Comments on the Riemann conjecture and index theory on Cantorian fractal space-time

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Abstract

An heuristic proof of the Riemman conjecture is proposed. It is based on the old idea of Polya-Hilbert. A discrete/fractal derivative self adjoint operator whose spectrum may contain the nontrivial zeroes of the zeta function is presented. To substantiate this heuristic proposal we show using generalized index-theory arguments, corresponding to the (fractal) spectral dimensions of fractal branes living in Cantorian-fractal space-time, how the required *negative* traces associated with those derivative operators naturally agree with the zeta function evaluated at the spectral dimensions. The $\zeta(0) = -1/2$ plays a fundamental role.

1 Introduction

Riemann's outstanding conjecture that the non-trivial complex zeroes of the zeta function $\zeta(s)$ must be of the form $s = 1/2 \pm i\nu;\ \nu > 0$, remains one of the open problems in pure mathematics. Starting from an heuristic study of the index theorem associated with the dynamics of fractal p-branes living in Cantorian-fractal space-time $\mathcal{E}^{(\infty)}$ [1] we found some suggestive relations with the Riemann conjecture.

The construction of Cantorian-fractal space-time [1], $\mathcal{E}^{(\infty)}$, contains an infinite number of sets $\mathcal{E}^{(i)}$, where the index i ranges from $-\infty, +\infty$. Such index labels the topological dimension of the smooth space into which the fractal set is packed densely. For example, the sand on the beach looks two-dimensional on the surface. This is due to a coarse-grain averaging/smoothing of the underlying $3D$-grains which comprise it. In a similar vein the Hausdorff dimensions of the fractal sets packed densely inside the smooth manifold of **integer** dimension can be **larger** than the actual topological dimension of the space into which is being packed.

The best representative of this is the random back-bone Cantor set, $\mathcal{E}^{(0)}$, a fractal dust which is packed densely into a set of topological dimension zero (a point), and whose Hausdorff dimension equals to the golden-mean $\phi > 0$, with probability **one**, according to the Mauldin-Williams theorem [2]. We set the golden mean to be $1/(1+\phi) = \phi = (\sqrt{5}-1)/2 = 0.618....$ Notice that our conventions differ from those by Connes in his book [3]. He chooses for $\phi = (\sqrt{5}+1)/2$. We hope this will not cause confusion.

Incidentally we noted that

$$\phi^k = (-1)^k F_{k-1} + (-1)^{k+1} F_k \phi, \quad \phi^{-k} = F_{k+1} + F_k \phi, \quad k = 0, \pm 1, \pm 2, ... \quad (1)$$

where F_k are Fibonacci numbers. In this way, ... $\phi^{-4} = 5 + 3\phi$, $\phi^{-3} = 3 + 2\phi = 4 + \phi^3$, $\phi^{-2} = 2 + \phi$, $\phi^{-1} = 1 + \phi$, $\phi^2 = 1 - \phi$, $\phi^3 = -1 + 2\phi$, $\phi^4 = 2 - 3\phi$, $\phi^5 = -3 + 5\phi$, $\phi^6 = 5 - 8\phi$, $\phi^7 = -8 + 13\phi$, ...

The negative values of the topological dimensions signify the degree of "emptiness" or voids inside $\mathcal{E}^{(\infty)}$. The simplest analog of this is Dirac's theory of holes to explain the negative energy solutions to his equations (positrons/antimatter). Negative entropies and negative dimensions [5] were of crucial importance to have a rigorous derivation of why the average dimension of the world (today) is very close to: $4 + \phi^3 = 4.236....$

Negative probabilities and the non-commutative properties of $\mathcal{E}^{(\infty)}$ were essential to explain the wave-particle duality of an indivisible quantum particle traversing the Young's double-slit [5]. The non-commutative geometry of the von Neumann's type associated with Cantorian-fractal space-time $\mathcal{E}^{(\infty)}$ is the appropriate geometry to formulate the new relativity theory [6] that has derived the string uncertainty relations, and the p-branes generalizations, from first fundamental principles [7]. Moreover, such new scale relativity, an extension of Nottale's original scale relativity [8], is devoid of **EPR** paradoxes [9] and it explains the origins of the holographic principle [10].

In the first part of section **2** we shall briefly discuss the basic features of Cantorian-fractal space-time and heuristically postulate the existence of trace formula linked to the index of a fractal/discrete derivative operator. In the final part of section **2** we present a rigorous derivation of the index-theoretic results based on the η invariant which is related to the spectral staircase associated with the spectral dimensions of the infinite number of hierarchical sets living inside the fractal strings.

2 Quantum chaos and index theory in $\mathcal{E}^{(\infty)}$

Our motivation was sparked originally by the quantum counterpart of the classical chaos linked to the "billiard ball" moving on hyperbolic surfaces (constant negative curvature). As is well known to the experts the Selberg trace formula is essential to count the primitive periodic orbits of classical dynamical systems. The spectrum of (minus) Laplace-Beltrami operator on such hyperbolic surfaces is linked to the zeroes of the Selberg zeta function.

Knowing the energy eigenstates of the Schrödinger equation allows to locate the location of the nontrivial zeroes of the Selberg zeta function. They also have the form of $s = 1/2 \pm ip_n$ where $E_n = p_n^2 + 1/4$.

