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ON THE RIEMANN HYPOTHESIS, AREA QUANTIZATION, DIRAC OPERATORS, MODULARITY, AND RENORMALIZATION GROUP

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Two methods to prove the Riemann Hypothesis are presented. One is based on the modular properties of Θ (theta) functions and the other on the Hilbert–Polya proposal to find an operator whose spectrum reproduces the ordinates ρ_n (imaginary parts) of the zeta zeros in the critical line: $s_n = \frac{1}{2} + i\rho_n$. A detailed analysis of a one-dimensional Dirac-like operator with a potential $V(x)$ is given that reproduces the spectrum of energy levels $E_n = \rho_n$, when the boundary conditions $\Psi_E(x = -\infty) = \pm\Psi_E(x = +\infty)$ are imposed. Such potential $V(x)$ is derived *implicitly* from the relation $x = x(V) = \frac{\pi}{2}(d\mathcal{N}(V)/dV)$, where the functional form of $\mathcal{N}(V)$ is given by the full-fledged Riemann–von Mangoldt counting function of the zeta zeros, including the *fluctuating* as well as the $\mathcal{O}(E^{-n})$ terms. The construction is also extended to self-adjoint Schroedinger operators. Crucial is the introduction of an energy-dependent cut-off function $\Lambda(E)$. Finally, the natural quantization of the phase space areas (associated to *nonperiodic* crystal-like structures) in *integer* multiples of π follows from the Bohr–Sommerfeld quantization conditions of Quantum Mechanics. It allows to find a physical reasoning why the average density of the primes distribution for very large $x(\mathcal{O}(\frac{1}{\log x}))$ has a one-to-one correspondence with the asymptotic limit of the *inverse* average density of the zeta zeros in the critical line suggesting intriguing connections to the renormalization group program.

Keywords: Quantum mechanics; Dirac operators; Riemann hypothesis; Hilbert–Polya conjecture; area quantization; modularity; renormalization group; string theory.

1. Introduction: Riemann Hypothesis, Scaling, and Modular Invariance

Riemann's outstanding hypothesis [1] that the nontrivial complex zeros of the zeta-function $\zeta(s)$ must be of the form $s_n = 1/2 \pm i\rho_n$, is one of the most important open problems in pure mathematics. The zeta-function has a relation with the number of prime numbers less than a given quantity and the zeros of zeta are deeply connected with the distribution of primes [1]. Karatsuba, Voronin, Patterson, Edwards and Titchmarsh [2] are devoted to the mathematical properties of the zeta-function.

The Riemann hypothesis (RH) has also been studied from the point of view of mathematics and physics [6, 9, 14, 18, 11, 19] among many others. A novel physical interpretation of the location of the nontrivial Riemann zeta zeros which corresponds to the presence of tachyonic-resonances/tachyonic-condensates in bosonic string theory was found in [7]: if there were zeros outside the critical line violating the RH these zeros do not correspond to any poles of the string scattering amplitude. The spectral properties of the ρ_n's are associated with the random statistical fluctuations of the energy levels (quantum chaos) of a classical chaotic system [14]. Montgomery [8] has shown that the two-level correlation function of the distribution of the ρ_n's coincides with the expression obtained by Dyson with the help of random matrices corresponding to a Gaussian unitary ensemble.

In [10] by constructing of a continuous family of scaling-like operators involving the Gauss–Jacobi theta series and logarithmic derivatives, and after invoking a $\mathcal{CT}$-symmetry corresponding to a judicious charge conjugation $\mathcal{C}$ and time reversal $\mathcal{T}$ operation, we were able to show that the RH follows. The charge conjugation operation $\mathcal{C}$ is related to scalings transformations, and time reversal $\mathcal{T}$ operation, is related to the *inversions* $t \to (1/t)$ such that $\log(t) \to -\log(t)$. A "Wick rotation" of variables $t = iz$ furnishes $z \to -(1/z)$ which is a modular $SL(2, Z)$ transformation $z \to (az + b/cz + d)$ with unit determinant $ad - bc = 1$.

For these reasons, before entering into the next two sections we deem it very important to review the results [10, 16] based on a family of scaling-like operators in one dimension involving the Gauss–Jacobi theta series and an infinite parameter family of theta series where the inner product of their eigenfunctions $\Psi_s(t; l)$ is given by $(2/l)Z[\frac{2}{l}(2k - s^* - s)]$, where $Z(s)$ is the Riemann completed zeta-function and the l, k parameters are constrained to obey $(l + 4)/8 = k$ in order to have $\mathcal{CT}$-invariance.

There is a one-to-one correspondence among the zeta zeros $s_n(Z[s_n] = 0 \Rightarrow \zeta(s_n) = 0)$ with the eigenfunctions $\Psi_{s_n}(t; l)$ (of the latter scaling-like operators) when the latter are orthogonal to the "ground" reference state $\Psi_{s_o}(t; l)$; where $s_o = \frac{1}{2} + i0$ is the center of symmetry of the location of the nontrivial zeta zeros. We shall present a concise review [10] and show why the RH follows from a $\mathcal{CT}$ invariance when the pseudo-norm of the eigenfunctions $\langle \Psi_s | \mathcal{CT} | \Psi_s \rangle$ *is not null*. Had the pseudo-norm $\langle \Psi_s | \mathcal{CT} | \Psi_s \rangle$ been *null*, the RH would have been false.

The Scaling Operators related to the Gauss–Jacobi Theta series and the Riemann zeros [16] are given by

$$D_1 = -\frac{d}{d \ln t} + \frac{dV}{d \ln t} + k, \tag{1.1}$$

such that its eigenvalues s are complex-valued, and its eigenfunctions are given by

$$\psi_s(t) = t^{-s+k} e^{V(t)}. \tag{1.2a}$$

D_1 is *not* self-adjoint since it is an operator that does not admit an adjoint extension to the whole real line characterized by the *real* variable t. The parameter k is

also real-valued. The eigenvalues of D_1 are complex-valued numbers s. The charge conjugation operation $\mathcal{C}$ acting on the eigenfunctions is defined as

$$\begin{aligned}\psi_s(t) = t^{-s+k}e^{V(t)} \to \psi_{s^*}(t) = t^{-s^*+k}e^{V(t)} \\ = t^{-s^*+s}\psi_s(t),\end{aligned} \tag{1.2b}$$

which is related to scalings transformations of $\psi_s(t)$ by t-dependent (local) scaling factors

$$t^{-s^*+s} = e^{(-s^*+s)\ln t} = e^{2i\mathcal{I}m(s)\ln t} \Rightarrow \text{a phase rotation}, \tag{1.2c}$$

where $\mathcal{I}m(s)$ is the imaginary part of s. Since local t-dependent ($\ln t$ dependent to be precise) phase rotations resemble $U(1)$ gauge transformations one can then interpret the $(dV/d\ln t)$ term in D_1 as a gauge field (potential) in one dimension that gauges the scalings transformations. V is the pre-potential and $A = (dV/d\ln t)$ is the potential. Thus, charge conjugations (1.2b) can be recast as scaling transformations (1.2c).

We also define the "mirror" operator to D_1 as follows,

$$D_2 = \frac{d}{d\ln t} - \frac{dV(1/t)}{d\ln t} + k, \tag{1.3}$$

that is related to D_1 by the substitution $t \to 1/t$ and by noticing that

$$\frac{dV(1/t)}{d\ln(1/t)} = -\frac{dV(1/t)}{d\ln t}, \tag{1.4}$$

where $V(1/t)$ is not equal to $V(t)$ and D_2 is *not* self-adjoint either. When $l = 4(2k-1)$, the eigenfunctions of the D_2 operator are $\Psi_s(\frac{1}{t})$ (with eigenvalue s), and which can be shown to be equal to $\Psi_{1-s}(t)$ [16]. This results from the properties of the Gauss–Jacobi theta series under the $x \to 1/x$ transformations. Since $V(t)$ can be chosen arbitrarily, we chose it to be related to the Bernoulli string spectral counting function, given by the Jacobi theta series,

$$e^{2V(t)} = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t^l} = 2\omega(t^l) + 1, \tag{1.5}$$

this is where the l parameter appears in (1.5); the k parameter appears in (1.1) and (1.3). The condition $l = 4(2k-1)$ [16] is required so the orthogonal states $\Psi_{s_n}(t)$ (parametrized by the complex eigenvalues s_n) to the ground state $\Psi_{s=1/2}(t)$ have a one-to-one correspondence to the zeta zeros z_n in such a way that the quartets of numbers $\{s_n\}$ are symmetrically located w.r.t. the critical line, and real axis, in the same way that the zeta zeros z_n are: the quartets are $\{s_n\} = s_n; 1-s_n; s_n^*; 1-s_n^*$. Furthermore, the condition $l = 4(2k-1)$ is required in order to construct $\mathcal{CT}$-invariant (but not Hermitian) Hamiltonians as we describe below.

The related theta function defined by Gauss was

$$G\left(\frac{1}{x}\right) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2/x} = 2\omega\left(\frac{1}{x}\right) + 1, \tag{1.6}$$

where $\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. The Gauss–Jacobi series obeys the relation

$$G\left(\frac{1}{x}\right) = \sqrt{x}G(x), \tag{1.7}$$

resulting from the Poisson re-summation formula. The $V(t)$ is defined as $e^{2V(t)} = G(t^l)$ where $x = t^l$. The pair of mirror Hamiltonians $H_A = D_2D_1$ and $H_B = D_1D_2$, when $l = 4(2k-1)$ obey

$$H_A\Psi_s(t) = s(1-s)\Psi_s(t), \quad H_B\Psi_s\left(\frac{1}{t}\right) = s(1-s)\Psi_s\left(\frac{1}{t}\right), \tag{1.8}$$

due to the relation $\Psi_s(1/t) = \Psi_{1-s}(t)$ based on the modular properties of the Gauss–Jacobi series, $G(\frac{1}{x}) = \sqrt{x}G(x)$. Therefore, despite that H_A, H_B are *not* Hermitian they have the same spectrum $s(1-s)$ which is *real*-valued only in the critical line *and* in the real line. Equation (1.8) is the one-dimensional version of the eigenfunctions of the two-dimensional hyperbolic Laplacian given in terms of the Eisenstein's series.

Had H_A, H_B been Hermitian one would have had an immediate proof of the RH. Hermitian operators have a real spectrum, hence if $s(1-s)$ is real this means that $s = \frac{1}{2} + i\rho$, and/or $s = real$. The trivial zeta zeros are located at the negative even integers (real) and the nontrivial zeta zeros are located in the critical line $s = \frac{1}{2} + i\rho$. From Eq. (1.8) and using the properties of the Gauss–Jacobi series $G(\frac{1}{x}) = \sqrt{x}G(x)$ it follows that under the "time reversal" $\mathcal{T}$ operation $t \to \frac{1}{t}$ the eigenfunctions $\Psi_s(t)$ behave as

$$\mathcal{T}\Psi_s(t) = \Psi_s\left(\frac{1}{t}\right) = \Psi_{1-s}(t), \tag{1.9}$$

such that the Hamiltonian operators $H_A = D_2D_1, H_B = D_1D_2$ transform as

$$\mathcal{T}H_B\mathcal{T}^{-1} = H_A, \quad \mathcal{T}H_A\mathcal{T}^{-1} = H_B, \tag{1.10a}$$

the combined action of $\mathcal{CT}$ transformations is implemented on the states as follows

$$\begin{aligned} \mathcal{CT}H_A[\mathcal{CT}]^{-1}\Psi_s(t) &= H_A\Psi_s(t), \\ \mathcal{CT}H_B[\mathcal{CT}]^{-1}\Psi_s(t) &= H_B\Psi_s(t), \end{aligned} \tag{1.10b}$$

since $\Psi_s(t)$ span a continuum of eigenfunctions, for a continuum of s values, Eq. (1.10b) result in the vanishing of the commutators $[H_A, \mathcal{CT}] = [H_B, \mathcal{CT}] = 0$. When the operators H_A, H_B commute with $\mathcal{CT}$, there exits new eigenfunctions $\Psi_s^{\mathcal{CT}}(t)$ of the H_A operator with eigenvalues $s^*(1-s^*)$. Let us focus only in the H_A operator since similar results follow for the H_B operator. Defining

$$|\Psi_s^{\mathcal{CT}}(t)\rangle \equiv \mathcal{CT}|\Psi_s(t)\rangle. \tag{1.11}$$

One can see that it is also an eigenfunction of H_A with eigenvalue $s^*(1-s^*)$:

$$\begin{aligned} H_A|\Psi_s^{\mathcal{CT}}(t)\rangle &= H_A\mathcal{CT}|\Psi_s(t)\rangle = H_A|\Psi_{1-s^*}(t)\rangle \\ &= s^*(1-s^*)|\Psi_{1-s^*}(t)\rangle = s^*(1-s^*)\mathcal{CT}|\Psi_s(t)\rangle \\ &= (E_s)^*|\Psi_s^{\mathcal{CT}}(t)\rangle, \end{aligned} \tag{1.12}$$

where we have defined $(E_s)^* = s^*(1-s^*)$. The $\mathcal{CT}$ action on $s(1-s)\Psi_s$ is defined to be linear: $s(1-s)\mathcal{CT}\Psi_s$ since $\mathcal{C}$ acts only on the states $\Psi_s(t)$ as scalings (1.2b), and *not* on the numbers $s(1-s)$. Therefore, one has

$$\begin{aligned} [H_A,\mathcal{CT}] = 0 &\Rightarrow \langle\Psi_s|[H_A,\mathcal{CT}]|\Psi_s\rangle = 0 \\ &\Rightarrow \langle\Psi_s|H_A\mathcal{CT}|\Psi_s\rangle - \langle\Psi_s|\mathcal{CT}H_A|\Psi_s\rangle \\ &= (E_s)^*\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle - E_s\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle \\ &= (E_s^* - E_s)\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle = 0. \end{aligned} \tag{1.13}$$

Similar results follow for the H_B operator. From (1.13), one has two cases to consider.

