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Eigenfunctions of the transfer operators and the period functions for modular groups

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Abstract. We extend the transfer operator approach to the period functions of Lewis and Zagier for the group $PSL(2,\mathbb{Z})$ [**LZ97**] to general subgroups of $PSL(2,\mathbb{Z})$ with finite index. Thereby we derive functional equations for the eigenfunctions of the transfer operators with eigenvalue $\lambda = 1$ which generalize the one derived by J. Lewis for $PSL(2,\mathbb{Z})$. For special congruence subgroups we find that Lehner and Atkins theory of old and new forms [**AL70**] is realized also on the level of the eigenfunctions of the transfer operators for these groups. It turns out that the old eigenfunctions can be related through Lewis' transform for $PSL(2,\mathbb{Z})$ to the old forms for these subgroups, whereas a similar relation for new eigenfunctions and the corresponding new forms is not yet known.

1. Introduction

In the Eichler-Manin-Shimura theory of periods [**Eic57**] to every holomorphic modular cusp form one can associate a period polynomial with certain cocycle properties under the action of the group elements. This theory was extended to general holomorphic forms for $PSL(2,\mathbb{Z})$ by Zagier in [**Zag91**]. Regularizing the corresponding integrals of Eichler in the case of the holomorphic Eisenstein series for $PSL(2,\mathbb{Z})$ leads to certain rational functions. Recently, J. Lewis found a further extension of this theory in which to every Maaß wave form for $PSL(2,\mathbb{Z})$ there is related a certain holomorphic function in the cut z-plane which fulfills a functional equation depending on a complex parameter β. This equation combines the two cocycle relations for the polynomials of the two generators of $PSL(2,\mathbb{Z})$ [**Lew97**], [**LZ97**], [**LZ**]. The polynomial solutions of this equation are just the period polynomials and its rational solutions are the rational functions of Zagier. Zagier found another discrete series of solutions of the Lewis-Zagier functional equation which are transforms of the nonholomorphic Eisenstein series of $PSL(2,\mathbb{Z})$ for parameter values β corresponding to the resonances of the Laplacian on the modular surface [**CM99**] given by the nontrivial zeros of Riemann's zeta function $\zeta_R(2\beta)$.

It was found [**CM99**], [**LZ97**] that all the above mentioned solutions of the Lewis functional equation are eigenfunctions to the eigenvalues $\lambda = +1$ or $\lambda = -1$

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of the so called transfer operator $\mathcal{L}_\beta$ of the geodesic flow on the modular surface, that means, they are closely related to the zeros of the function $Z_S(\beta) = \det(1-\mathcal{L}_\beta)\,\det(1+\mathcal{L}_\beta)$. This function on the other hand is nothing else than Selberg's zeta function for the group $PSL(2,\mathbb{Z})$ [**CM99**], [**May91b**]. Selberg's zeta function relates in a highly non trivial way classical mechanics to quantum mechanics: the geodesic flow describes a classical point particle moving freely on the corresponding surface with constant velocity, the spectrum of the Laplace-Beltrami operator on the other hand describes the same free particle in quantum mechanics. This relation between classical and quantum mechanics is the central problem within the theory of quantum chaos in physics. Selberg's theory, based on his trace formula, relates the energy spectrum of the free quantum particle to the length spectrum of the classical particle. The new approach through the transfer operator on the other hand gives also a connection between the bound and scattering states of the quantum system and the eigenfunctions of the transfer operator which obviously themselves are purely classical objects defined by the geodesic flow and its dynamical properties. In this sense in this special case the classical system completely determines the quantum system.