One of the most important features of the fractal/discrete operator is that it has **negative index** as we shall intend to show. This is just a result of the negative dimensions/holes/voids of $\mathcal{E}^{(\infty)}$.

These voids behave like absorption lines in the spectra of the Hamiltonian associated with the fractal operator $\mathcal{D}_f$. Connes already gave a detailed analysis

of the necessity for the trace to be **negative** (absorption lines) to account for the zeroes of the zeta function [11].

The old idea of Polya-Hilbert, is for example, the search for an equation of the type, Schrödinger equation on a hyperbolic space (constant negative curvature):

$$-\mathcal{D}_f(\mathcal{D}_f-1)\Psi = s(1-s)\Psi = s\bar{s}\Psi = E_n\Psi \Rightarrow s = \frac{1}{2}+ip_n = \frac{1}{2}+i\sqrt{E_n-\frac{1}{4}}. \quad (2)$$

The zeroes of the zeta function are then linked to the energy eigenvalues E_n in such a way that $\zeta(1/2 \pm ip_n) = 0$ and therefore the zeroes of zeta must lie in the critical line: $\Re(s) = 1/2$ and the Riemann conjecture could be proven, at least heuristically.

Quantum groups emerged as a result of inverse scattering methods. The search is to find now whether such operators can be constructed. If they can, then their spectrum will pick up the imaginary part of the zeroes of the zeta.

The sought-after self-adjoint properties of the fractal derivative, with respect a suitable inner product, are:

$$[\mathcal{D}_f]^+ = \mathcal{D}_f-1. \quad [\mathcal{D}_f-1]^+ = \mathcal{D}_f. \quad [\mathcal{D}_f(\mathcal{D}_f-1)]^+ = [(\mathcal{D}_f-1)^+(\mathcal{D}_f)^+] = \mathcal{D}_f(\mathcal{D}_f-1). \quad (3)$$

Then the Laplace-Beltrami operator is self-adjoint and $-\mathcal{D}_f(\mathcal{D}_f-1)$ has a positive definite energy spectrum. If this fractal/discrete derivative operator satisfies the properties above, and the operator is trace-class, then the Riemann conjecture could be proven following the arguments of Polya and Hilbert.

In general the Riemann-Roch theorem corresponding to a Riemann surface of genus g is associated with a family of derivative operators $\nabla_z^{(n)}$. The former is the derivative operator of "conformal" $U(1)$ weight "n" acting on the family of tensors $T^{(n)}$ with "q" holomorphic indices and "p" antiholomorphic indices such that $q-p=n$.

The index of the operator $\nabla_z^{(n)}$ is defined as the (complex) dimension of its kernel minus the (complex) dimension of its cokernel and is equal to $-(n-\frac{1}{2})$ times the Euler number of the Riemann surface of genus g which is given by $2-2g$.

In particular, when $n=0$ then $\nabla_z^{(0)} = \partial_z$ and the index of ∂_z is defined as the (complex) dimension of the kernel of ∂_z minus the (complex) dimension of the cokernel of ∂_z. The Riemann-Roch theorem becomes then for $n=0$

$$Index[\partial_z] = \frac{1}{2}Euler\ number = \frac{1}{2}(2-2g) = 1-g, \quad (4)$$

the reason is that the complex dimension is 1/2 the real dimension, so the alternative sum of Betti numbers multiplied by an over all factor of 1/2 will select the complex dimension in compliance with the Hirzebruch-Riemann-Roch Index theorem. The index in this case depends on the genus of the surface: $g=0$ corresponds to a sphere, $g=1$ to a torus and so forth. For details we refer to Nakahara's book [13].

We are generalizing these results to the case when the Euler number is given by the alternative sums of Betti numbers, alternative sums of all dimensions of

the possible cycles. In the case of a two dimensional surface one has three terms, and only three terms, in the sums only: Euler number of a two dimensional Riemann surface $= 1 - 2g + 1 = 2 - 2g$.

We will show that the index corresponding to the fractal/discrete derivative operator on the fractal world sheet $\mathcal{E}^{(2)}$, whose fractal dimension is $1 + \phi$, is given by the value of the zeta function evaluated on the spectral dimension of the infinite-cycle intersection $E_{infinity}$ which equals $dim\ E_{infinity} = (s) = 0$.

Spinors on Riemann surfaces are defined in terms of the square roots of the $n = 1$ line bundles: The weight $1/2$ corresponds to positive chirality spinors and the weight $-1/2$ corresponds to negative chirality ones.

In Cantorian fractal space-time we may follow by analogy similar arguments if one takes into account that the intrinsic fractal dimension of a bosonic random walk is $(1 + \phi)$. This means that dimensions are counted in basic units of the latter dimension. In particular, the intrinsic dimension of a fermionic random walk is $1/2$ the bosonic one: $1/2(1 + \phi)$.