- *Case A*: If the pseudo-norm is null

$$\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle = 0 \Rightarrow (E_s - E_s^*) \neq 0, \tag{1.14}$$

  then the *complex* eigenvalues $E_s = s(1-s)$ and $E_s^* = s^*(1-s^*)$ are *complex* conjugates of each other. In this case, the RH would be false and there are quartets of nontrivial Riemann zeta zeros given by $s_n, 1-s_n, s_n^*, 1-s_n^*$.
- *Case B*: If the pseudo-norm is *not* null:

$$\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle \neq 0 \Rightarrow (E_s - E_s^*) = 0, \tag{1.15}$$

  then the eigenvalues are *real* given by $E_s = s(1-s) = E_s^* = s^*(1-s^*)$ and which implies that $s = real$ (location of the trivial zeta zeros) and/or $s = \frac{1}{2} + i\rho$ (location of the nontrivial zeta zeros). In this case, the RH would be true and the nontrivial Riemann zeta zeros are given by $s_n = \frac{1}{2} + i\rho_n$ and $1 - s_n = s_n^* = \frac{1}{2} - i\rho_n$. We are going to prove next why Case A does and cannot occur, therefore the RH is true because we are left with case B.

The inner product are defined as follows,

$$\langle f|g\rangle = \int_0^\infty f^* g \frac{dt}{t}.$$

Based on this definition the inner product of two eigenfunctions of D_1 is

$$\begin{aligned} \langle\psi_{s_1}|\psi_{s_2}\rangle &= \int_0^\infty e^{2V} t^{-s_{12}+2k-1} dt \\ &= \frac{2}{l} Z\left[\frac{2}{l}(2k - s_{12})\right], \end{aligned} \tag{1.16}$$

where we have denoted $s_{12} = s_1^* + s_2 = x_1 + x_2 + i(y_2 - y_1)$ and used the expressions for the Gauss–Jacobi theta function and the definition of the completed zeta-function $Z[s]$ resulting from the Mellin transform as shown below.

We notice that

$$\langle \psi_{s_1} \,|\, \psi_{s_2} \rangle = \langle \psi_{s_o} \,|\, \psi_s \rangle, \tag{1.17}$$

thus, the inner product of ψ_{s_1} and ψ_{s_2} is equivalent to the inner product of ψ_{s_o} and ψ_s, where $s_o = 1/2 + i0$ and $s = s_{12} - 1/2$. The integral is evaluated by introducing a change of variables $t^l = x$ (which gives $dt/t = (1/l)dx/x$) and using the result provided by the Gauss–Jacobi Theta given in Karatsuba and Voronin's book [2]. The completed function $Z[s]$ in Eq. (1.16) can be expressed in terms of the Jacobi theta series, $\omega(x)$ defined by Eqs. (1.5) and (1.6) as

$$\begin{aligned} \int_0^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{s/2-1} dx &= \int_0^\infty x^{s/2-1} \omega(x) dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty [x^{s/2-1} + x^{(1-s)/2-1}] \omega(x) dx \\ &= Z(s) = Z(1-s), \end{aligned} \tag{1.18}$$

where the completed zeta-function is

$$Z(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \tag{1.19}$$

which obeys the functional relation $Z(s) = Z(1-s)$, which is a self-duality relation [26].

In [10], we recurred to an *infinite* family of H_A, H_B operators associated with an *infinite* family of potentials $V_{jm}(t)$ corresponding to an infinite family of theta series with the advantage that *no regularization* of the inner products is necessary. Another salient feature is that the pseudo-norm $\langle \Psi_s^{jm} | \mathcal{CT} | \Psi_s^{jm} \rangle$ is *not* null (see Eq. (1.28)) as a result that the zeta-function $\zeta(s)$ has *no* zeros at $s = \frac{1}{2} - 2m, m = 1, 2, 3, \ldots, \infty$. The relevance of the behavior of $\zeta(\frac{1}{2} - 2m) \neq 0, m = 1, 2, 3, \ldots, \infty$ is that it automatically avoids looking at the behavior of zeta at $s = 1/2$. Armitage [3] has found a zeta-function $\zeta_L(s)$ defined over the *algebraic* number field L that has a zero at $s = 1/2$ and presumably satisfies the RH. This finding would not be compatible with the result of Eq. (1.16) and which was based on a regularized inner product. Therefore, the well-defined inner product where *no regularization* is needed leads to the result (see Eq. (1.28)) $\langle \Psi_s^{jm} | \mathcal{CT} | \Psi_s^{jm} \rangle \sim \zeta(\frac{1}{2} - 2m) \neq 0, m = 1, 2, 3, \ldots, \infty$ and which is no longer in variance with the behavior of the zeta-function $\zeta_L(s)$ defined over the *algebraic* number field L and that has a zero at $s = 1/2$ [3].

Analogous results follow if we had defined a new family of potentials $V_{2j}(t)$ in terms of a *weighted* theta series $\Theta_{2j}(t)$ and whose Mellin transform yields the

infinite family of extended zeta-functions of Keating [4] and their associated completed zeta-functions as shown by Coffey [5]. The Hermite polynomials weighted theta series associated to $2j$ = even-degree polynomials are defined by

$$e^{2V_{2j}(t)} = \Theta_{2j}(t) \equiv \sum_{n=-\infty}^{n=\infty} (8\pi)^{-j} H_{2j}(n\sqrt{2\pi t}) e^{-\pi n^2 t}, \tag{1.20}$$

and are related to the potentials $V_{2j}(t)$ which appear in the definitions of the differential operators (1.1) and (1.2). The weighted theta series obeys the relation

$$\frac{(-1)^j}{\sqrt{t}} \Theta_{2j}\left(\frac{1}{t}\right) = \Theta_{2j}(t). \tag{1.21}$$

Only when j = even in (1.21) one can implement $\mathcal{CT}$ invariance in the new family of Hamiltonians H_A, H_B associated with the potentials $V_{2j}(t)$ of (1.20) because $H_A\Psi_s(t) = s(1-s)\Psi(t)$ and $H_B\Psi_s(\frac{1}{t}) = s(1-s)\Psi_s(\frac{1}{t})$ would only be valid when j = even as a result of the relations (1.1)–(1.3), (1.20), and (1.21).

The Mellin transform based on the weighted $\Theta_{2j}(t)$ [5] requires once again to *extract* the zero mode $n = 0$ contribution of $\Theta_{2j}(t)$ (to *regularize* the divergent integrals) in order to arrive at

$$\int_0^\infty \frac{1}{2}[\Theta_{2j}(t) - (8\pi)^{-2j} H_{2j}(0)] t^{s/2-1} dt = P_j(s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \quad Re\, s > 0, \tag{1.22}$$

in the definition of the (regularized) inner products of the eigenstates associated to the new potentials (1.20). The polynomial pre-factor in front of the completed Riemann zeta $Z(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is given in terms of a terminating hypergeometric series [5]

$$P_j(s) = (8\pi)^{-j}(-1)^j \frac{(2j)!}{j!} {}_2F_1\left(-j, \frac{s}{2}; \frac{1}{2}; 2\right). \tag{1.23}$$

The orthogonal states $\Psi_{s_n}(t)$ to the ground state $\Psi_{s_o}(t)$ ($s_o = \frac{1}{2} + i0$) will now be enlarged to include the nontrivial zeta zeros and the zeros of the polynomial $P_j(s)$.

The polynomial $P_j(s)$ has simple zeros on the critical line $Re\, s = \frac{1}{2}$, obeys the functional relation $P_j(s) = (-1)^j P_j(1-s)$ and in particular $P_j(s = \frac{1}{2}) = 0$ when j = odd [5]. It is only when j = even that $P_j(s = \frac{1}{2}) \neq 0$ and when we can implement $\mathcal{CT}$ invariance resulting from the relation (1.21) and which is consistent with the results of Eqs. (1.8), (1.10a) and (1.10b).

In order to avoid the regularization of the integrals involving the Mellin transform (1.22), we proposed another family of theta series where *no* regularization is needed in the construction of the inner products. There is a two-parameter family of theta series $\Theta_{2j,2m}(t)$ that yield well-defined inner products *without* the need to extract the zero mode $n = 0$ divergent contribution. Given

$$e^{2V_{2j,2m}(t)} = \Theta_{2j,2m}(t) \equiv \sum_{n=-\infty}^{n=\infty} n^{2m} H_{2j}(n\sqrt{2\pi t}) e^{-\pi n^2 t}. \tag{1.24}$$

When $m \neq 0$, the zero mode $n = 0$ does *not* contribute to the sum and the Mellin transform of $\Theta_{2j,2m}(t)$, after exploiting the symmetry of the even-degree Hermite polynomials, is [4, 5]

$$\int_0^\infty \left[2 \sum_{n=1}^{n=\infty} n^{2m} H_{2j}(n\sqrt{2\pi t}) e^{-\pi n^2 t} \right] t^{s/2-1} dt$$
$$= 2(8\pi)^j P_j(s) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s-2m); \quad Re\, s > 1 + 2m, \quad m = 1, 2, \ldots . \tag{1.25}$$

In order to find the analytical continuation of the Mellin transform (1.25) for *all* values of s in the complex plane we must use the analytical continuation of $\zeta(s)$ was found by Riemann in his celebrated paper. A Poisson re-summation formula for $\Theta_{2j,2m}(t)$ (1.24) leads to similar modular behavior as Eq. (1.21) and only when j = even one can implement $\mathcal{CT}$ invariance in the new family of Hamiltonians H_A, H_B associated with the new potentials $V_{2j,2m}(t)$ of (1.24).