In a recent paper [**CMct**] we showed how the transfer operator approach to the geodesic flow can be extended to subgroups Γ of the modular group $PSL(2,\mathbb{Z})$ of finite index. It was shown that for such groups Selberg's zeta function can be expressed, as in the case $PSL(2,\mathbb{Z})$, as the Fredholm determinant of the corresponding transfer operator $\tilde{\mathcal{L}}_\beta$ such that its zero's and poles are related to those β values where 1 is an eigenvalue of $\tilde{\mathcal{L}}_\beta$. It was found that the operator $\tilde{\mathcal{L}}_\beta$ can be interpreted also as the transfer operator of $PSL(2,\mathbb{Z})$ together with the representation of this group induced from the trivial representation of the subgroup Γ. This gives a new proof of the well known fact that Selberg's zeta function for $\Gamma \subseteq PSL(2,\mathbb{Z})$ is identical to the generalized zeta function for $PSL(2,\mathbb{Z})$ with the above mentioned induced representation.

In the present paper we continue our investigations of the transfer operators for subgroups of the group $PSL(2,\mathbb{Z})$. In chapter 2 we briefly recall the explicit form of the transfer operator $\tilde{\mathcal{L}}_\beta$ for an arbitrary subgroup $\Gamma \subseteq PSL(2,\mathbb{Z})$ of finite index, the theory of Atkin and Lehner of old and new forms for congruence subgroups and Eichler's definition of period polynomials for cusp forms. In chapter 3 we derive, starting from the transfer operator for Γ, a functional equation for its eigenfunctions and discuss which solutions of this equation define eigenfunctions of the transfer operator with eigenvalue 1 for the special groups $\Gamma(1)$, Γ_2, $\Gamma_0(2)$, $\Gamma^0(2)$, Γ_ϑ and $\Gamma(2)$. It turns out that in these cases the eigenfunctions of the transfer operator fall into two classes, old ones and new ones, reflecting in a nice way the theory of old and new forms for these groups: The old eigenfunctions of $\tilde{\mathcal{L}}_\beta$ are related to the old forms of the subgroup Γ through the transformations of Lewis for $PSL(2,\mathbb{Z})$. A similar relation between the new eigenfunctions and the new forms however is not yet known. Also the general relation between the polynomial eigenfunctions and Eichler's period polynomials for general subgroups Γ is not yet clear.

2. The transfer operators for modular groups

2.1. Modular groups and modular surfaces. The hyperbolic plane $\mathbb{H}$ is the complex upper half plane $\mathbb{H} = \{x+iy \,|\, x,\, y \in \mathbb{R}, y > 0\}$ equipped with the

Poincaré metric $d\,s^2 = y^{-2}\,(\,d\,x^2 + d\,y^2\,)$. The group

$$PSL(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\, b,\, c,\, d, \in \mathbb{R},\; ad - bc = 1\}/\{\pm\, id\}$$

is the isometry group of this space acting on $\mathbb{H}$ as $g\,z = \frac{az+b}{cz+d}$. The (full) modular group

$$\Gamma(1) := PSL(2,\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\, b,\, c,\, d, \in \mathbb{Z},\, ad - bc = 1\}/\{\pm\, id\}$$

is a discrete subgroup of $PSL(2,\mathbb{R})$ with two generators:

$$Q := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

which fulfill the relations

$$Q^2 = id \quad \text{and} \quad (QT)^3 = id. \tag{2.2}$$

The subgroups $\Gamma \subseteq \Gamma(1)$ with finite index $\mu = [\Gamma(1) : \Gamma]$ are called modular groups. Examples are the principal congruence subgroup of order N, $N \in \mathbb{N}$,

$$\Gamma(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; mod\; N\,\}$$

which is a normal subgoup of the modular group $\Gamma(1)$ or the congruence subgroups

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c = 0 \; mod\, N,\;\; N \in \mathbb{N}\}\,,$$

$$\Gamma^0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid b = 0 \; mod\, N,\;\; N \in \mathbb{N}\}$$

which are not normal. Of interest for the following discussion is also the theta group

$$\Gamma_\vartheta := \Gamma(2) \cup \Gamma(2)Q$$

which has index three in $\Gamma(1)$ and is conjugated to $\Gamma_0(2)$ and $\Gamma^0(2)$. The group $\Gamma_2 := \Gamma(2) \cup \Gamma(2)QT \cup \Gamma(2)(QT)^2$ is another normal group of $\Gamma(1)$ with index two. The modular surface defined by such a modular group Γ is the quotient space $M_\Gamma = \Gamma\backslash\mathbb{H}$ [**Ter85**].