The analog of the higher genus Riemann surfaces corresponds to spaces of negative dimensions. In Cantorian-fractal space-time, the totally void set, $\mathcal{E}^{(-\infty)}$, is the one whose fractal dimension is equal to zero and is embedded in a space of $-\infty$ topological dimension. The index associated with the analog of the $n = 0$ derivative operator ∂_z in ordinary Riemann surfaces of genus g, in Cantorian-fractal space-time is no longer an integer!, and is defined in basic units of $(1+\phi)$. It is evaluated on the infinite-cycle-intersection $E_{infinity}$ space and is given by the analog of the Riemann-Roch theorem:

$$\begin{aligned} \text{“}Index\text{”}\mathcal{D} &= \text{“}Trace\text{”}[\mathcal{D}^{-(s)}] = -(0 - 1/2)(1 + \phi)\ Euler\ [E_{infinity}] \\ &= \frac{1}{2}(1 + \phi)(-\phi) = -\frac{1}{2} = \zeta(0) \end{aligned} \tag{5}$$

and this agrees exactly with the value of $\zeta(0)$ since the dimension of the infinite-intersection cycle $E_{infinity}$ is precisely $(s) = 0$:

$$dim\ E_{infinity} = (s) = 1 \cdot \phi \cdot \phi^2 \cdot \phi^3 \cdot \ ... \phi^{s-1} = 0 \tag{6}$$

in the $s = \infty$ limit.

Therefore, using fractal derivatives and/or discrete derivatives like they occur in quantum-groups, q-calculus, and in p-adic QM, and studying the spectrum of fractal strings/branes in $\mathcal{E}^{\infty}$ space-times one may try to use the analog of the Riemann-Roch theorem:

$$\text{“}Index\text{”}[\mathcal{D}_f](E_{(s)}) = \text{“}Trace\text{”}[\mathcal{D}^{-(s)}_{fractal}] = \zeta(s) \sim Euler(E_{(s)}), \tag{7}$$

where the subspace $E_{(s)}$ (where the index is restricted on) of the world sheet $\mathcal{E}^{(2)}$ is some suitable **intersection** of a collection of sets, or cycles of $\mathcal{E}^{(\infty)}$ living **inside** the world sheet $\mathcal{E}^{(2)}$.

$$\mathcal{E}^1 \wedge \mathcal{E}^0 \wedge \mathcal{E}^{-1} ... \wedge \mathcal{E}^{-s+1}, \tag{8}$$

whose dimension is

$$dim\ E_{(s)} = 1 \cdot \phi \cdot \phi^2 ... \phi^{s-1} = \phi^{s(s-1)/2}, \tag{9}$$

where $(s) \equiv s(s-1)/2$ and s counts the number of cycles involved in the intersections, and whose Euler number is given by the usual formulae (alternating sums of Betti numbers):

$$Euler(E_{(s)}) = \sum_{k=-s}^{1} (-1)^k \phi^{-k+1} = \frac{-1+(-1)^s \phi^{s+2}}{1+\phi} = \frac{-1+F_{1+s}-\phi F_{2+s}}{1+\phi}. \tag{10}$$

Equation (10) is the explicit expression of the generalized Euler number as an alternative sum of the dimensions of the higher-dimensional voids/ holes ("genus" of the fractal string). Notice that in the asymptotic limit the alternating series converges exactly to: $Euler(E_{(\infty)}) = -\phi < 0$! which is a clear indication that Cantorian-fractal space-time is **left-handed**. This asymmetry between right/left chirality is also very natural in Penrose's twistor theory.

Higher dimensional (than one) sets: dim $= (1+\phi)^k$ correspond to fractal p-branes, are those corresponding to the values of $p = (1+\phi)^k > 1$. The sets of negative topological dimension are higher-dimensional *holes/voids*: They play the role of the *higher genus* surfaces in Cantorian-fractal space-time. The backbone set $\mathcal{E}^0$ will play the role of the discrete/fractal flow of time associated with the fractal string which is represented by an open subset of the normal set $\mathcal{E}^1$ whose dimension is equal to 1. Such discrete/fractal temporal evolution of the fractal string has for fractal world-sheet, $\mathcal{E}^{(2)}$ of fractal dimension $1+\phi$, given by direct product of the back bone set $\mathcal{E}^0$ times $\mathcal{E}^{(1)}$ and whose higher-dimensional voids/holes or higher "genus Riemannian surfaces" are nothing but the rest of the sets of negative topological dimensions.

In (super) strings, the multi-loop scattering amplitudes depend crucially on the suitable integrals over the (super) moduli space of the higher-genus surfaces. The Selberg zeta function plays an essential role in providing proper counting of the number of the primitive closed geodesics that tasselate the hyperbolic space, providing with a single cover of the (super) moduli space. This occurs for genus higher than 1 (the torus).

The parameter k which defines the lower bound of the alternating sum for the Euler number is nothing but the **analog** of the "genus" of the world-sheet associated with this **fractal "string"** "living" in $\mathcal{E}^\infty$. A p-brane spans a $p+1$ world volume generated by its motion in time which accounts for the extra dimension: A string spans a 2-dim world-sheet; a membrane a 3-volume and so forth. All this is naturally related to the statistical properties of random matrix models in lower dimensions in the large N limit; irrational conformal field theories; irrational values of the central charges; the monster group, etc.

From an elementary numerical calculation of equation (10), we can see that the Euler number of the multiple cycle-intersection $E_{(s)}$, as a function of the powers of ϕ^k (the genus-analog) **oscillates** about the golden mean. It is well known that the distribution of primes oscillates abruptly like the spectral-staircase levels in quantum chaos.