Therefore, one has now at our disposal a well-defined inner product of the states $\Psi_s(t)$ (*without* the need to regularize it by extracting out the zero $n = 0$ mode of the theta series). In particular, the inner product of the states $\Psi_s(t)$ with the *shifted* "ground" states $\Psi_{\frac{1}{2}+2m}(t), m = 1, 2, \ldots$ corresponding to the potentials in (1.24), by recurring to the result (1.25) and following similar steps as in (1.16) is

$$\langle \Psi_{\frac{1}{2}+2m}(t) \,|\, \Psi_s(t) \rangle = -2(8\pi)^j P_j(s+2m) \pi^{-(s+2m)/2} \Gamma\left(\frac{s+2m}{2}\right) \zeta(s), \tag{1.26}$$

this result requires *fixing* uniquely the values $l = -2; k = \frac{1}{4}$. The nontrivial zeta zeros s_n correspond to the states $\Psi_{s_n}(t)$ *orthogonal* to the shifted "ground" states $\Psi_{\frac{1}{2}+2m}(t)$ in Eq. (1.26):

$$\langle \Psi_{\frac{1}{2}+2m}(t) \,|\, \Psi_{s_n}(t) \rangle = -2(8\pi)^j P_j(s_n+2m) \pi^{-(s_n+2m)/2}$$
$$\times \Gamma\left(\frac{s_n+2m}{2}\right) \zeta(s_n) = 0; \quad m = 1, 2, 3, \ldots . \tag{1.27}$$

It remains to prove when $l = -2, k = \frac{1}{4}$, and $s_{12} = s_1^* + s_2 = s_1^* + (1 - s_1^*) = 1$ that

$$\langle \Psi_s | \mathcal{CT} | \Psi_s \rangle = \langle \Psi_s || \Psi_{1-s^*} \rangle$$
$$= \int_0^\infty \left[2 \sum_{n=1}^{n=\infty} n^{2m} H_{2j}(n\sqrt{2\pi t}) e^{-\pi n^2 t} \right] t^{\frac{2(-s_{12}+2k)}{2l} - 1} dt$$
$$= -2(8\pi)^j P_j\left(s = \frac{1}{2}\right) \pi^{-1/4} \Gamma\left(\frac{1}{4}\right) \zeta\left(\frac{1}{2} - 2m\right)$$
$$\neq 0; \quad j = \text{even}, \quad m = 1, 2, 3, \ldots . \tag{1.28}$$

Hence, one arrives at a definite solid conclusion based on a well-defined inner product: because $\zeta(\frac{1}{2} - 2m) \neq 0$ when $m = 1, 2, \ldots$, and $P_j(\frac{1}{2}) \neq 0$ when j = even

in Eq. (1.28), the pseudo-norm $\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle \neq 0$, and this rules out Case A in Eq. (1.14), and singles out Case B in Eq. (1.15) leading to the conclusion that $E_s = s(1-s) = \text{real} \Rightarrow s = \frac{1}{2} + i\rho$ (and/or $s = \text{real}$), and consequently the RH follows if, and only if, $\mathcal{CT}$ invariance holds. The key reason why the RH follows is due to the $\mathcal{CT}$ invariance of the Hamiltonians H_A, H_B and that the pseudo-norm $\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle$ *is not null*. Had the pseudo-norm $\langle\Psi_s|\mathcal{CT}|\Psi_s\rangle$ been *null*, the RH would have been false. It remains to be seen whether our procedure is valid to prove the grand-RH associated to the L-functions.

2. The Dirac and Schroedinger Operators that Reproduce the Zeta Zeros

The previous section was devoted to a family of scaling operators needed in the construction of a pair of non-Hermitian Hamiltonians, involving $\Theta(t)$ functions, whose spectrum $E_s = s(1-s) = E_s^* = s^*(1-s^*)$ was shown to be real-valued resulting from $\mathcal{CT}$ invariance, and whose solutions for s are $s = \frac{1}{2} + i\rho$ and/or $s = \text{real}$, showing how the RH is a physical realization of $\mathcal{CT}$ invariant quantum mechanics (QM). In this section, we will find the Dirac-like operator (with a potential $V(x)$) in one dimension whose spectrum reproduces *exactly* the imaginary parts (ordinates) $\rho_n = E_n$ of the zeta zeros: $\zeta(\frac{1}{2} \pm iE_n) = 0$. At the end, we will also provide a *different* potential $V(x)$ associated with a Schroedinger operator in one dimension that provides the same spectrum $\rho_n = E_n$.

The Dirac-like equation in one dimension in the presence of a potential $V(x)$ is

$$\left\{-i\frac{\partial}{\partial x} + V(x)\right\}\Psi_E(x) = E\Psi_E(x); \quad \hbar = c = 1, \tag{2.1}$$

where the one-dim Clifford algebra with $2^1 = 2$ generators is realized in terms of the unit element $\mathbf{1}$ and a 1×1 matrix γ whose entry is $-i$. Equation (2.1) is the operator representation of the (constraint) dispersion relation

$$\mathcal{P}(x) + V(x) = E, \quad \text{when } \mathcal{P} \to -i\frac{\partial}{\partial x}, \tag{2.2}$$

such that the Bohr–Sommerfield quantization condition yields the number of energy levels $E_1, E_2, \ldots, E_n$ (the number of the first n zeta zeros on the critical line)

$$\begin{aligned}\frac{2}{\pi}\int_{x_o=0}^{x_n}\mathcal{P}(x)dx &= \frac{2}{\pi}\int_{x_o=0}^{x_n}[E_n - V(x)]dx \\ &= \frac{2}{\pi}\int_{V_o}^{E_n}(E_n - V)\frac{dx}{dV}dV = \mathcal{N}(E_n) - \mathcal{N}(V_o).\end{aligned} \tag{2.3}$$

We have set the lower integration limits at $x = 0$ because we assume that the potential is symmetric. In fact, the potential will also turn out to be *multi-valued*. The potential used by Wu–Sprung [13] for the Schroedinger operator

$$\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{WS}(x)\right\}\Psi = E\Psi; \quad \hbar^2 = 2m = 1, \tag{2.4}$$

turned out to be symmetric $V_{WS}(-x) = V_{WS}(x)$ for the choice of the average energy level counting function given by

$$N(E) = \frac{E}{2\pi}\left[\log\left(\frac{E}{2\pi}\right) - 1\right] + \frac{7}{8}, \tag{2.5}$$

(where log is the natural Neper logarithm in the Euler number base) after recurring to the solutions to Abel's integral equation of the first kind obtained after differentiation w.r.t. the E parameter of the Bohr–Sommerfield quantization condition

$$\frac{2}{\pi}\int_0^E \sqrt{E-V}dx = \frac{2}{\pi}\int_{V_o}^E \sqrt{E-V}\frac{dx}{dV}dV = N(E) - N(V_o), \tag{2.6}$$

where $V_o = V(x=0)$ was chosen to obey the boundary condition $N(V_o) = 0$

$$N(V_o) = \frac{V_o}{2\pi}\left[\log\left(\frac{V_o}{2\pi}\right) - 1\right] + \frac{7}{8} = 0 \Rightarrow V_o \sim 3.10073\pi. \tag{2.7}$$

A differentiation of Eq. (2.6) w.r.t. to E (using Leibnitz rule) gives

$$\begin{aligned}\frac{2}{\pi}\frac{d}{dE}\int_{V_o}^E \sqrt{E-V}\frac{dx}{dV}dV &= \frac{1}{\pi}\int_{V_o}^E \frac{1}{\sqrt{E-V}}\frac{dx}{dV}dV \\ &= \frac{dN(E)}{dE} = \frac{1}{2\pi}\log\left(\frac{E}{2\pi}\right).\end{aligned} \tag{2.8}$$

The above equation belongs to the family of Abel's integral equations associated with the unknown function $f(V) \equiv (dx/dV)$

$$\mathcal{J}^\alpha\left[\frac{dx}{dV}\right] = \frac{1}{\Gamma(\alpha)}\int_0^E \frac{(dx/dV)}{(E-V)^{1-\alpha}}dV = \frac{1}{2\pi}\log\left(\frac{E}{2\pi}\right); \quad 0<\alpha<1. \tag{2.9}$$

The reason one had to differentiate Eq. (2.6) w.r.t. E is to enforce the condition $0<\alpha<1$. Abel's integral equation is basically the action of a fractional derivative operator $\mathcal{J}^\alpha$ [12], for the particular value $\alpha = \frac{1}{2}$, on the unknown function $f(V) = (dx/dV)$. Inverting the action of the fractional derivative operator (fractional anti-derivative) yields the solution for

$$\frac{dx}{dV} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dV}\int_0^V \frac{1}{2\pi}\log\left(\frac{E}{2\pi}\right)\frac{1}{(V-E)^\alpha}dE. \tag{2.10a}$$

After setting the value $\alpha = \frac{1}{2}$ in (2.10a), Wu–Sprung [13] found the solution in terms of *quadratures* for the dx/dV function, and a subsequent integration w.r.t. V, gives

$$x = x(V) = \frac{1}{\pi}\sqrt{V-V_o}\log\left(\frac{V_o}{2\pi e^2}\right) + \frac{1}{\pi}\sqrt{V}\log\left[\frac{\sqrt{V}+\sqrt{V-V_o}}{\sqrt{V}-\sqrt{V-V_o}}\right], \tag{2.10b}$$

where $e = 2.71828\ldots$ is Euler's number and when $V = V_o \Rightarrow x_o = x(V_o) = 0$ consistent with the condition $V(x=0) = V_o \sim 3.10073\pi$. The sought-after potential $V_{WS}(x)$ that reproduces the *average* level density of zeta zeros (energy eigenvalues)

given by $N(E)$ is *implicitly* given from the relation $x(V)$ upon inverting the function. It was after the fitting process of the first 500 Riemann zeta zeros on the critical line when Wu–Sprung found numerically that a *fractal-shaped* potential (obtained as a perturbation of the smooth $V_{WS}(x)$) of dimension $d = 1.5$ was needed. A further fitting of the first 4000 zeros furnished identical results for the fractal dimension $d = 1.5$ [15] associated with the shape of the potential.

Based on these findings, we proposed within the context of supersymmetric quantum mechanics (SQM) a Weierstrass fractal function [16] as the fractal-shaped corrections to the smooth potential (2.10b) and consistent with the numerical findings by [13, 15] in order to model the fractal behavior of the potential that fitted those zeta zeros. Later on, Slater [17] performed an exhaustive detailed numerical analysis of our Weierstrass fractal function (and other fractal functions) to find a numerical fit for the first $n = 25, 50, 75, \ldots$ zeta zeros.

The relevant feature of the expression for $x(V)$ (2.10b) is that it is explicitly given in terms of square roots (quadratures), such that changing the signs of the square roots containing the variable V will yield a change of sign: $x(-\sqrt{V}) = -x(\sqrt{V})$ consistent with the assumption that V was symmetric $V(-x) = V(x)$. One can verify why this must be so from Abel's solution (2.10a). If (dx/dV) changes to $-(dx/dV)$ by replacing $x \to -x$, leaving V fixed, one must take the *minus* sign of the $1/\sqrt{V-E}$ terms appearing in the r.h.s. of (2.10a) when $\alpha = \frac{1}{2}$.

In general, when one replaces the average level counting function $N(E)$ by the more general expression $\mathcal{N}(E)$ obtained by Riemann–von Mangoldt formula (given below in Eq. (2.12)) one has the solution (for $\alpha = \frac{1}{2}$)

$$\frac{dx}{dV} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dV} \int_0^V \frac{d\mathcal{N}(E)}{dE} \frac{1}{(V-E)^\alpha} dE$$

$$\Rightarrow x(V) - x(V_o) = \frac{1}{\Gamma(1-\alpha)} \int_0^V \frac{d\mathcal{N}(E)}{dE} \frac{1}{(V-E)^\alpha} dE; \quad \alpha = \frac{1}{2}. \tag{2.11}$$

At the end of this section, we shall return to solution (2.11) corresponding to the Schroedinger operator. Solutions to Eq. (2.11) for a *truncated* version of the Riemann–von Mangoldt formula (2.12), where the oscillatory terms and the integral terms were dropped, was given by Slater [17] using the Mathematica Integrator package.

By recurring to a Dirac-like operator it allows to use the *full* expression for number of zeros (energy levels) $\mathcal{N}(E)$, including the fluctuating/oscillatory terms, leading to a *symmetric* and multi-valued potential due to the oscillatory terms in $\mathcal{N}(E)$. The coordinate function $x(V)$ is assumed to be single-valued, but its inverse, the potential $V(x)$ is not necessarily single-valued, and in fact, it will turn out to be multi-valued. The typical example is the sine function $x = \sin(V)$ (single-valued) whose inverse $V = \arcsin(x)$ is multi-valued.