2.2. The geodesic flow on M_Γ.

2.2.1. *The Poincaré map of the geodesic flow on M_Γ.* The physical phase space of a free particle on M_Γ with unit velocity is the unit tangent bundle T_1M_Γ. The dynamics on T_1M_Γ can be described by the geodesic flow

$$\begin{aligned} \hat\phi_t \quad &: \quad T_1M_\Gamma \to T_1M_\Gamma \\ & \quad\;\; (\hat\gamma_{M_\Gamma}(0), \dot{\hat\gamma}_{M_\Gamma}(0))(\hat\gamma_{M_\Gamma}(t), \dot{\hat\gamma}_{M_\Gamma}(t)) \end{aligned} \tag{2.3}$$

where $\hat\gamma_{M_\Gamma} : \mathbb{R} \to M_\Gamma$ denotes the geodesic in M_Γ, parameterized by the time t through the initial position $\hat\gamma_{M_\Gamma}(0)$ with the initial tangent vector $\dot{\hat\gamma}_{M_\Gamma}(0)$. When choosing an appropriate Poincaré section X_Γ for this flow [**CMct**] its Poincaré return map can be described as

$$P_\Gamma \quad : \quad [0,1] \times [0,1] \times \mathbb{Z}_2 \times \Gamma\backslash\Gamma(1) \to [0,1] \times [0,1] \times \mathbb{Z}_2 \times \Gamma\backslash\Gamma(1)$$

with

$$P_\Gamma(x_1,\,x_2,\,\varepsilon,\,\{g\}) = (T^{-n\varepsilon}Qx_1,\,QT^{n\varepsilon}x_2,\,-\varepsilon,\,\{gT^{n\varepsilon}Q\})$$

$$= (T_G x_1, \frac{1}{[\frac{1}{x_1}]+x_2}, -\varepsilon, \{gT^{n\varepsilon}Q\})\,, \tag{2.4}$$

where $T_G : [0,1] \to [0,1]$ denotes the Gauss map $T_G\, z = \frac{1}{z} - [\frac{1}{z}]$ and $n = n(x) = [\frac{1}{x_1}]$ denotes the integer part for $\frac{1}{x_1}$ with $x := (x_1, x_2, \varepsilon, \{g\})$. The ergodic properties of this map are determined by its expanding directions, namely

$$P_\Gamma|_{ex}(z,\varepsilon,\{g\}) = (T_G\, z, -\varepsilon, \{gT^{n\varepsilon}Q\}) = (\frac{1}{z} - [\frac{1}{z}], -\varepsilon, \{gT^{n\varepsilon}Q\})\,, \tag{2.5}$$

with $n = n(x) = [\frac{1}{z}]$ and $x = (z, \varepsilon, \{g\})$.

2.2.2. *The transfer operator for* Γ. The transfer operator for the map (2.5) has the explicit form [**CMct**]

$$\tilde{\mathcal{L}}_\beta^{\Gamma(1),\,\chi^\Gamma} \underline{f}(z,\varepsilon) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2\beta} \chi^\Gamma(QT^{n\varepsilon}) \underline{f}(\frac{1}{z+n}, -\varepsilon)\,, \tag{2.6}$$

where χ^Γ denotes the representation of $\Gamma(1)$ induced from the trivial representation of the subgroup Γ. Let $B(D)$ be the Banach space of all holomorphic functions on the disk

$$D := \{z | z \in \mathbb{C},\ |z-1| < \frac{3}{2}\} \tag{2.7}$$

which are continuous on the closure $\bar{D}$ of the disk. Then the transfer operator (2.6) is well defined on $\oplus_{i=1}^{2\mu} B(D)$, where $\mu = [\Gamma(1) : \Gamma]$:

$$\tilde{\mathcal{L}}_\beta^{\Gamma(1),\,\chi^\Gamma} : \oplus_{i=1}^{2\mu} B(D) \to \oplus_{i=1}^{2\mu} B(D)\,.$$