In the limit of infinite "genus", the Euler number, the alternating oscillatory sum will converge to the golden mean with a **negative sign**, consistent with the

nature of the absorption lines linked to the holes/voids/genus of the world sheet of the fractal string. As remarked earlier, Connes emphasized the importance that a **negative** value of the index theorem in non-commutative geometry must have to understand the location of the zeroes of zeta function as absorption spectral lines.

To sum up what has been said so far: Using $fractal/discrete$ derivatives $\mathcal{D}_f$ which are the more appropriate ones for $\mathcal{E}^{(\infty)}$ we make contact with the "Riemann zeta" function associated with such Cantorian-fractal space-time. As said previously, discrete derivatives are very natural in the q-calculus used in quantum groups (Jackson's calculus) and there is a deep relation between p-adic quantum mechanics and quantum groups as well.

The initial reasons why we believe a trace formula may be valid in Cantorian-fractal space-time is the following argument, despite the fact that we don't get a perfect matching of numbers. But they are close.

Looking at equation (10) for the first entry associated with the **triple** cycle-intersection of the three sets: $\mathcal{E}^0; \mathcal{E}^1; \mathcal{E}^{-1}$, we find for the Euler number associated with the alternating sum of dimensions:

$$Euler(E_{(3)}) = -1 + \phi - \phi^2 = -1 - 1 + 2\phi = -1 + \phi^3 = \frac{\phi^3}{2} - (1 - \frac{\phi^3}{2}). \quad (11)$$

The fractal dimension of the intersection of these three sets, intersection of three **cycles** is:

$$dim\ E_{(3)} = (1)(\phi)(\phi^2) = \phi^3 = 0.236068. \quad (12)$$

Now evaluate the "index" for this particular case:

$$\begin{aligned} \text{“}Index\text{”}[\mathcal{D}_{fractal}](E_{(s)}) &= \text{“}Trace\text{”}[\mathcal{D}^{-(s)}_{fractal}] = \zeta(s = \phi^3) = -0.790068 \\ &\sim Euler(E_{(3)}) = -0.763932 < 0. \end{aligned} \quad (13)$$

Notice how close our answer was: $-0.790068 \sim -0.763932$. Although not a perfect match, this is a good sign that we are on the right track. Looking down, one can see that the numbers do not differ much. The index of the exterior derivative operator associated with the de Rham elliptic complex coincides with the Euler number, for example.

The "trace" in non-commutative geometry as Connes has emphasized many times:

$$\text{“}Trace\text{”}[\mathcal{D}^{-(s)}_{fractal}] \leftrightarrow volume\ of\ the\ space\ that\ has\ dimension = (s). \quad (14)$$

One must be very careful not to confuse the label "s" with the label "(s)", they are not the same. Only in the special case $E_{(3)}$.

To justify further this proposal, for example, lets take now a look at the quadruple intersection (for real dimensions). The quadruple intersection of four cycles:

$$dim\ E_{(4)} = 1 \cdot \phi \cdot \phi^2 \cdot \phi^3 = \phi^6 = \phi^{4(4-1)/2} = 0.0557281. \quad (15)$$

So evaluating the Euler number from the alternating sum/Betti numbers from k = -2,-1, 0, 1:

$$\begin{aligned} & \text{“}Index\text{”}\ [\mathcal{D}_{fractal}](E_{(4)}) = \text{“}Trace\text{”}[\mathcal{D}^{-(s)}] = \zeta(\phi^6) = -0.55451 \\ \sim\quad & Euler\ [E_{(4)}] = 2\phi^3 - 1 = \phi^3 - (1-\phi^3) = -0.527864 < 0. \end{aligned} \tag{16}$$

Notice once again that the numbers are not so far off

$$-\,0.55451 \sim -0.527864\ ! \tag{17}$$

If one looks at the asymptotic infinite-dimensional voids/holes (“genus”) limit, to extract non-perturbative information, when the number of intersections of the sets of negative dimensions goes to infinity, the Euler number (for real dimensions) converges to the golden mean. Therefore in the asymptotic limit we have by looking at the last entries of our tables and at the limit of formula (10):

$$\begin{aligned} \zeta(0) &= -\frac{1}{2}, \quad \text{but} \\ Euler\ [E_{(infinity)}] &= -\phi = -0.618033. \end{aligned} \tag{18}$$

In this limit one can see that the space clearly is left/right chiral **asymmetric** like twistors! The number of self dual modes is not equal to the number of anti-self-dual ones ...

The quantum field theory associated with the geometrical excitations of Cantorian-fractal space-time is related to a Braided-Hopf-quantum-Clifford algebra: Braided statistics, etc... The Clifford-lines in **C**-space (Clifford spaces) are the Clifford-algebra valued (hyper-complex number-valued) lines which are the extensions of Penrose twistor based on complex numbers.

In this asymptotic limit the departure between the $\zeta(dim\ E_{(\infty)})$ and the Euler number is the greatest.