A knowledge of the functional form of the number of zeros $\mathcal{N}(E)$ in the above integral-differential equation (2.3) gives the potential $V(x)$ implicitly. Let us write the functional form for $\mathcal{N}(E)$ to be given by the Riemann–von Mangoldt formula which is valid for $E \geq 1$

$$\mathcal{N}_{RvM}(E) = \frac{E}{2\pi}\left[\log\left(\frac{E}{2\pi}\right) - 1\right] + \frac{7}{8} + \frac{1}{\pi}\arg\left[\zeta\left(\frac{1}{2} + iE\right)\right] + \frac{1}{\pi}\delta(E), \qquad (2.12)$$

where the (infinitely many times) strongly *oscillating* function is given by the argument of the zeta-function evaluated in the critical line

$$S(E) = \frac{1}{\pi}\arg\left[\zeta\left(\frac{1}{2} + iE\right)\right] = \lim_{\epsilon\to 0}\frac{1}{\pi}\mathcal{I}m\log\left[\zeta\left(\frac{1}{2} + iE + \epsilon\right)\right]. \qquad (2.13)$$

The argument of $\zeta(\frac{1}{2} + iE)$ is obtained by the continuous extension of $\arg\zeta(s)$ along the broken line starting at the point $s = 2 + i0$ and then going to the point $s = 2 + iE$ and then to $s = \frac{1}{2} + iE$. If E coincides with the imaginary part of a zeta zero, then

$$S(E_n) = \lim_{\epsilon\to 0}\frac{1}{2}[S(E_n + \epsilon) + S(E_n - \epsilon)]. \qquad (2.14)$$

An extensive analysis of the behavior of $S(E)$ can be found in [20]. In particular, the property that $S(E)$ is a piecewise smooth function with *discontinuities* at the ordinates E_n of the complex zeros of $\zeta(s_n = \frac{1}{2} + iE_n) = 0$. When E passes through a point of discontinuity, E_n, the function $S(E)$ makes a jump equal to the sum of multiplicities of the zeta zeros at that point. The zeros found so far in the critical line are simple [22]. In every interval of continuity (E, E'), where $E_n < E < E' < E_{n+1}$, $S(E)$ is monotonically decreasing with derivatives given by

$$S'(E) = -\frac{1}{2\pi}\log\left(\frac{E}{2\pi}\right) + \mathcal{O}(E^{-2}); \quad S''(E) = -\frac{1}{2\pi E} + \mathcal{O}(E^{-3}). \qquad (2.15)$$

The most salient feature of these properties is that the derivative $S'(E)$ blows up at the location of the zeta zeros E_n due to the discontinuity (jump) of $S(E)$ at E_n. Also, the strongly oscillatory behavior of $S(E)$ forces the potential $V(x)$ to be a multi-valued function of x.

The expression for $\delta(E)$ is [20]

$$\delta(E) = \frac{E}{4}\log\left(1 + \frac{1}{4E^2}\right) + \frac{1}{4}\arctan\left(\frac{1}{2E}\right) - \frac{E}{2}\int_0^\infty \frac{\rho(u)du}{(u + 1/4)^2 + (E/2)^2}, \qquad (2.16)$$

with $\rho(u) = \frac{1}{2} - \{u\}$, where $\{u\}$ is the fractional part of u and which can be written as $u - [u]$, where $[u]$ is the integer part of u. In this way, one can perform the integral involving $[u]$ in the numerator by partitioning the $[0, \infty]$ interval in intervals of unit length: $[0, 1], [1, 2], [2, 3], \ldots, [n, n + 1], \ldots$. The definite integral when the upper

limit is bounded by an ultraviolet regulator Λ is

$$-\frac{E}{2}\int_0^{\Lambda}\frac{\rho(u)du}{(u+1/4)^2+(E/2)^2}=\frac{E}{4}\log\left[\frac{(E/2)^2+(\Lambda+1/4)^2}{(E/2)^2+(1/4)^2}\right]$$
$$-\sum_{n=1}^{[\Lambda]}n\left[\arctan\left(\frac{4n+5}{2E}\right)-\arctan\left(\frac{4n+1}{2E}\right)\right]$$
$$-\frac{3}{4}\left[\arctan\left(\frac{4\Lambda+1}{2E}\right)-\arctan\left(\frac{1}{2E}\right)\right]. \tag{2.17a}$$

It is the Euler–Maclaurin summation formula

$$\sum_{k=1}^{N-1}f_k=\int_0^N f(k)dk-\frac{1}{2}[f(0)+f(N)]$$
$$+\frac{1}{12}[f'(N)-f'(0)]-\frac{1}{720}[f'''(N)-f'''(0)]+\cdots \tag{2.17b}$$

that permits the exact evaluation of the $\Lambda\to\infty$ limit of the expression (2.17a): the divergent terms $E\log(\Lambda)$ in (2.17a) cancel out exactly leading to the $\delta(E)$ terms of (2.16)

$$\delta(E)=-\frac{E}{4}+\frac{E^2}{8}\left[\arctan\left(\frac{5}{2E}\right)-\arctan\left(\frac{1}{2E}\right)\right]$$
$$+\frac{1}{32}\left[33\arctan\left(\frac{1}{2E}\right)-25\arctan\left(\frac{5}{2E}\right)\right]$$
$$+\frac{E}{16}\left[5\log\left(1+\frac{25}{4E^2}\right)-\log\left(1+\frac{1}{4E^2}\right)\right]$$
$$+\frac{1}{12}\left[\arctan\left(\frac{5}{2E}\right)-\arctan\left(\frac{1}{2E}\right)\right]+\cdots. \tag{2.17c}$$

For large E, a Taylor expansion of $\delta(E)$ gives

$$\delta(E)=\frac{1}{48}\frac{1}{E}+\mathcal{O}\left(\frac{1}{E^3}\right). \tag{2.17d}$$

Thus the leading term of $\delta(E)$ is of the order $(1/E)$ as expected. However, it is important to keep *all* the terms involving E given by (2.17c) when E is not large.

We must search now for solutions to the integral equation associated with the Dirac-like operator in one dimension

$$\frac{1}{\pi}\int_{V_o}^{E}(E-V)\frac{dx}{dV}\,dV=\mathcal{N}(E)-\mathcal{N}(V_o), \tag{2.18}$$

associated with the unknown function $f=f(V)\equiv\frac{dx}{dV}$ and subject to the boundary condition $V(x=0)=V_o$, i.e. the integral transform of $f(V)$ defined by Eq. (2.18) is the counting function $\mathcal{N}(E)$. The solutions to Abel's integral equations will not be

necessary in our case to find dx/dV. What should the choice of V_o be? To answer this question we need to discuss the following points. The integral (2.18) is trivially zero when the upper limit E coincides with V_o, which is consistent with the trivial fact: $\mathcal{N}(E = V_o) - \mathcal{N}(V_o) = 0$.

Despite that the Riemann–von Mangoldt expression (2.12) is only valid for $E \geq 1$, one can still verify by inspection that when $V_o = 0 \Rightarrow \mathcal{N}(E = V_o = 0) = 0$. This can be seen if one chooses the argument of $\zeta(1/2) = -1.46$ to be given by $-\pi$, instead of π. With this choice for the argument and taking $\arctan(\infty) = \frac{\pi}{2}$, then (2.12) becomes

$$\mathcal{N}(E = 0) = \frac{7}{8} + \frac{1}{\pi}(-\pi) + \frac{1}{4\pi}\frac{\pi}{2} = 0. \tag{2.19}$$

Had one chosen the argument π one would have $\mathcal{N}(E = 0) = 2$ which is the wrong answer since there are no zeros at $\zeta(1/2)$. The choice $V_o = 0$ is a very natural one from the physical point of view and compatible with the $E = 0$ ground state of SQM in one dimension. The super-potential $W(x)$ in SQM vanishes at $x = 0$ if supersymmetry is not broken.

Upon taking derivatives on both sides of Eq. (2.18) w.r.t. to E gives

$$\frac{2}{\pi}\int_0^E \frac{dx(V)}{dV}dV = \frac{2}{\pi}\int_0^x dx = \frac{2x(E)}{\pi} = \frac{d\mathcal{N}(E)}{dE}. \tag{2.20}$$

Notice that despite the derivatives $\mathcal{N}'(E)$ *blow* up at the location of the zeta zeros $E = E_n$, due to the discontinuity (jump) of $S(E)$ at E_n, the expression (2.20) is nevertheless correct because it just means that the function $x(E)$ also blows up $x(E_n) = \infty$ when $E = E_n$. Therefore, the fact that $\mathcal{N}'(E)$ blows up at a discrete number of locations $E = E_n$ does not preclude us from differentiating both sides of Eq. (2.18) w.r.t. to E.

From the above relation (2.20), we will show that the solution to the integral equation (2.18) is

$$\frac{2x}{\pi} = \frac{2x(V)}{\pi} = \frac{d\mathcal{N}(V)}{dV} \equiv \rho(V), \tag{2.21}$$

where $\mathcal{N}(V)$ has the *same functional form* as $\mathcal{N}(E)$. The physical interpretation of (2.21) is that the coordinate function $x = x(V)$ is just proportional to the density of zeros $\rho(V)$ (times $\pi/2$): the number of zeros per unit of energy. On dimensional grounds this makes sense, since length has the dimensions of *inverse* of energy when $\hbar = c = 1$. Therefore, one can infer that when $V = E_n \Rightarrow x = x(E_n) = x_n = \infty$ for all values of $n = 1, 2, 3, \ldots$ due to the *singular* behavior of the derivatives $\mathcal{N}'(V = E_n)$ at the ordinates of the zeta zeros resulting from the discontinuity of the argument of the zeta-function at E_n. The last expression (2.21) for $x(V)$ furnishes the sought-after potential $V = V(x)$ in *implicit* form. $x(V)$ is single-valued but $V(x)$ is multi-valued. In particular, $V(x = \infty) = E_n, n = 1, 2, 3, \ldots$.

Equipped with the known expression for the functional form of $N(V)$ (2.12) (after replacing E for V) the quantization condition (2.18) reads

$$\int_{V_o}^{E}(E-V)\frac{d^2\mathcal{N}(V)}{dV^2}dV=\mathcal{N}(E)-\mathcal{N}(V_o). \tag{2.22}$$

Taking derivatives on both sides of (2.22) w.r.t. to E and using the most general Leibnitz formula for differentiation of a definite integral when the upper $b(E)$ and lower $b(E)$ limits are functions of a parameter E:

$$\begin{aligned}\frac{d}{dE}\int_{a(E)}^{b(E)}f(V;E)dV=&\int_{a(E)}^{b(E)}\left(\frac{\partial f(V;E)}{\partial E}\right)dV\\&+f(V=b(E);E)\left(\frac{db(E)}{dE}\right)\\&-f(V=a(E);E)\left(\frac{da(E)}{dE}\right),\end{aligned} \tag{2.23}$$

leads to

$$\int_{V_o}^{E}\frac{d^2\mathcal{N}(V)}{dV^2}dV=\frac{d\mathcal{N}(E)}{dE}. \tag{2.24}$$

Since the lower limit V_o is taken to be independent of E and the integrand vanishes in the upper limit $V=b(E)=E$. The integral (2.24) is straightforward

$$\frac{d\mathcal{N}(V)}{dV}(V=E)-\frac{d\mathcal{N}(V)}{dV}(V_o)=\frac{d\mathcal{N}(E)}{dE}, \tag{2.25}$$

from which one infers that one must have $\mathcal{N}'(V_o)=\frac{d\mathcal{N}(V)}{dV}(V_o)=0$ since the *functional* forms of $\mathcal{N}(V)$ and $\mathcal{N}(E)$ are the *same*.

However, there is a potential problem because there *is no assurance* that the function $\mathcal{N}(V)$ *obeys* the condition $\mathcal{N}'(V_o)=0$ unless one chooses the value of V_o to be the solution to the transcendental equation

$$\begin{aligned}&\frac{1}{2\pi}\log\left(\frac{V_o}{2\pi}\right)+\frac{1}{\pi}\mathcal{I}m\frac{i\zeta'(\frac{1}{2}+iV_o)}{\zeta(\frac{1}{2}+iV_o)}+\frac{1}{4\pi}\log\left(1+\frac{1}{4V_o^2}\right)-\frac{3}{2\pi}\frac{1}{1+4V_o^2}\\&-\frac{1}{2\pi}\int_0^\infty\frac{\rho(u)du}{(u+1/4)^2+(V_o/2)^2}+\frac{V_o}{2\pi}\int_0^\infty\frac{\rho(u)(V_o/2)du}{[(u+1/4)^2+(V_o/2)^2]^2}=0.\end{aligned} \tag{2.26}$$

If, and only if, there is a real-valued solution V_o to Eq. (2.26) that fixes the value of the zero-point energy V_o and that falls in the range $1\leq V_o<E_1$, then one has that $x(V)=\frac{\pi}{2}(d\mathcal{N}(V)/dV)$ yields the potential $V(x)$ in implicit form reproducing the ordinates of the zeta zeros for the spectrum E_n.