Denoting the operator $\tilde{\mathcal{L}}_\beta^{\Gamma(1),\,\chi^\Gamma}$ simply as $\tilde{\mathcal{L}}_\beta$, the decomposition

$$\tilde{\mathcal{L}}_\beta = \tilde{\mathcal{A}}_\beta^{(\kappa)} + \tilde{\mathcal{L}}_\beta^{(\kappa)} \qquad \text{for } \ \kappa \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \tag{2.8}$$

with

$$\tilde{\mathcal{A}}_\beta^{(\kappa)} \underline{f}_\beta(z,\varepsilon) = \sum_{l=0}^{\kappa} (\frac{1}{r})^{2\beta+l} \sum_{m=1}^{r} \chi^\Gamma(QT^{m\varepsilon})\, \frac{\underline{f}_\beta^{(l)}(0,-\varepsilon)}{l\,!}\, \zeta(2\beta+l, \frac{z+m}{r}) \tag{2.9}$$

and

$$\tilde{\mathcal{L}}_\beta^{(\kappa)} \underline{f}_\beta(z,\varepsilon) = \sum_{n=1}^{\infty} \sum_{m=1}^{r} \chi^\Gamma(QT^{m\varepsilon}) \Big(\frac{1}{z+m+r(n-1)}\Big)^{2\beta} \times \Bigg[\underline{f}\Big(\frac{1}{z+m+r(n-1)}, -\varepsilon\Big) - \sum_{l=0}^{\kappa} \frac{\underline{f}_\beta^{(l)}(0,-\varepsilon)}{l\,!} \Big(\frac{1}{z+m+r(n-1)}\Big)^{l}\Bigg]\,, \tag{2.10}$$

shows that $\tilde{\mathcal{L}}_\beta$ is meromorphic in the half plane $\Re\beta > -\frac{\kappa}{2}$ with possible singularities at $\beta = \beta_l := \frac{1-l}{2}$, $l = 0,\, 1,\, \cdots,\, \kappa$ [**CMct**]. The number r is the minimal natural number with $\chi^\Gamma(T^r) = 1$. Since κ is an arbitrary natural number, the operator $\tilde{\mathcal{L}}_\beta$ is meromorphic in the entire complex β-plane.

The operator $\tilde{\mathcal{L}}_\beta$ can be written as [**CMct**]

$$\tilde{\mathcal{L}}_\beta = \begin{pmatrix} 0 & \mathcal{L}_{\beta,-} \\ \mathcal{L}_{\beta,+} & 0 \end{pmatrix} \tag{2.11}$$

with

$$\mathcal{L}_{\beta,\pm} : \oplus_{i=1}^{\mu} B(D) \to \oplus_{i=1}^{\mu} B(D) .$$

where

$$(\mathcal{L}_{\beta,\pm} \underline{f}_{\beta,\pm})(z) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2\beta} \chi^\Gamma(QT^{\pm n}) \underline{f}_{\beta,\mp}(\frac{1}{z+n}) , \tag{2.12}$$

and

$$\underline{f}_\beta(z) := \begin{pmatrix} \underline{f}_\beta(z,+1) \\ \underline{f}_\beta(z,-1) \end{pmatrix} = \begin{pmatrix} \underline{f}_{\beta,+}(z) \\ \underline{f}_{\beta,-}(z) \end{pmatrix} .$$

If the induced representation χ^Γ has the property $\chi^\Gamma(T^2) = 1$, then $\mathcal{L}_{\beta+} = \mathcal{L}_{\beta-} := \mathcal{L}_\beta$. In this case the eigenvalues and the eigenfunctions of $\mathcal{L}_\beta$ determine the eigenvalues and the eigenfunctions of $\tilde{\mathcal{L}}_\beta$ [**CMct**].

