|F(1-\sigma)|}{\sigma})$$

and from $L(\sigma,\chi) \to_{\sigma\to+\infty} 1$ we see that the exponential type of $F(s)$ is at most

$$\max(\log(\Lambda) - \log(\sqrt{q}), \log(\Lambda) - \log(\sqrt{q})) = \log(\Lambda) - \log(\sqrt{q})$$

This concludes the proof when $\Lambda < \sqrt{q}$. When $\Lambda = \sqrt{q}$, we see that $q^{-(s-1/2)/2}\theta_1(s)$ has minimal exponential type. But from $\widehat{f}(s) = L(s,\overline{\chi})\theta_1(s)$ we see that it is square-integrable on the line $\mathrm{Re}(s) = 2$. By the Paley-Wiener theorem it thus vanishes identically. □

We turn now to some speculative ideas concerning the zeta function, the GUE hypothesis and the Riemann hypothesis. When we wrote "The Explicit formula and a propagator" we had already spent some time trying to think about the nature of the zeta function. Our conclusion, which had found some kind of support with the conductor operator $\log|x| + \log|y|$, stands today. The spaces H_Λ and especially Theorem 7.2 have given us for the first time a quite specific signal that it may hold some value. What is more Theorem 7.2 has encouraged us into trying to encompass in our speculations the GUE hypothesis, and more daring and distant yet, the Riemann Hypothesis Herself.

We are mainly inspired by the large body of ideas associated with the Renormalization Group, the Wilson idea of the statistical continuum limit, and the unification it has allowed of the physics of second-order phase transitions with the concepts of quantum field theory. Our general philosophical outlook had been originally deeply framed through the Niels Bohr idea of complementarity, but this is a topic more distant yet from our immediate goals, so we will leave this aside here.

We believe that the zeta function is analogous to a multiplicative wave-field renormalization. We expect that there exists some kind of a system, in some manner rather alike the Ising models of statistical physics, but much richer in its phase diagram, as each of the L-function will be associated to a certain universality domain. That is we do not at all attempt at realizing the zeta function as a partition function. No the zeta function rather corresponds to some kind of symmetry pattern appearing at low temperature. But the other L-functions too may themselves be the symmetry where the system gets frozen at low temperature.

Renormalization group trajectories flow through the entire space encompassing all universality domains, and perhaps because there are literally fixed points, or another more subtle mechanism, this gives rise to sets of critical exponents associated with each domain: the (non-trivial) zeros of the L-functions. So there could be some underlying quantum dynamics, but the zeros arise at a more classical level, at the level of the renormalization group flow.

The Fourier transform as has been used constantly in this manuscript will correspond to a simple symmetry, like exchanging all spins up with all spins down. The functional equations reflect this simple-minded symmetry and do not have a decisive significance in the phase picture.

But we do believe that some sort of a much more hidden thing exist, a Kramers-Wannier like duality exchanging the low temperature phase with a single hot temperature phase, not number-theoretical. If this were really the case, some universal properties would hold across all phases, reflecting the universality examplified by the GUE hypothesis. Of course the hot phase is then expected to be somehow related with quantities arising in the study of random matrices. In the picture from Theorem 7.2, Λ seems to play the rôle of an inverse temperature (coupling constant).

We expect that if such a duality did reign on our space it would interact in such a manner with the renormalization group flow that this would give birth to scattering processes. Indeed the duality could be used to compare incoming to outgoing (classical) states. Perhaps the constraints related with this interaction would result in a property of causality equivalent to the Riemann Hypothesis.

Concerning the duality at this time we can only picture it to be somehow connected with the Artin reciprocity law, the ideas of class field theory and generalizations thereof. So here our attempt at being a revolutionary ends in utmost conservatism.

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