How do we reconcile the fact that the $\zeta(0) = -\frac{1}{2}$ differs from the Euler number of $E_{infinity}$ given by $-\phi$? This is where we invoke the analog of the Riemann-Roch theorem associated to the derivative operator ∂_z, where now one has holomorphic/antiholomorphic differentials of fractional weight, in units of the intrinsic fractal dimension of a fractal world-sheet which coincides precisely with the dimensions of a bosonic random walk so that:

$$\begin{aligned} Index &= Trace[\mathcal{D}^{-(s)}](E_{infinity}) = \zeta(0) = -\frac{1}{2} \\ &= -(0-\frac{1}{2})(1+\phi)Euler\ [E_{infinity}] = \frac{1}{2}(1+\phi)(-\phi) = -\frac{1}{2} \end{aligned} \tag{19}$$

Roughly speaking, we are taking the infinite cycle intersections in the Grassmanian space that encodes the “Riemann surfaces” of arbitrary “genus”. Each “point” of the ordinary Grassmanian represents a Riemann surface of a given genus. Cantorian-fractals space-time is endowed with a p-adic topology where

every disc is either contained inside another disc or it is disjoint from the latter. Every point inside a disc is a center, meaning that the center of a disc can occupy two different places simultaneously.

Similarly one could define spinors as the square root of the line bundles, $n = 1$ in units of $(1+\phi)$ so the spin bundles will be characterized by the following values of conformal weights: $|\zeta(0)|/\phi = |\zeta(0)|(1+\phi) = 1/2(1+\phi)$. As strangely as it may seem, fractional spin and fractal statistics is something which is not so farfetched. For references see [14]. Fractional spins are natural ingredients in the quantum Hall effect.

The analog of the Riemann-Roch index theorem applied to this operator would be proportional to the Euler number, in the infinite "genus" case, in the infinite cycle-intersections, if this operator corresponds to the conformal weight $n = 0$ derivative operator ∂_z, in units of the intrinsic fractal dimension of a bosonic random walk, $(1+\phi)$:

$$Index = \zeta(0) = -(0 - \frac{1}{2})(1+\phi)(-\phi) = -\frac{1}{2} \tag{20}$$

This index theoretic part of the paper is the main result. Besides, something very interesting occurs. The intrinsic dimension of a fractal Brownian walk of a boson is $1+\phi$ (and the fractal world-sheet as well). The dimension of a fermion random walk is $1/2$ (dimension of a bosonic random walk): $(1+\phi)/2$. This is precisely what we get. The extrinsic/embedding fractal dimension of a bosonic random walk is 2. The extrinsic/embedding dimension of the fermionic random walk is $1/2(2) = 1$.

In ordinary Conformal Field Theory, the central charge of a boson is 1 which is equal to the topological dimension of a path. The central charge of a fermion is $1/2$.

What about spin statistics when we have fractal spin dimension? The dimension appear in units of $(1+\phi)$. One has then an effective Planck constant $\hbar$ (See [6]) that will be then:

$$\hbar_{eff} = (1+\phi)\hbar. \tag{21}$$

So a fermion, for example, will have spin $= 1/2$ in units of $\hbar_{eff}$ instead of in units of the usual $\hbar$.

Notice how crucial is the fact that $\zeta(0) = -1/2$ in all these results. So essentially, the index theoretic results only make sense in the critical strip where real part of s lies between 0 and 1.

We need to examine the validity of the Riemann-Roch theorem for fractal Riemann surfaces and for higher dimensional surfaces as well, etc ... It is all heuristic but it may lead to a plausible clue which may shed some light in proving the Riemann conjecture.

What we shall explore the opposite scenario: What if there are non-trivial zeroes violating the Riemann conjecture?

We get approximate numbers, when the number of cycle intersections is finite, but not exact. This has a simple explanation. The zeta function for

real values of s lying between $0, 1$ with $s = \phi, \phi^2, \phi^3, ...$ is a slowly decreasing function while the Euler number **oscillates**.

A natural thing is to evaluate the zeta function at **complex** dimensions and check whether a matching with the Euler numbers (for complex dimensions) occurs; i.e to see if one can extend the analog of the Riemann-Roch theorem. Before we evaluate the Euler numbers for complex dimensions it is important to emphasize that there are no zeroes of zeta (besides the trivial ones $s = -2N$) in the region $\Re(s) < 0$. The argument goes as follows, the functional equation obeyed by the $\zeta(s)$ is of the form:

$$\pi^{-s/2}\zeta(s)\Gamma(s/2) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma(\frac{1-s}{2}). \tag{22}$$

Since the gamma function has trivial poles at $s = -2N$, the $\Gamma(s/2) = \Gamma(-N) = \infty$, this implies that $\zeta(s)$ will have trivial zeroes when $s = -2N$. This is due to the fact that the right hand side of the previous equation is well defined as functions of $(1-s)$. When $s = 0$, $\zeta(0) = -1/2$ and the pole of the $\Gamma(0) = \infty$ corresponds precisely to the only pole of $\zeta(1-0) = \infty$ in the right hand side of the equation.

Notice that the critical zeroes of the Riemann zeta function: $s = \frac{1}{2} + i\nu$ lie **exactly** in the vertical line between the following vertical lines of complex dimensions:

$$\phi + i\nu. \quad \phi^2 + i\nu \Rightarrow \frac{1}{2} + i\nu = \frac{1}{2}[\phi + i\nu + \phi^2 + i\nu] = s = \frac{1}{2} + i\nu \equiv critical\ zeroes. \tag{23}$$

For plausible violations of the Riemann conjecture, **inside** the critical strip, it would be interesting to look at the behavior of the following points which are symmetrically distributed about the vertical critical line $\Re(s) = 1/2$:

$$\zeta(\phi + i\nu) \text{ and } \zeta(\phi^2 + i\nu) = \zeta(1 - \phi + i\nu) \tag{24}$$

for example, where $\nu = \Im(s)$, imaginary part of the non trivial zeroes of zeta, $s = 1/2 + i\nu$.