In [**CMct**] we showed, that Selberg's zeta function $Z_S(\beta;\Gamma)$ can be written as

$$Z_S(\beta;\Gamma) = Z_S(\beta;\Gamma(1);\chi^\Gamma) = \det(1 - \tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi^\Gamma}) . \tag{2.13}$$

Due to this relation, the β values where the operator $\tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi^\Gamma}$ has eigenvalue $+1$ play a very special role, since these β's are related to the zero's of Selberg's zeta function. The zeros of Selberg's zeta function fall into three classes, namely the trivial zeros at $\beta = \beta_l = \frac{1-l}{2}$, $l \in \mathbb{N}_0$, the Riemann zeros on the line $\Re\beta = \frac{1}{4}$ (if Riemann's conjecture is true) and the spectral zeros on the line $\Re\beta = \frac{1}{2}$, which are related to the quantum energies $\beta(1-\beta)$ of a free particle on M_Γ [**Hej76**]. All these zeros are closely related to the automorphic forms and the Maaß wave forms for the group Γ, whose definition we briefly recall for the readers convenience.

2.3. Automorphic forms and Maaß wave forms for Γ.

2.3.1. *Automorphic forms.* Let k be a nonnegative even integer. A complex function f on $\mathbb{H}$ is called an automorphic form of weight k (or degree $-k$) for Γ if f is holomorphic on $\mathbb{H}$ and at every cusp of Γ [**Ter85**] and $f(z)|_k\,\sigma = f(z)$ for all $\sigma \in \Gamma$, where

$$f(z)\,|_k\,\sigma := (cz+d)^{-k} f(\sigma z) . \tag{2.14}$$

For $\Gamma \subseteq \Gamma(1)$ the automorphic forms are also called modular forms.

Remark 2.1. Some authors define the automorphic forms and the modular forms as meromorphic functions [**Miy89**], [**Shi71**]. However, we stay to the definition in [**Ter85**] and [**Zag92a**].

Denote by $A_k(\Gamma)$ the space of all automorphic forms of weight k for the group Γ. A function in $A_k(\Gamma)$ is called a holomorphic cusp form if $f(z)$ vanishes at every cusp of Γ [**Lan76**]. The set of cusp forms is denoted by $S_k(\Gamma)$. Formulas for the dimensions of the spaces $S_k(\Gamma)$ for different Γ can be found for instance in [**Shi71**], [**Miy89**]. For $\Gamma = \Gamma(1)$ one has

$$dim\, S_k(\Gamma(1)) = \begin{cases} 0 & (k=2) , \\ \text{[k/12]-1} & (k>2,\, k \equiv\, 2\, mod\, 12) , \\ \text{[k/12]} & (k \not\equiv\, 2\, mod\, 12) , \end{cases} \tag{2.15}$$

where $[x]$ is the integer part of x.

The holomorphic Eisenstein series are modular forms which don't vanish at the cusps. The Eisenstein series of weight k for the full modular group $\Gamma(1)$ reads

$$E_k(z) = \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} (mz+n)^{-k}\,. \tag{2.16}$$

For examples of Eisenstein series for subgroups of $\Gamma(1)$ see [**Kob93**] p.131, [**Ran77**] p.174 and [**Sar90**] p.16. The number of Eisenstein series for $\Gamma \subseteq \Gamma(1)$ is equal to the number of Γ-inequivalent cusps in $\Gamma\backslash\mathbb{H}$ [**Ran77**].

2.3.2. *The Maaß wave forms.* Let z_i, $i = 1, \cdots, n$ be the Γ-inequivalent cusps of the hyperbolic surface M_Γ and s_i be a generator of the stablizer S_i of z_i, i.e., the subgroup $S_i \subseteq \Gamma$ such that $s_i\, z_i = z_i$ for $s_i \in S_i$. The Laplace-Beltrami operator

$$-\Delta = -y^2\,(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\,, \qquad z = x + i\,y \in \mathbb{H} \tag{2.17}$$

for $\Gamma \subseteq \Gamma(1)$ can be defined on the Hilbert space

$$\begin{aligned}\mathcal{H}(\Gamma) \quad := \quad & \Big\{\underline{f} : \mathbb{H} \to V \,|\, \underline{f}(\sigma\, z) = \chi(\sigma)\, \underline{f}(z),\, \forall\sigma \in \Gamma,\, \forall z \in \mathbb{H}, \\ & \qquad ||\underline{f}||^2 = \int_{\mathcal{F}_\Gamma} |\underline{f}(z)|^2\, dm(z) < \infty\Big\}\,,\end{aligned}$$

where χ is a representation of Γ in the finite dimensional Hilbert space V and $dm = \frac{dx\,dy}{y^2}$ is the Γ-invariant hyperbolic measure. The representation χ is called singular respectively regular at the cusp z_i if $\chi(s_i) = 1$ respectively $\chi(s_i) \neq 1$ for $s_i \in S_i$ [**BV97**].