However, if there is *no* real-valued solution V_o to the transcendental equation (2.26) that falls in the range $1\leq V_o<E_1$, then one can go ahead and truncate the upper limit of the definite integral appearing in the definition of $\delta(E)$ in Eqs. (2.16) and (2.17) by introducing an E-dependent ultraviolet cut-off $\Lambda=\Lambda(E)$, such that

the zero derivative condition is modified from the form given by (2.26) to the one given by

$$\mathcal{N}'(V_o,\Lambda(V_o)) = \frac{1}{2\pi}\log\left(\frac{V_o}{2\pi}\right) + \frac{1}{\pi}\mathcal{I}m\frac{i\zeta'(\frac{1}{2}+iV_o)}{\zeta(\frac{1}{2}+iV_o)} + \frac{1}{4\pi}\log\left(1+\frac{1}{4V_o^2}\right)$$
$$-\frac{3}{2\pi}\frac{1}{1+4V_o^2} - \frac{1}{2\pi}\int_0^{\Lambda(V_o)}\frac{\rho(u)du}{(u+1/4)^2+(V_o/2)^2}$$
$$+\frac{V_o}{2\pi}\int_0^{\Lambda(V_o)}\frac{\rho(u)(V_o/2)du}{[(u+1/4)^2+(V_o/2)^2]^2}$$
$$-\frac{V_o}{2\pi}\frac{\rho(\Lambda(V_o))}{[(\Lambda(V_o))+(1/4)]^2+(V_o/2)^2}\left(\frac{d\Lambda(V)}{dV}\right)(V=V_o)=0. \tag{2.27}$$

Therefore, the above condition $\mathcal{N}'(V_o,\Lambda(V_o))=0$ provides the necessary constraint between V_o and $\Lambda(V_o)$ to satisfy our goal. It is customary in the Renormalization Group (RG) process in Quantum Field Theories (QFT) to introduce an energy cut-off; here $\Lambda(E)$ is a running and increasing function of E which tends to infinity when $E\to\infty$.

To sum up, if there is a real-valued solution V_o to Eq. (2.26) that fixes the value of the zero-point energy V_o and that falls in the range $1\leq V_o<E_1$, then $x(V)=\frac{\pi}{2}(d\mathcal{N}(V)/dV)$ yields the potential $V(x)$ in implicit form. If there is *no* real-valued solution V_o to the transcendental equation (2.26) that falls in the range $1\leq V_o<E_1$, then one truncates the upper limit of the definite integral leading to a modified Eq. (2.27) and $x(V)=\frac{\pi}{2}\frac{d\mathcal{N}(V,\Lambda(V))}{dV}$ determines the potential implicitly in terms of the cut-off function $\Lambda(V)$.

Next we describe how one determines the functional form of the cut-off function $\Lambda(V)$ in such a case. Because $\Lambda(E)$ is a cut-off function that runs with energy E, one has now enough freedom to impose the exact conditions

$$\mathcal{N}(E_n;\Lambda(E_n))-\mathcal{N}(V_o;\Lambda(V_o))=n;\quad n=1,2,3,\ldots;\quad 1\leq V_o<E_1, \tag{2.28a}$$

when $E_1,E_2,E_3,\ldots,E_n,E_{n+1},\ldots$ are the (positive) imaginary parts (ordinates) of the zeta zeros in the critical line. In order to evaluate $\mathcal{N}(V)$ at $V=E_n$ due to the discontinuity of the fluctuating term $S(E)$ of (2.12) at E_n one must take the arithmetic mean as described by Eq. (2.14). The upper limit of the values of V_o should be bounded by the first zero E_1 avoiding having potential singularities in Eq. (2.27) due to the zeros of zeta appearing in the denominator of second term. The lower bound of V_o should be 1 since the domain of validity of the Riemann–von Mangoldt expression is $E\geq 1$.

If one were to replace the values $\Lambda(E_n)=\lambda_n$ for $\Lambda=\infty$ one may rewrite Eq. (2.28) as follows

$$\mathcal{N}(E_n;\lambda_n)-\mathcal{N}(V_o;\Lambda(V_o)) = [\mathcal{N}(E_n;\Lambda=\infty)+\Delta(\lambda_n,E_n)]-\mathcal{N}(V_o;\Lambda(V_o))$$
$$=(n-\delta_n)+\Delta(\lambda_n,E_n)-\mathcal{N}(V_o;\Lambda(V_o))=n, \tag{2.28b}$$

where the number of levels (zeros) just below the nth zero given by the Riemann–von Mongoldt expression are $\mathcal{N}(E_n;\Lambda=\infty)=n-\delta_n$ (if the RH is true), and δ_n is a fraction such that $0<\delta_n<1$. The positive-definite quantity $\Delta(\lambda_n,E_n)$ is the deficit value of the integral appearing in Eqs. (2.16) and (2.17) given by $(E_n/2)\int_{\lambda_n}^{\infty}(\ldots)$.

Finally, one can derive implicitly the potential of the Dirac-like operator that reproduces the zeta zeros from

$$\frac{2x}{\pi}=\frac{2x(V)}{\pi}=\frac{d\mathcal{N}(V;\Lambda(V))}{dV}=\frac{\partial\mathcal{N}(V;\Lambda(V))}{\partial V}+\frac{d\Lambda(V)}{dV}\frac{\partial\mathcal{N}(V;\Lambda(V))}{\partial\Lambda(V)}$$

$$\Rightarrow x_o=x(V_o)=\frac{\pi}{2}\frac{d\mathcal{N}(V;\Lambda(V))}{dV}(V=V_o)=0.$$

Next we describe how to solve the system of Eqs. (2.27) and (2.28). First, one begins by truncating the series expansion for the cut-off function $\Lambda(V)$ as follows

$$\Lambda(V)=\sum_{k=0}^{k=\infty}a_kV^k\Rightarrow\Lambda(V;N)=\sum_{k=0}^{k=N}a_kV^k$$

$$\Rightarrow\left(\frac{d\Lambda(V,N)}{dV}\right)=\sum_{k=0}^{k=N}a_kkV^{k-1}. \tag{2.29}$$

We are going to display two case scenarios on how to solve Eqs. (2.27) and (2.28). In the first case, we are going to drastically *simplify* these equations by choosing $V_o=0$, despite that the domain of values for V_o in the definition of $\mathcal{N}(V_o)$ given by the Riemann–von Mangoldt formula should be $1\leq V_o<E_1$. In the second case scenario, we shall enforce the latter conditions on V_o. Hence, Eqs. (2.27) and (2.28) are automatically simplified by setting $V_o=0\Rightarrow\mathcal{N}(V_o=0;\lambda_o)=0$, which follows from Eq. (2.19), so now the value of $\lambda_o=\Lambda(V_o=0)$ is determined from the relation (2.27) for $V_o=0$

$$-\frac{1}{2\pi}\log(4\pi)+\frac{1}{\pi}\mathcal{I}m\frac{i\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})}-\frac{3}{2\pi}-\frac{1}{2\pi}\int_0^{\lambda_o}\frac{\rho(u)du}{(u+1/4)^2}=0, \tag{2.30}$$

after noticing a cancellation between the singular $\log(V_o=0)-\log(V_o=0)$ terms associated with the first and third terms of (2.27) resulting from the relation $\lim_{V_o\to 0}(1/4\pi)\log(1+\frac{1}{4V_o^2})\to-(1/2\pi)\log(V_o=0)=+\infty$. The value of the integral in (2.30) can be inferred from $(-2/E)\times$ the value of the integral in Eq. (2.17). When $E\to 0$, it requires the use of L'Hopital's rule giving a finite result for a finite value of λ_o. The values $\zeta'(\frac{1}{2})=-3.92265$; $\zeta(\frac{1}{2})=-1.46$ in Eq. (2.30) yields a negative value for λ_o given by

$$-0.0787=\frac{1}{2}\int_0^{\lambda_o}\frac{\rho(u)du}{(u+1/4)^2}=-\frac{1}{2}\int_{\lambda_o}^{0}\frac{\rho(u)du}{(u+1/4)^2};\quad\text{with}\ -\frac{1}{4}<\lambda_o<0.$$

The quantization conditions (2.28a) and (2.28b) corresponding to $\mathcal{N}(V_o = 0; \lambda_o) = 0$, and $\Lambda(V_o = 0) \equiv \lambda_o < 0$ given by the solution to Eq. (2.30), and by writing $\Lambda(E_n) \equiv \lambda_n$, become

$$\mathcal{N}(E_n; \lambda_n) - \mathcal{N}(V_o = 0; \lambda_o) = \mathcal{N}(E_n; \lambda_n) = n; \quad n = 1, 2, 3, \ldots, \infty$$
$$\Rightarrow \Delta(\lambda_n, E_n) = \frac{E_n}{2} \int_{\lambda_n}^{\infty} \frac{\rho(u) du}{[(u + 1/4)^2 + (E_n/2)^2]} = \delta_n = n - \mathcal{N}(E_n; \Lambda = \infty). \tag{2.31}$$

$\Delta(\lambda_n, E_n)$ is positive definite because the contributions to the integral in all of the intervals $[[u], [u] + 1]$ which do *not* contain λ_n are all positive due to the increasing values of the denominator, whereas the magnitude of the values $\rho(u) = \frac{1}{2} - \{u\}$ are symmetrically distributed about the midpoint of the intervals while being positive and negative definite in the intervals $[[u], [u] + \frac{1}{2}]$, $[[u] + \frac{1}{2}, [u] + 1]$, respectively. The integral in the interval $[[\lambda_n], [\lambda_n] + 1]$ is positive, negative, or zero depending on the location of λ_n. One can always choose the value of λ_n to obey the relations in Eq. (2.31).

To sum up, given a set of $N + 1$ integers $n = 0, 1, 2, 3, \ldots, N$ bounded by N, Eqs. (2.30) and (2.31) yield a system of $N + 1$ equations which in principle determine the values of the $N + 1$ unknown cut-off parameters $\lambda_o, \lambda_1, \lambda_2, \ldots, \lambda_N$ in terms of the ordinates of the zeta zeros $E_1, E_2, E_3, \ldots, E_N$. The value of λ_o has already been fixed from Eq. (2.30). Equation (2.31) determines the remaining one $\lambda_n, n = 1, 2, 3, \ldots, N$.

Finally, after solving Eqs. (2.30) and (2.31), the defining relations

$$\Lambda(V_o; N) = \sum_{k=0}^{k=N} a_k (V_o)^k = \lambda_o \Rightarrow \Lambda(V_o = 0; N) = a_o = \lambda_o \tag{2.32}$$

and

$$\Lambda(V = E_n; N) = \sum_{k=0}^{k=N} a_k (E_n)^k = \lambda_n, \tag{2.33}$$

yield a *linear* system of algebraic equations for the N coefficients $a_1, a_2, \ldots, a_N$ associated with the truncating series (2.29), and whose solutions are given in terms of the imaginary parts of the zeta zeros E_n and the values of the cut-off parameters $\Lambda(V_o = 0) = \lambda_o$, $\Lambda(E_n) = \lambda_n$. The latter cut-off parameters have themselves been determined from the solutions to Eqs. (2.27) and (2.28). The solutions for the coefficients a_k can be written compactly in terms of the van der Monde determinant Δ of the $(N + 1) \times (N + 1)$ matrix comprised of N rows whose entries in the nth row are $1, E_n, E_n^2, E_n^3, \ldots, E_n^N$, for $n = 1, 2, 3, \ldots, N$. The first row has entries $1, 0, 0, 0, \ldots, 0$ since $V_o = E_o = 0$, so the total number of rows and columns is $N + 1$. The van der Monde determinant is

$$\Delta = \prod (E_i - E_j), \quad \text{for } i > j. \tag{2.34}$$

The other determinants involved in the solutions Δ_k correspond to the $(N+1)\times(N+1)$ matrices obtained by replacing the kth column by a column comprised of the entries $\lambda_o, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. The solutions for the coefficients that define the cut-off function $\Lambda(E, N)$ at level N are compactly written as

$$a_k^{(N)} = \frac{\Delta_k(\lambda_o, \{\lambda_n\}; V_o = 0; \{E_n\})}{\prod(E_i - E_j)}, \quad i > j, \quad i, j = 0, 1, 2, 3, \ldots, N, \tag{2.35}$$

with $E_o = V_o = 0$. If the large N limit converges

$$\lim_{N\to\infty} a_k^{(N)}(\lambda_o, \lambda_1, \lambda_2, \ldots, \lambda_N; E_1, E_2, \ldots, E_N) \to a_k^*, \quad \text{a fixed point}, \tag{2.36}$$

to a fixed point, the full-fledged energy-dependent cut-off function $\Lambda(E)$ is determined by the infinite series

$$\Lambda(E) = \sum_{k=0}^{k=\infty} a_k^* E^k, \tag{2.37}$$

which is defined in terms of the infinite number of coefficients given by the infinite number of *fixed* points a_k^*. The spectral statistics of Random Matrix Models in the large N limit have been known for a long time to have deep connection to the zeta zeros since Montgomery–Dyson found the pair-correlation functions of the ordinates of the zeta zeros with normalized spacings in terms of an integrand involving the function $1 - (\frac{\sin \pi x}{\pi x})^2$. This function is the pair-correlation function for the eigenvalues of very large Random Hermitian matrices measured with a Gaussian measure (the Gaussian Unitary Ensemble) [21].