It would be interesting to examine the behavior of the zeta evaluated at these points for all values of ν, the imaginary parts of the nontrivial zeroes and/or other values of ν. And to see what is the behavior of such values of zeta for large ν. How fast they approach or depart from zero, etc...

Let's start now with complex-valued dimensions and evaluate ζ there:

$$\zeta(\phi + i\nu). \quad \zeta(\phi^2 + i\nu). \quad \zeta(\phi^3 + i\nu), ... \zeta(\phi^n + i\nu), ... \tag{25}$$

Let us **foliate** this critical strip (between ϕ and ϕ^2) by an infinite family of **horizontal lines** passing through each single one of the imaginary parts of the critical zeroes: The horizontal lines at $\pm i\nu$ will do the job. The critical strip is comprised of the two vertical lines in the complex-dimension-plane which are symmetrically distributed with respect the critical line: $\Re(s) = 1/2$.

The "index" $[\mathcal{D}_{fractal}](E_{(s)})$ is evaluated now for a particular family of **complex** dimensions of the spaces given by the **triple** cycle intersections of $\mathcal{E}^{(1)}, \mathcal{E}^{(0)}, \mathcal{E}^{(-1)}$:

$$1 \pm i\nu. \quad \phi \pm i\nu. \quad \phi^2 \pm i\nu. \tag{26}$$

Riemann discovered [12] that $\zeta(s)$ has an analytic continuation to the whole complex plane except for one simple pole at $s = 1$ with residue equal to one. These properties have immediate consequences on the zero distribution of $\zeta(s)$: There are no zeros in the half-plane $\Re(s) > 1$, there are the so-called **trivial** zeros at $s = -2N$ for every positive integer N, but no others zeros in $\Re(s) < 0$. Therefore, **nontrivial** zeros can only occur in the critical strip $0 \leq \Re(s) \leq 1$.

Equation (26) furnishes these hypothetic complex dimensions where we could test the validity of the Riemann-Roch index theorem: Located at the complex plane points

$$(1 + i\nu)(\phi + i\nu)(\phi^2 + i\nu), \quad \nu > 0, \quad \zeta(1/2 + i\nu) = 0. \tag{27}$$

One must also add the complex conjugates.

This triple product of dimension equals to,

$$(-1 + 2\phi - 2\nu^2) + i(2\phi\nu - \nu^3) \tag{28}$$

Two comments are in order:

• The lowest ν is bigger than 14. Then we encounter that $(1 + i\nu)(\phi + i\nu)(\phi^2 + i\nu)$, for every ν, has a negative real part. Then, ζ **cannot** have zeroes at $(1 + i\nu)(\phi + i\nu)(\phi^2 + i\nu)$.

• Because the ζ evaluated in the following regions where $\Re(s) < 0$ is large: $\zeta[(1+i\nu)(\phi+i\nu)(\phi^2+i\nu)]$ is exponentially very large and the values of the Euler numbers corresponding to these complex dimensions are not of that magnitude, one cannot longer relate the values of the zeta evaluated at complex dimensions with the Euler number times the fractal dimension of a bosonic random walk. Hence, the analog of the Riemann-Roch Index theorem seems to be valid only in the critical strip: $0 < \Re(s) < 1$.

Concluding this section: If, and only if, a fractal/discrete derivative operator $\mathcal{D}_{fractal}$ is found to obey the properties described in this section then the Riemann conjecture could be proven heuristically.

3 The Atiyah-Patodi-Singer index theorem and fractal strings

So far our arguments have been heuristic. It is desirable to have a more rigorous derivation of these results. The physical picture we are proposing is that of a string (one dimensional object) with a discrete fractal flow of time. The discrete/fractal time is now represented by the random Cantor set $S^0 \equiv \mathcal{E}^{(0)}$ or fractal dust of dimension equal to the golden mean ϕ embedded in a space of topological dimension equal to 0, a "point". The one dimensional string is

represented by the normal set $S^1 \equiv \mathcal{E}^{(1)}$ where its fractal dimension equals 1 and is embedded in space of topological dimension equal to unity as well.

The world-sheet spanned by this fractal string has the topology of $S^1 \times S^0 \equiv \mathcal{E}^{(2)}$ and corresponds to a space of fractal dimension equal to $1+\phi$ embedded in a space of topological dimension equal to 2. This is what we referred earlier as the intrinsic/extrinsic dimension of a fractal bosonic random walk respectively.

Since the flow of time is discrete/fractal we can represent the topology of $S^1 \times S^0$ as that of an "even" dimensional manifold (since the fractal world sheet is embedded in a two-dimensional space) with **boundaries**. The boundaries are represented precisely by the discrete/fractal flow of time and the string itself corresponds to the *odd* dimensional manifold S^1.