The Maaß wave forms (or non-holomorphic modular forms) $u(z)$ for the group Γ are the eigenfunctions of the Laplace-Beltrami operator $-\Delta$ on $\mathbb{H}$ fulfilling $u(\,\sigma z\,) = u(z)$ for all $\sigma \in \Gamma$ and $z \in \mathbb{H}$ which increase at every cusp at most like a polynomial. The Maaß wave forms which vanish at infinity are called Maaß cusp forms. Besides the constant function $u(z) = c$ the only explicitly known Maaß wave forms are the non-holomorphic Eisenstein series which don't vanish at infinity. These Eisenstein series can be defined as follows: Let z_i, $i = 1, \cdots, n$ denote the Γ-inequivalent cusps of $\Gamma\backslash\mathbb{H}$ with $z_1 = \infty$ and S_1 be the stablizer of z_1, i.e., $gz_1 = z_1$ for all $g \in S_1$. Suppose $g_i \in PSL(2,\mathbb{R})$ is chosen such that $g_i\infty = z_i$. Then $S_i := g_i S_1 g_i^{-1}$ is the stablizer of z_i. The Eisenstein series $E_s^{(i)}(z)$ for the cusp z_i of Γ is then defined as [**Kub73**]

$$E_s^{(i)}(z) = \sum_{\sigma\in S_i\backslash\Gamma} y^s\,(g_i^{-1}\sigma z)\,, \quad \Re s > 1\,, \tag{2.18}$$

with $y(z) := \Im z$. For $\Gamma(1)$ the point $z = \infty$ is the only cusp. The Eisenstein series in this case reads

$$E_s(z) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} y^s(\sigma z) \quad \text{with} \quad \Gamma_\infty = \Big\{\begin{pmatrix}\pm 1 & n\\ 0 & \pm 1\end{pmatrix} \Big|\, n \in \mathbb{Z}\Big\}.$$

Another definition for the Eisenstein series of $\Gamma(1)$ is

$$\tilde{E}_s(z) = \frac{1}{2} y^s \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} |mz+n|^{-2s}\,, \quad \Re\, s > 1\,. \tag{2.19}$$

It differs from (2.18) only by the factor $\zeta(2s)$ [**Ter85**] p.207. A more detailed discussion of the non-holomorphic Eisenstein series can be found in [**Kub73**] and [**Ter85**].

2.3.3. *The theory of old and new forms.* The theory of old and new modular forms goes back to A. Atkin and J. Lehner [**AL70**]. It is a classification of modular forms of Γ depending on whether they are related to modular forms for some group Γ' with $\Gamma \subset \Gamma'$ or not. In the literature this theory is usually developped for congruence subgroups [**AL70**], [**Zag92a**] or the principal congruence subgroups [**Lan76**]. We restrict ourselves to the congruence subgroups:

Let $\Gamma_0(N)$ be a subgroup of $\Gamma_0(N')$ with $N,\, N' \in \mathbb{N}$ and $N = kN'$, $k > 1$. Then obviously $\begin{pmatrix} a & bm \\ c/m & d \end{pmatrix} \in \Gamma_0(N')$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, when m is a divisor of k. Suppose $f(z)$ is a modular form of $\Gamma_0(N')$. Then it follows that $c/m = l\,N/m = l\,k\,N'/m = n\,N'$ with $n,\, l \in \mathbb{N}$ and one gets

$$f(z) = (\frac{c}{m}z + d)^{-k}\, f\Big(\begin{pmatrix} a & bm \\ c/m & d \end{pmatrix} z\Big).$$