Since fixed points in RG techniques in QFT are ubiquitous, it is warranted to explore the connections among the putative fixed points a_k^* with the fixed points associated with the beta function in QFT. A RG analysis was performed by Peterman [23] to shed some light as to why the density of prime numbers decreases as $1/\log x$. The zeta-function has also been used extensively in regularization methods (of infinities) in QFT, see [24] and references therein. The "Russian Doll Renormalization" group has been found to have connections to the RH [19].

Finally, once the cut-off function $\Lambda(V)$ is constructed from the definition

$$\Lambda(V) = \sum_{k=0}^{k=\infty} a_k^*(\lambda_o, \{\lambda_n\}; V_o = 0; \{E_n\}) V^k, \tag{2.38}$$

for *all* values $n = 1, 2, 3, \ldots, \infty$, the sought-after potential is *implicitly* determined from the fundamental result

$$\frac{2x}{\pi} = \frac{2x(V)}{\pi} = \frac{d\mathcal{N}(V; \Lambda(V))}{dV}, \tag{2.39}$$

where the functional form of $\mathcal{N}(V; \Lambda(V))$ is given by the Riemann–von Mangoldt formula (2.12) by replacing $E \to V$ and by inserting the energy-dependent cut-off $\Lambda(V)$ found in Eq. (2.38) into the *upper* limit of the integral appearing in the definition of the $\delta(E)$ terms in Eq. (2.16). Naturally, due to numerical

limitations, the potential can only be constructed *iteratively*, level by level, N, $N+1, N+2, \ldots, \infty$. Another salient feature to look for is to verify that the family of coefficients $a_k^{(N)}$ does indeed converge to the fixed point values a_k^* when $N \to \infty$. This is where the results of large N Random Matrices methods are relevant.

The second case scenario is more complicated to solve if one forces V_o to lie in the domain $1 \leq V_o < E_1$. Choosing a particular value of $V_o = V_o^*$ in that range, one has for Eq. (2.29)

$$\Lambda(V_o^*; N) = \sum_{k=0}^{k=N} a_k (V_o^*)^k = \lambda_o^* \tag{2.40}$$

and

$$\Lambda(V = E_n; N) = \sum_{k=0}^{k=N} a_k (E_n)^k = \lambda_n, \tag{2.41}$$

when $V_o = V_o^* \neq 0$, one has new solutions for the coefficients $a_o, a_1, a_2, \ldots, a_N$

$$a_k = a_k(\lambda_o^*, \lambda_n; V_o^*, E_n) = \frac{\Delta_k(\lambda_o^*, \{\lambda_n\}; V_o^*, \{E_n\})}{\prod (E_i - E_j)}; \quad i > j, \tag{2.42}$$

for $i, j = 0, 1, 2, 3, \ldots, N$ and where the first row of the matrix involved in the van der Monde determinant now must include the $1, V_o^*, (V_o^*)^2, \ldots, (V_o^*)^N$ terms.

Equipped with the above solutions a_k (2.42) one inserts them into the expression for the derivative term

$$\left(\frac{d\Lambda(V, N)}{dV} \right) (V_o^*) = \sum_{k=0}^{k=N} a_k(\lambda_o^*, \lambda_n; V_o^*, E_n) k (V_o^*)^{k-1}, \tag{2.43}$$

which appears in Eq. (2.27). The latter Eqs. (2.27), (2.40)–(2.43) combined with Eq. (2.28), where now $\mathcal{N}(V_o^*, \lambda_o^*) \neq 0$, furnishes a system of $N+1$ equations which determines in principle the numerical values for the cut-off parameters λ_o^*, λ_n's. From these latter values, one reconstructs the a_k coefficients from Eq. (2.42), and which in turn, will determine the sought-after cut-off function $\Lambda(V; N)$ Eq. (2.29) (at level N), so that finally one can write the full explict form of $\mathcal{N}(V, \Lambda(V))$ (given by the Riemann–von Mangoldt formula). Its derivative yields $\frac{d\mathcal{N}(V,\Lambda(V))}{dV} = (2x(V)/\pi)$ which finally furnishes the form of the sought-after potential $V(x)$ (implicitly). Naturally, the equations to solve in this second case scenario are far more difficult than the ones when one simply chooses $V_o = 0$ simplifying drastically these calculations.

There is a more general third case scenario when one has $N+2$ undetermined parameters $V_o^{**}, \lambda_o^{**}, \lambda_n, n = 1, 2, 3, \ldots, N$ at each level N which are constrained to obey $N+2$ equations given by the $N+1$ Eqs. (2.27) and (2.28), plus an *additional* extra condition involving the second derivatives $\mathcal{N}''(V_o^{**}, \lambda_o^{**}) = 0$. The procedure to solve this far more complicated system of $N+2$ equations is still similar to the second case scenario, the only difference is that now V_o^{**} is *not* put in by hand, but instead is another unknown parameter to be determined from the solutions of

these $N+2$ equations. The open question remains whether or not the solutions for V_o^{**} fall in the range $0 \leq V_o^{**} < E_1$. Even perhaps, the solution for V_o^{**} might be negative. Out of these three cases, the simplest one to follow is the one when one takes $V_o = 0$ which simplifies drastically all the calculations and allows us to provide with actual numerical results. The full numerical analysis of Eqs. (2.27)–(2.29), ... is beyond the scope of this work. It requires very sophisticated computations.

To complete this subsection, we need to discuss the nature of the solutions $\Psi(x)$ to the one-dim Dirac-like equation (2.1) on the line $[-\infty, \infty]$ given by

$$\Psi_E(x) = \Psi_{o,E} \exp\left[i \int_{-\infty}^{x} P(x')dx' \right]$$
$$= \Psi_{o,E} \exp\left[i \int_{-\infty}^{x} (E - V(x'))dx' \right], \tag{2.44}$$

where E is a real-valued *continuous* parameter and $\Psi_{o,E}$ is a constant amplitude.

The operator $-i\partial/\partial x$ would be self-adjoint in the full line $[-\infty, \infty]$, or in the compact interval $[x_a, x_b]$, if one could impose suitable vanishing boundary conditions on the above $\Psi_E(x)$ solutions at $\pm\infty$ and/or at x_a, x_b. However, the solutions $\Psi_E(x)$ are just proportional to a pure phase factor so the Ψ's are nonvanishing for all values of x, unless one constrains the amplitudes $\Psi_{o,E}$ to zero that will render the solutions *trivial*. Therefore, since one cannot find nontrivial solutions $\Psi_E(x)$ obeying the boundary conditions $\Psi_E(x = \pm\infty) = 0$ and/or $\Psi_E(x) = 0$ at x_a, x_b, the operator $-i\partial/\partial x$ is not self-adjoint in $[-\infty, \infty]$, or in the compact interval $[x_a, x_b]$, for the space of solutions given by (2.44). If one had a second-order operator $\mathcal{D}^2 = \frac{-\hbar^2}{2m}(\partial^2/\partial x^2)$ like the Schroedinger operator, there are no problems with finding self-adjoint extensions. For example, a free particle inside a box of length $2L$ admits normalizable wave-functions $\Psi_n(x) \sim \sin(\frac{n\pi x}{L})$ obeying the suitable boundary conditions $\Psi_n(x = \pm L) = 0$.

For these reasons, we conclude that the self-adjointness property is *not* required to fulfill our goals. We saw in Sec. 1 how by working with a pair of *non*-Hermitian Hamiltonian operators was sufficient to show why it was the $\mathcal{CT}$ symmetry which *forced* the energy spectrum to be *real*: $E_s = s(1-s) = E_s^* = s^*(1-s^*)$, leading to the only possible solutions $s = \frac{1}{2} + i\rho$ and/or $s =$ real, and consistent with the fact that the Riemann zeta-function has trivial zeros on the negative even integers and nontrivial zeros in the critical line $\mathcal{R}es = \frac{1}{2}$. For the Dirac-like operator (2.1), all we need is to impose PT symmetry where this time by T reversal symmetry we do not mean inversion $t \to (1/t) \Rightarrow \log(t) \to -\log(t)$, but the standard $t \to -t$ symmetry used in PT symmetric QM.

The momentum $p = dx/dt$ is invariant under PT symmetry since x and t both reverse signs, this means that $i \to -i$ under PT symmetry so that the momentum operator remains invariant $\hat{p} = -i\hbar(\partial/\partial x)$. There is nothing strange by having i change sign under PT symmetry since Clifford algebras in $D = 1$ have two generators, the identity element $\mathbf{1}$ and the 1×1 matrix γ whose entry is just $-i$, so that

$\{\gamma,\gamma\}=2i^2=-2$, if one takes the metric of the one-dim space to be $g_{11}=-1$. Therefore, under PT symmetry $\gamma\to-\gamma$ which implies that $i\to-i$.

This leaves us with having to impose the condition $V(-x)=V(x)$ on the potential in order to have a PT symmetric Dirac-like operator $-i\partial/\partial x+V(x)$. In order to define $V(x)$ in the regions $x<0$ one must choose the *minus* sign in front of $x(V)=-\frac{\pi}{2}\frac{d\mathcal{N}(V,\Lambda(V))}{dV}$. The positive sign selects the solutions in the region $x>0$. Furthermore, it is important to emphasize that the one-dimensional Jacobian (from the change of variables) is (dx/dV) in the $x>0$ region, but it is $-(dx/dV)$ in the mirror $x<0$ region. There is a crucial sign change to ensure that the portion of the line-integral along the left region does not trivially cancel out the portion of the line-integral in the right region. The Bohr–Sommerfeld quantization rule involves the *closed* contour in phase space that in the case of a symmetric potential gives $\oint pdx=4\int_o^\infty pdx=2n\pi$, thus care must be taken with the signs of dx/dV.

To sum up this discussion: the self-adjointness (Hermitian) property is not required to prove the RH. What matters was the $\mathcal{CT}$ symmetry in Sec. 1 and PT symmetry in this section related to the spectrum of the Dirac-like operator in one dimension. To finalize, once we extend the domain of $V(x)$ to the region $x<0$ by taking the mirror image of the potential constructed in this section; the solutions associated with the *discrete* family of zeta zeros E_n (embedded in the continuum of solutions) are simply obtained by inserting the value of $E=E_n$ inside the integrand of (2.45)

$$\Psi_{E_n}(x)=\frac{1}{\sqrt{L}}\exp\left[i\int_{-\infty}^{x}(E_n-V(x'))dx'\right], \tag{2.45}$$

where in the region $x<0$ one must use the branch of the potential solution given by $x(V)=-\frac{\pi}{2}\frac{d\mathcal{N}(V,\Lambda(V))}{dV}$ to ensure that indeed we are selecting solutions which obey $V(-x)=V(x)$. L is an *infrared* cut-off that is required so that the wave-functions $\Psi_{E_n}(x)$ are square integrable on the line

$$\lim_{L\to\infty}\left\{\int_{-L/2}^{L/2}\Psi^{\dagger}_{E_n}(x)\Psi_{E_n}(x)dx\right\}=1. \tag{2.46}$$

One must take the $L\to\infty$ limit *after* the integration (2.47) is performed and not before, otherwise one gets the trivial result for the wave-functions $\Psi=0$.