The appropriate index theorem for Dirac operators in compact manifolds with boundaries is the Atiyah-Patodi-Singer Index theorem (spectral flow) and the relevant invariant is the so called η invariant given by the spectral asymmetry of the eigenvalues λ_k of the Dirac operator defined in an odd-dimensional manifold. After a suitable regularisation is made the η invariant is:

$$\eta = \sum_{\lambda_k > 0} 1 - \sum_{\lambda_k < 0} 1 = \sum_{\lambda_k}{}' sgn(\lambda_k)|\lambda_k|^{-2s}, \tag{29}$$

for $Re(s) > 0$. The prime in the second summation means that the zero modes have been omitted.

We will proceed by analogy along similar lines as the definition of the η invariant. The main difference is that we are concerned solely with the **spectral dimension** distribution of the infinite hierarchy of Cantor sets living inside the fractal string $\mathcal{E}^{(1)}$. These are the infinite hierarchy of sets: $S^1, S^0, S^{-1}, S^{-2}, ...S^{-\infty}$ of fractal dimensions $1, \phi, \phi^2, ...\phi^n, ...$ embedded in spaces of topological dimensions $1, 0, -1, -2, -3, ... - n, ... - \infty$ respectively.

Therefore we will perform the sums over all topological dimensions less than 1 and use the standard zeta function regularisation (analytic continuation) associated with the $\sum 1 = \infty$ summation:

$$\begin{aligned} \eta[\mathcal{E}^{(1)}] &= \sum_{dim>1} 1 - \sum_{dim<1} 1 = -\sum_{dim<1} 1 \\ &= -\left[1 + \sum_{d=-1}^{d=-\infty} 1\right] = -[1+\zeta(0)] = -1/2 = \zeta(0) \end{aligned} \tag{30}$$

This is the analog of the spectral staircase, where we are counting the dimensions from $-\infty$ to 0. This value of $\zeta(0)$ is precisely what corresponds to the index of the fractal derivative operator evaluated on the infinite -intersection-cycle $E_{infinity}$. Such infinite number of cycles are nothing but the infinite hierarchy of Cantor sets living inside $\mathcal{E}^{(1)}$ and whose topological dimension of their corresponding embedding spaces are $1, 0, -1, -2, -3, ... - \infty$ respectively. The higher-dimensional voids correspond to $-1, -2, -3, ... - \infty$.

Therefore:

$$\begin{aligned} Index(\mathcal{D})[E_{infinity}] &= Trace(\mathcal{D}^{-(s)}) \\ &= \zeta(dim\ E_{infinity}) = \zeta(0) = -1/2 = \eta[\mathcal{E}^{(1)}]. \quad (31) \end{aligned}$$

Now we can see why the analog of the Riemann-Roch theorem, associated with an operator of $n = 0$ conformal weight ∂_z, in the Cantorian-fractal world-sheet becomes then:

$$Index = \frac{1}{2}(1+\phi)(-\phi) = (dim_C[\mathcal{E}^{(2)}])Euler\ [E_{infinity}] = \eta[\mathcal{E}^{(1)}]. \quad (32)$$

As said previously one must not be alarmed by seeing a non-integer index value! Cantorian-fractal space-time corresponds naturally to irrational numbers (irrational conformal field theory). This result does in fact correspond to the generalized Euler number associated with the infinite-cycle intersection space $E_{infinity}$ (living inside the fractal string) and with a complex dimension equal to 1/2 the real dimension of the fractal world sheet $\mathcal{E}^{(2)}$ given by $1+\phi$. This is just the intrinsic dimension of a fractal bosonic random walk. The fermionic dimension equals 1/2 its value.

It is very important to emphasize that despite the fact that the fractal dimension of the infinite-cycle intersection space $E_{infinity}$ is 0 this does **not** mean that such space is made of a point. Cantorian-fractal space-time has no points. It corresponds to a von Neumann non-commutative **pointless** geometry with a natural p-adic topology: Every point is the center of a disc because the center of a disc can occupy many different places simultaneously. This is the key to understanding the wave-particle duality properties of an indivisible quantum particle in the double-slit Young experiment within the framework of negative probabilities in Cantorian-fractal spaces [5].

Due to the ring structure of the golden mean, $\phi^n = m\phi + n$ where m, n are integers, Cantorian-fractal space-time displays a Grassmanian nature. For a Grassmanian number θ such that $\theta^2 = 0$ any function of θ must be of the form $a\theta + b$. In a sense Cantorian-fractal space-time encodes"super-symmetry" and why the index formula is linked to a "super-trace": $n_+ - n_-$. The generalized Euler number of $E_{infinity}$ is equal to $0 - \phi = -\phi$ meaning that the Cantorian-fractal world sheet is left/right asymmetric. For example, the Euler number associated with the triple intersection of S^1, S^0, S^{-1} was

$$-1 + \phi - \phi^2 = \frac{\phi^3}{2} - (1 - \frac{\phi^3}{2}) = \phi^3 - 1 = 2\phi - 2, etc. \quad (33)$$