Finally, when one evaluates the *discrete* family of wave-functions at the cusp points $x_n=+\infty$ (where the values of the potential are $V(x=x_n=+\infty)=E_n$) one arrives at

$$\begin{aligned}\Psi_{E_n}(x=x_n=\infty)&=\frac{1}{\sqrt{L}}\exp\left[i\int_{-\infty}^{x_n=\infty}(E_n-V(x'))dx'\right]\\&=\frac{1}{\sqrt{L}}\exp\left[2i\int_{0}^{x_n=\infty}(E_n-V(x'))dx'\right]\end{aligned}$$

$$= \frac{1}{\sqrt{L}} \exp\{i\pi[\mathcal{N}(E_n;\Lambda(E_n)) - \mathcal{N}(V_o;\Lambda(V_o))]\}$$
$$= \frac{1}{\sqrt{L}} \exp[in\pi] = \frac{(-1)^n}{\sqrt{L}}, \tag{2.47}$$

as a direct result of the conditions in Eqs. (2.28) and (2.3). Therefore, at the cusp points $x = x_n = +\infty$ the wave-functions $\Psi_{E_n}(x_n)$ *alternate* in sign. This changing of sign is related to the presence of Gram points in the Riemann–Siegel formula, with the only difference that the phase factors in (2.47) involve the full-fledged zeros (discrete energy levels E_n) counting function $\mathcal{N}(E,\Lambda(E))$ whereas in the Riemann–Siegel formula only the average energy level counting function is used given by the first two terms of Eq. (2.12) [2].

The values of the wave-functions at the $x = -\infty$ are simply $\Psi_{E_n}(x = -\infty) = \frac{1}{\sqrt{L}}$. For even n the wave-functions are periodic (with an *infinite* period) in the sense that $\Psi_{E_n}(x = -\infty) = \Psi_{E_n}(x = +\infty)$. For odd values of n the wave-functions are anti-periodic in the sense $\Psi_{E_n}(x = -\infty) = -\Psi_{E_n}(x = +\infty)$. Therefore, the imaginary parts of the zeta zeros E_n in the critical line are the only values among the E-continuum of values which obey the boundary conditions $\Psi_E(x = -\infty) = \pm\Psi_E(x = +\infty)$. This physical interpretation of the discrete values E_n among the E-continuum must have bearing on the periodic orbits associated with the "chaotic" Riemann dynamics whose periods are multiples of logarithms of prime numbers as described by Berry and Keating, and based on their classical Hamiltonian $H = xp$ [14].

Notice that if one were to replace the values λ_n and $\Lambda(V_o)$ for $\Lambda = \infty$, one would have for phases in the wave-functions (2.47) the following values

$$\pi\{\mathcal{N}(E_n;\Lambda = \infty) - \mathcal{N}(V_o;\Lambda = \infty)\}$$
$$= \pi\{n - \delta_n - \mathcal{N}(V_o;\Lambda = \infty)\} = \pi n - \phi_n - \phi_o, \tag{2.48}$$

where now the zero-point energy V_o is the one determined from the solution to the transcendental equation (2.26). Therefore, if no cut-offs are set in the upper limits of the integral defining the $\delta(E)$ terms (2.16) and (2.17) there would be a nontrivial *phase-shift* in the wave-functions as shown in Eq. (2.48) and one would no longer have the nice periodicity (anti-periodicity) behavior as before; i.e. one would have now a *quasi*-periodicity behavior.

To finalize this section, we return to the solution of Abel's integral equation (2.11) ($\alpha = \frac{1}{2}$) in the Schroedinger operator case

$$x(V) - x(V_o) = \frac{1}{\Gamma(1/2)} \int_0^V \frac{d\mathcal{N}(E)}{dE} \frac{1}{\sqrt{(V-E)}} dE. \tag{2.49}$$

To control the divergences in the integral once again we may introduce a cut-off function $\Lambda(E)$ in the counting function and have $\frac{d\mathcal{N}(E,\Lambda(E))}{dE}$ inserted into the above integral where the cut-off function $\Lambda(E)$ is defined by the series expansion of Eq. (2.38). The coefficients a_k^* are the *fixed* points of the large N limit of the family

$a_k^{(N)}(\lambda_o, V_o = 0; \{\lambda_n\}, \{E_n\})$ given explicitly by the relations (2.35) involving the van der Monde determinant.

The values of the λ_n's as functions of E_n are obtained by solving Eq. (2.28); however, now the value of λ_o is *no* longer determined from Eq. (2.27), because that equation *no* longer applies, but it is left out as a free parameter that is related to the integration constant $x(V_o = 0) = x_o$ in the l.h.s. of Eq. (2.48). Therefore, one has now, at each level $N, N+1, N+2, \ldots$ a well-behaved cut-off function $\Lambda(E, N)$ in the expression $\mathcal{N}(E, \Lambda(E))$ given by Eq. (2.12) that can be inserted into the above integral (2.48), and provide solutions for $x(V) - x_o(x(V_o = 0) = x_o)$, and which defines *implicitly*, the potential $V(x)$ of the self-adjoint Schroedinger operator defined in the whole real line that reproduces the zeta zeros.

3. Area Quantization in Phase Space, Duality, Space–Time Singularities, Renormalization Group, and Distribution of Primes and Zeta Zeros

To finalize this work, we will derive the Area Quantization condition in Phase Space $\mathcal{A}_n = n\pi$ of the intervals $[0, E_n]$ for $n = 1, 2, 3, \ldots, \infty$ and show why *area quantization* is one physical reason why the *average* distribution of primes density for very large x given by $\mathcal{O}(\frac{1}{\log x})$, has a one-to-one *correspondence* with the *inverse* average density of zeta zeros in the critical line. As the number density of primes decreases asymptotically with large x as $(1/\log x)$, the average density of zeta zeros in the critical line increases asymptotically (for very large E) as $\frac{1}{2\pi}\log(\frac{E}{2\pi})$. This finding is consistent with the results of Petermann [23] who found the $1/\log x$ behavior to be connected to the RG program in QFT.

In the previous section, we found that the potential function $V(x)$ obtained implicitly from Eq. (2.40) turns out to be a multi-valued function of x which requires splitting the energy regions into different bands, branches, like a nonperiodic crystal lattice

$$[0, E_1], [E_1, E_2], [E_2, E_3], \ldots, [E_{n-1}, E_n], \ldots \tag{3.1}$$

such that at the boundaries of those bands: $x(V = E_n) = \frac{\pi}{2}\mathcal{N}'(V = E_n) = \infty$ due to the discontinuity of the $S(E)$ term (2.12) at E_n.

The left and right derivatives of $x(V)$ at $V = E_n$ are $(dx/dV) = \pm\infty$ which is also consistent with taking the second derivatives of the Heaviside step function $\Theta''(E - E_n) = \delta'(E - E_n)$, since the counting function is defined by $\mathcal{N}(E_N) = \sum_1^N \Theta(E - E_n)$. In the infinitesimal region $V = E_n \pm \epsilon_n$, for a suitable infinitesimal $\epsilon_n(E_n) > 0$, one expects a *sudden jump* of the function $\mathcal{N}(V, \Lambda(V))$, from values *less* than n, to values *greater* than n, while reaching the *precise* value of n at $V = E_n$ due to the conditions imposed in (2.28). This sudden jump is provided for by the $S(E)$ term in the Riemann–von Mangoldt formula.

On a separate problem, we exploited this singular behavior of the derivatives of the Heaviside step function to construct a different solution to the static spherically

symmetric gravitational field produced by a point mass M at $r = 0$ than the standard text book solution. The solutions for the metric [32] were continuous *except* at the location $r = 0$ of the point mass, leading to a delta function for the scalar curvature $\mathcal{R} = (2GM\delta(r)/r^2)$ instead of $\mathcal{R} = 0$. The Euclideanized Einstein–Hilbert action coincided precisely with the Black Hole Entropy where the area of the horizon which has now been *displaced* at the location $r = 0^+$ (due to the *discontinuity* of the metric at $r = 0$) is the usual value $4\pi(2GM)^2$. The area-radial function chosen was $\rho(r) = r + 2G|M|\Theta(r)$ so that $\rho(r = 0) = 0; \rho(r = 0^+) = 2G|M|; \rho(r = 0^-) = -2G|M|$ due to the definition of the Heaviside step function: it is 1 for $r > 0$; -1 for $r < 0$, and is 0 for $r = 0$. This discontinuity has the same form as the discontinuity of the argument of $\zeta(\frac{1}{2} + iE)$. For this reason, we believe that John Nash's approach to the RH based on space–time singularities was on the right path.

Between two consecutive cusps where the coordinate function blows up $x(V) = \frac{\pi}{2}\mathcal{N}'(V; \Lambda(V)) = \infty$ at $V = E_n, E_{n+1}$, lies a "valley" region where there are *inflection* points of $\mathcal{N}(V, \Lambda(V))$ at the locations $E_*^{(n)}$, within the intervals $E_n < E_*^{(n)} < E_{n+1}$, such that $\mathcal{N}''(E_*^{(n)}, \Lambda(E_*^{(n)})) = 0$; i.e. $(dx/dV) = 0$ at the bottom of the valleys $V = E_*^{(n)}$, while $(dx/dV) = \pm\infty$ at the cusps E_n. One can visualize the coordinate graph function $x(V)$ as an infinitely long suspension bridge (from $V = 0$ to $V = \infty$) with infinitely high poles/spikes $x(V) = \infty$ at the specific locations $V = E_1, E_2, E_3, \ldots$, and with the suspension cables falling into the U-shaped valley regions in between.

With this picture in mind, the areas $\mathcal{A}_n$ in the Phase Space comprised by this *nonperiodic crystal lattice* of peaks and valleys, are quantized in multiples of π as follows

$$\begin{aligned} 2\int_0^{x_n=\infty} P(x)dx &= 2\int_0^{E_n} (E_n - V)\frac{dx}{dV}dV \\ &= 2\int_0^{E_1} (E_n - V)\frac{dx}{dV}dV + 2\int_{E_1}^{E_2} (E_n - V)\frac{dx}{dV}dV \\ &\quad + 2\int_{E_2}^{E_3} (E_n - V)\frac{dx}{dV}dV + \cdots + 2\int_{E_{n-1}}^{E_n} (E_n - V)\frac{dx}{dV}dV \\ &= \mathcal{A}_1^{(n)} + \mathcal{A}_2^{(n)} + \mathcal{A}_3^{(n)} + \cdots + \mathcal{A}_n^{(n)} = n\pi; \quad n = 1, 2, 3, \ldots . \end{aligned} \tag{3.2}$$

For a given value of $n = 1, 2, 3, \ldots$ the sum of each one of these n-aperiodic-crystal-like bands contributes to a *net* value of area $\mathcal{A}_n = n\pi$. This is *not* to say that the areas in (3.2) are equally partitioned in one unit of π! It is the whole sum which adds up to $n\pi$. For any given value of n one can take the ratios of areas to obtain a sequence of fractions

$$\frac{\mathcal{A}_1^{(n)}}{n\pi}, \frac{\mathcal{A}_2^{(n)}}{n\pi}, \frac{\mathcal{A}_3^{(n)}}{n\pi}, \ldots, \frac{\mathcal{A}_n^{(n)}}{n\pi}; \quad n = 1, 2, 3, \ldots \tag{3.3}$$

(one should take the *magnitude* of the areas in the case of negative contributions in the integrals). It is known that the self-similarity of the Farey sequence of fractions possess remarkable *fractal* properties [27] that is very relevant to the validity of the RH based on Farey fractions and the Franel–Landau shifts [28]. Do the area-fractions (3.3) follow a Farey sequence when $2x(V)/\pi = \mathcal{N}'(V, \Lambda(V))$?

A fractal SUSY QM model to fit the spectrum of the imaginary parts of the zeta zeros ρ_n was studied in [16] based on a Hamiltonian operator that admits a factorization into two factors involving fractional derivative operators whose fractional (irrational) order is one-half of the fractal dimension ($d = 1.5$) of the fractal potential found by Wu–Sprung [13]. A model of *fractional* spin has been constructed by Wellington da Cruz [29] in connection to the Fractional Quantum Hall effect based on the filling factors associated with the Farey fractions. This approach based on a *fractional* Quantum Hall should be contrasted with the ordinary Quantum Hall Effect approach to the RH [19].

The integral depicting the phase space area of a domain $\mathcal{D}$ can be written as a result of Stokes theorem as

$$\int_{\mathcal{D}} dX \wedge dP = \frac{1}{2} \oint_C (PdX - XdP), \tag{3.4}$$

where the *line* (contour) integral is defined along the boundary region of the domain $\mathcal{D}$ of phase space. Since the potential is *symmetric* $V(-x) = V(x)$ another way of obtaining the same result for the *net* areas is to compute the areas from the line-integral (3.4) using the equality due to the symmetry of the potential

$$\frac{1}{2} \oint_C PdX = -\frac{1}{2} \oint XdP. \tag{3.5}$$

The relation $P + V = E_n \Rightarrow P = E_n - V$ yields

$$\begin{aligned} \mathcal{A}_n &= -\oint XdP = -\oint Xd(E_n - V) \\ &= 2\int_0^{E_n} XdV = \pi \int_0^{E_n} \frac{d\mathcal{N}(V, \Lambda(V))}{dV} dV \\ &= \pi[\mathcal{N}(E_n, \Lambda(E_n)) - \mathcal{N}(V = 0, \Lambda(V = 0))] = n\pi. \end{aligned} \tag{3.6}$$

Therefore, in general, one has the *dual* or *reciprocal* forms for the same phase space area $\mathcal{A}_n$ as a direct consequence of Stokes theorem

$$\begin{aligned} \mathcal{A}_n &= 2\int_0^{E_n} X(V)dV = 2\int_0^{E_1} X(V)dV + 2\int_{E_1}^{E_2} X(V)dV \\ &+ 2\int_{E_2}^{E_3} X(V)dV + \cdots + 2\int_{E_{n-1}}^{E_n} X(V)dV \\ &= \mathcal{I}_1^{(n)} + \mathcal{I}_2^{(n)} + \mathcal{I}_3^{(n)} + \cdots + \mathcal{I}_n^{(n)} = n\pi; \quad n = 1, 2, 3, \ldots \end{aligned} \tag{3.7}$$

where we have re-written x as X. Because of the relationship $2X(V) = \pi \frac{d\mathcal{N}(V,\Lambda(V))}{dV}$ derived in the previous section, and by setting $\mathcal{N}(V = 0, \Lambda(V = 0)) = 0$, one has

$$\mathcal{A}_n = 2\int_0^{E_n} X(V)dV = \pi \int_0^{E_n} \frac{d\mathcal{N}(V,\Lambda(V))}{dV} dV = \pi \mathcal{N}(E_n, \Lambda(E_n)) = n\pi. \quad (3.8)$$

From Eqs. (3.7) and (3.8) one can write the values of the integrals associated to each one of the respective intervals $[0, E_1], [E_1, E_2], [E_2, E_3], \ldots, [E_{n-1}, E_n]$ as

$$\pi + \pi(2-1) + \pi(3-2) + \cdots + \pi(n-(n-1)) = \pi + \pi + \pi + \cdots + \pi = n\pi, \quad (3.9)$$

which is a direct consequence of the quantization conditions (2.28) when $\mathcal{N}(V = 0, \Lambda(V = 0)) = 0$

$$\mathcal{N}(E_1, \Lambda(E_1)) = 1, \mathcal{N}(E_2, \Lambda(E_2)) = 2, \ldots, \mathcal{N}(E_n, \Lambda(E_n)) = n. \quad (3.10)$$

Therefore, from this decomposition of the areas in terms of $\int X dV$ integrals, one has now an *equipartition* of the area $\mathcal{A}_n = n\pi$ into n-single bits and whose quantum of area is π

$$\mathcal{I}_1^{(n)} = \pi, \quad \mathcal{I}_2^{(n)} = \pi, \quad \mathcal{I}_3^{(n)} = \pi, \ldots, \mathcal{I}_n^{(n)} = \pi. \quad (3.11)$$

A full cycle requires starting at $-\infty$, going to $+\infty$ and back to $-\infty$, thus the full cycle will generate $2n\pi$ area-bits, consistent why the nth-winding number of the orbit associated with the nth zeta zero E_n. This is where one can make contact with the work by Berry and Keating [14] on the periodic orbits associated with the "chaotic" Riemann dynamics whose periods are multiples of logarithms of prime numbers based on the classical Hamiltonian $H = xp$ [14] and the Gutzwiller trace formula.

Now we are ready to find the relationship between area quantization, and the distribution of primes and zeta zeros. The following integrals Y_n, for $n = 1, 2, 3, \ldots$ also give the same values of $n\pi$, after the renaming of variables $y = \frac{V}{2\pi n}$

$$\begin{aligned} Y_n &= -\frac{1}{2}\int_0^{2\pi n} \log\left(\frac{V}{2\pi n}\right) dV = -n\pi \int_0^{2\pi n} \log\left(\frac{V}{2\pi n}\right) d\left(\frac{V}{2\pi n}\right) \\ &= -n\pi \int_0^1 \log(y)dy = \{-n\pi y(\log y - 1)\}_0^1 = n\pi, \end{aligned} \quad (3.12)$$

because $0 \log 0 \to 0$. Upon equating the three *integrals* (3.2), (3.7), and (3.12), and after using the results of the previous section $x(V) = \frac{\pi}{2}\mathcal{N}'(V) \Rightarrow x'(V) = \frac{\pi}{2}\mathcal{N}''(V)$, the area quantization in phase space reads

$$\begin{aligned} \frac{\mathcal{A}_n}{\pi} &= \int_{V=0}^{E_n} (E_n - V)\frac{d^2\mathcal{N}}{dV^2} dV = \int_0^{E_n} \frac{d\mathcal{N}(V,\Lambda(V))}{dV} dV \\ &= -\frac{1}{2\pi}\int_0^{2\pi n} \log\left(\frac{V}{2\pi n}\right) dV = -n\int_0^1 \log(y)dy = n; \quad n = 1, 2, 3, \ldots. \end{aligned} \quad (3.13)$$

The fact that the *integrals* are equal does *not* mean the *integrands* are equal, nevertheless one can still establish the following one-to-one correspondence of the integrands and domains of integration as follows

$$-\frac{1}{2\pi}\log\left(\frac{V}{2\pi n}\right)\leftrightarrow\frac{d\mathcal{N}(V,\Lambda(V))}{dV}\equiv\rho(V);\quad 2\pi n\leftrightarrow E_n, \tag{3.14}$$

$$\frac{d\mathcal{N}(V,\Lambda(V))}{dV}\leftrightarrow(E_n-V)\frac{d^2\mathcal{N}(V;\Lambda(V))}{dV^2}. \tag{3.15}$$

From the correspondence (3.14) one learns that the irregularly spaced zeta zeros E_n has a correspondence with the evenly spaced energy levels given by $2\pi n$. While the logarithmic integrand $-\frac{1}{2\pi}\log(\frac{V}{2\pi n})=-\frac{1}{2\pi}\log(y)$, which has the *same* functional form as the *inverse* average density of primes $\log(x)$ (up to a sign and numerical factor) has a correspondence to the density of zeta zeros $\rho(V)$ in the critical line. The negative sign $-\frac{1}{2\pi}\log(\frac{V}{2\pi n})$ has a connection to Connes work on the RH and Noncommutative Trace formula where the location of the zeta zeros were interpreted as *absorption* lines of the spectrum, instead of emission lines [18].

The prime number theorem states the number of primes $\mathcal{P}(N)$ in the interval $[0,N]$, for large N, is of the order of $\mathcal{P}(N)\sim(N/\log N)$. The average number density of primes is $\mathcal{P}(N)/N\sim(1/\log N)$, so its inverse is $\log(N)$. The density of primes is instead $d\mathcal{P}(N)/dN=(1/\log N)-(1/\log N)^2$. We believe this is no coincidence for the even harmonious spacing of the energy levels $2\pi n$ is related to the imaginary parts of the zeros of:

$$\sin(iz)=i\sinh(z)=i\sinh(x+iy)=i\sinh(x)\cos(y)-\cosh(x)\sin(y)=0$$
$$\Rightarrow\sinh(x)\cos(y)=0,\quad\text{and}\ \cosh(x)\sin(y)=0. \tag{3.16}$$

The solutions to these last two equations is $x=0,y=\pm 2\pi n$. Therefore, the zeros of the function $\sin(iz)=i\sinh(z)=0$ are $z_n=0\pm i2\pi n$, which satisfy an analog of the RH. They all line in the vertical line $\mathcal{R}e(z)=0$, with the main difference being that the latter zeros are all evenly and harmoniously spaced in intervals of 2π along the imaginary axis. Thus, the $E_n\leftrightarrow 2\pi n$ correspondence would be another reflection of the irregular but "harmonious" distribution of the primes.

It is warranted to explore the connections to the area quantization of the quantum droplets in the Quantum Hall Effect. Studies of the Lowest Landau Levels in the quantum mechanical model for a charged particle on a plane in a constant uniform perpendicular magnetic field by Sierra and Townsend [19], have shown to yield the absorption level spectrum of the zeta zeros by Connes [18] and related to the Berry–Keating [14] model of the average level density of the zeta zeros based on the classical Hamiltonian $H=xp$. It was conjectured [19] that the *fluctuating* part $S(E)$ of the counting function $\mathcal{N}(E)$ (2.12) might be accounted for by the higher Landau levels. The upshot of our results is that we have been involved with the full-fledged Riemann–von Mangoldt expression $\mathcal{N}(E)$ in Eq. (2.12) which not only has the fluctuating part $S(E)$, but also the higher order $\mathcal{O}(E^{-n})$ corrections as well, the $\delta(E)$ terms, in addition to the standard terms $\frac{E}{2\pi}(\log\frac{E}{2\pi}-1)+7/8$.

To finalize, we should add that since we are dealing with Dirac-like operators one must not forget the existence of anti-particles with negative energy states $-E_n$, although we are not working in four dimensions where the CPT theorem applies. Negative energy states is consistent with the fact that the zeta zeros in the critical line appear in pairs of complex conjugates $\frac{1}{2} \pm iE_n$. The absence of a positive-energy electron behaves as if a positron of positive charge and negative energy were created. Equation (2.1) admits the analog of negative energy in one dimension (or 0+1-dim), by simply writing the dispersion relation $(P+V)^2 = \mathcal{M}^2 \Rightarrow (P+V) = \pm\mathcal{M} = \pm E$. Therefore, the existence of $\pm E$ eigenvalues is compatible with the zeros appearing in pairs of complex conjugates $\frac{1}{2} \pm iE_n$ along the critical line. In Electro-Magnetism (EM) the canonical momentum is defined by the replacement $p_\mu \rightarrow p_\mu - eA_\mu$, where A_μ is the EM potential and $-e$ is the negative charge of the electron. Thus having the generalized momentum $P+V$ bears some relation to the canonical momentum in EM, which brings up again the connection to the work on Landau Lowest Levels, Quantum Hall Effect ... by Sierra and Townsend [19].

In future work, we will explore the relations of our work to

- Chaotic RG Flows, Universal Mandelbrot Set, Phase transitions, attractors, Julia sets, ... by Dolotin and Morozov [31].
- Fractal strings, fractal membranes, noncommutative spaces, Dirac-like operators, spectral triples, quasi-crystals, modular flows of the moduli space of fractal membranes, adeles, arithmetic geometries, ... in connection to the flows of zeros of zeta-functions towards the critical line, by Lapidus *et al.* [26].
- Connes noncommutative trace formula [18]. The fermionic Trace Formula, supersymmetry, Witten index, and the Mobius function [30].
- Black Hole entropy and area quantization in Loop Quantum Gravity; Farey sequences, fractal statistics, and the fractional Quantum Hall Effect [29].
- Cyclotomy, Phase quantization, Ramanujan sums, ... by Planat *et al.* [25].

To summarize this work, in Secs. 1 and 2 we have presented two plausible methods to prove the RH. One was based on the modular properties of Θ functions and the other on the Hilbert–Polya proposal to find an operator whose spectrum reproduces the ordinates of the zeta zeros in the critical line. We described in detail how the Dirac-like operator with a potential $V(x)$ in Eq. (2.1) reproduces the spectrum after the boundary conditions $\Psi_E(x = -\infty) = \pm\Psi_E(x = +\infty)$ are imposed. Such potential $V(x)$ was derived implicitly from the relation $x = x(V) = \frac{\pi}{2}\rho(V) = \frac{\pi}{2}(d\mathcal{N}(V, \Lambda(V))/dV)$. At the end of Sec. 2, we explained how to provide the implicit form of the potential $V(x)$ for the self-adjoint Schroedinger operator that also reproduces the zeta zeros. Crucial in the construction, was the introduction of an energy-dependent cut-off function $\Lambda(E)$. In the final Sec. 3, the natural quantization of the phase space areas (associated to *nonperiodic* crystal-like structures) in integer multiples of π follows from the Bohr–Sommerfeld quantization conditions of Quantum Mechanics. It allows to find a physical reasoning as to why the average density of the primes distribution for very large x: $\mathcal{O}(\frac{1}{\log x})$, has a

one-to-one correspondence with the asymptotic limit of the *inverse* average density of zeta zeros in the critical line.

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