Let's imagine that we want to generalize this result to fractal p-branes. Imagine the volume of the p-brane is given by the direct product $\mathcal{E}^{(n)} \times \mathcal{E}^{(0)}$, where the first term represents the spatial dimensions given by $(1+\phi)^{n-1}$ and the second one corresponds to the discrete fractal flow of time represented by dimension of $\mathcal{E}^{(0)}$ which is the golden mean. The total dimension of the world volume is given by the sum $(1+\phi)^{n-1} + \phi$. The complex dimension is one half

that value. The spectral staircase relation associated with the intersection of the following cycles:

$$\mathcal{E}^n \wedge \mathcal{E}^{n-1} ... \wedge \mathcal{E}^{-s+1} ..., \tag{34}$$

is

$$\begin{aligned} \eta[\mathcal{E}^{(n)}] &= \sum_{dim>n} 1 - \sum_{dim<n} 1 \\ &= -(n-1) + \zeta(0) = \frac{1}{2}\left[(1+\phi)^{n-1} + \phi\right] \cdot \frac{(-1)^n \phi^{1-n}}{1+\phi}. \end{aligned} \tag{35}$$

However such relation is **not** always valid for an arbitrary value of n!. In the last equation, the last factor denotes the generalized Euler number of the infinite-intersection cycle. We have seen that for fractal strings, $n = 1$, is valid. Are there other **odd** values of n obeying such relation? A careful study shows that **the only** solution is the fractal string case $n = 1$.

Now let's make a further generalization of the p-branes case. The volume of the p-brane is given by the direct product $\mathcal{E}^{(n)} \times \mathcal{E}^{(m)}$, where the first term represents the spatial dimensions given by $(1+\phi)^{n-1}$ and the second one corresponds to the discrete fractal flow of time represented by dimension of $\mathcal{E}^{(m)}$ which is $(1+\phi)^{m-1}$. The total dimension of the world volume is given by the sum $(1+\phi)^{n-1} + (1+\phi)^{m-1}$. The complex dimension is one half that value. The spectral staircase relation associated with the intersection of the following cycles:

$$\mathcal{E}^{n+m} \wedge \mathcal{E}^{n+m-1} ... \wedge \mathcal{E}^{-s+1} ..., \tag{36}$$

is

$$\begin{aligned} \eta[\mathcal{E}^{(n+m)}] &= \sum_{dim>n+m} 1 - \sum_{dim<n+m} 1 = -(n+m-1) + \zeta(0) \\ &= \frac{1}{2}\left[(1+\phi)^{n-1} + (1+\phi)^{m-1}\right] \cdot \frac{(-1)^{n+m} \phi^{1-n-m}}{1+\phi}. \end{aligned} \tag{37}$$

A careful analysis shows that the only solution, as before, is the fractal string cases ($n = 1$, $m = 0$) or ($n = 0$, $m = 1$).

4 Concluding remarks

In section 2 we outlined the steps towards an heuristic proof of the Riemann conjecture. The conjecture could be proven heuristically, if and only if, a fractal/discrete derivative operator is found obeying the requisites outlined. The analog of the Riemann-Roch theorem in Cantorian-fractal space-time was furnished. The index formula for the derivative operator of zero "conformal" $U(1)$ weight, *restricted* in the infinite-intersection cycle, living inside the fractal world sheet, was found to agree **exactly** with the $\zeta(0) = -1/2$. The conformal weights were given in units of $(1+\phi)$ which is the intrinsic fractal dimension of a bosonic

random walk. Fermionic random walks have 1/2 the dimension of the bosonic ones. The η invariant which is related to the spectral staircase associated with the spectral dimensions of the infinite hierarchy of Cantor sets living inside the (odd dimensional) fractal string, required a zeta function regularisation $\sum 1 = \zeta(0) = -1/2$. The flow of time was discrete and fractal and corresponded to the extra dimension of ϕ yielding a world sheet $\mathcal{E}^{(2)}$ of fractal dimension $1+\phi$.

Next, we entertained the opposite idea. And was found that ζ **cannot** have zeroes at

$$(1\pm i\nu)(\phi\pm i\nu)(\phi^2\pm i\nu), \quad (1\pm i\nu)(\frac{1}{\phi}\pm i\nu)(\frac{1}{\phi^2}\pm i\nu). \quad \nu = \{\Im(s)\,|\zeta(s)=0\}. \quad (38)$$

Nevertheless, it is warranted to check if zeroes at $\phi+i\nu$ and $(1-\phi)+i\nu$ exist. These points lie inside the critical strip, $0 < \Re(s) < 1$ and are **symmetrically** distributed with respect to the critical line $Re(s) = 1/2$; i.e iff s is a non-trivial zero then $1-s$ must be as well as a direct consequence of the functional equation obeyed by the zeta function inside the critical strip (no poles in the gamma function). Since $\phi + \phi^2 = 1$, it is plausible that there could be zeroes of zeta at some values along the vertical lines beginning at those points. which values of the imaginary part? This is the question ... We checked some values but not all of them. Perhaps the imaginary values do not correspond to the imaginary values of $s = 1/2 + i\nu$ but to some other unknown ones ...

On a closing note, we honestly feel that theoretical physics in the next century may dwell on the following partial list: The new relativity theory $\leftrightarrow$ fractal p-branes in Cantorian space-time $\leftrightarrow$ irrational conformal field theory $\leftrightarrow$ number theory $\leftrightarrow$ non-commutative (non-associative) geometry $\leftrightarrow$ quantum chaos (quantum computing) $\leftrightarrow$ quantum groups $\leftrightarrow$ p-adic quantum mechanics.

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