However this implies

$$f(mz) = (\frac{c}{m}mz + d)^{-k}\, f\Big(\frac{amz + bm}{\frac{c}{m}mz + d}\Big) = (cz + d)^{-k}\, f\Big(m \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\Big).$$

That is, $f(mz)$ is a modular form of $\Gamma_0(N)$ if $f(z)$ was a modular form of $\Gamma_0(N')$. In this sense the function $f(mz)$ is called an old form of $\Gamma_0(N)$. We define $A_k(\Gamma)^{old}$ as the space of old forms and $A_k(\Gamma)^{new}$ as the space of new forms which is the complement of $A_k(\Gamma)^{old}$, i.e., $A_k(\Gamma) = A_k(\Gamma)^{old} \oplus A_k(\Gamma)^{new}$ with respect to the Petersson scalar product [**Lan76**]. The concept of old and new modular forms can be generalized to Maaß wave forms [**Iwa95**] and to arbitrary subgroups Γ of $\Gamma(1)$.

2.4. Period polynomials and period functions associated to automorphic forms. In the classical theory of Eichler-Shimura and Manin and its extension by Zagier [**Zag91**] to every holomorphic form for a Fuchsian group Γ one associates period polynomials or rational period functions which fulfill certain cocycle relations under the action of the group Γ. This theory was recently generalized in the case of the modular group $\Gamma(1)$ to nonholomorphic automorphic forms by J. Lewis and D. Zagier [**LZ97**]. The resulting period functions are holomorphic in the complex plane cut along the negative real axis and fulfill a certain functional equation depending on a complex parameter β. For negative integer β the solutions of this functional equation are just the aforementioned classical period polynomials of Eichler respectively rational period functions of Zagier for $\Gamma(1)$. For preparing our discussion of this theory for subgroups $\Gamma \subseteq \Gamma(1)$ we briefly recall the results of Zagier and Lewis for $\Gamma(1)$.

2.4.1. *Period polynomials in the Eichler-Shimura-Manin theory.* M. Eichler studied in [**Eic57**] the $(k-1)$-th indefinite integral of cusp forms of weight k ($k \geq 2$ even number) leading to so called 'period polynomials' for $u(z)$ a holomorphic cusp form for a Fuchsian group $\Gamma \subseteq PSL(2, \mathbb{R})$ of weight k. He considered the integral

$$\Phi(\tau) = \frac{1}{(k-2)\,!} \int_{\tau_0}^{\tau} (\tau - z)^{k-2} u(z)\, dz\,, \tag{2.20}$$

for τ_0, $\tau \in \mathbb{H}$. This integral is obviously independent of the path in $\mathbb{H}$ connecting τ_0 and τ. It can be easily verified that

$$\frac{d^{k-1}}{d\tau^{k-1}}\Phi(\tau) = u(\tau)\,. \tag{2.21}$$

Obviously an arbitrary polynomial $\Theta(\tau)$ of degree $k-2$ can be added to $\Phi(\tau)$ in (2.21) as an 'integration constant' without destroying relation (2.21). That is, instead of (2.20) one can also consider the function

$$\Phi(\tau) = \frac{1}{(k-2)\,!}\int_{\tau_0}^{\tau}(\tau - z)^{k-2}u(z)\,dz + \Theta(\tau)\,. \tag{2.22}$$

Changing the integration variable z to $\sigma\, z$ with $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq PSL(2,\mathbb{R})$ one finds

$$\Phi(\tau)\,|_{2-k}\,\sigma = \Phi(\tau) + \Omega_\sigma(\tau)\,, \tag{2.23}$$

with

$$\begin{aligned} \Omega_\sigma(\tau) \;&=\; \Theta(\tau)\,(\,|_{2-k}\,\sigma - |_{2-k}\,1\,) \\ &\quad + \frac{1}{(k-2)\,!}\int_{\tau_0}^{\sigma\,\tau_0}(a\tau + b - (c\tau + d)z)^{k-2}u(z)\,dz\,, \end{aligned} \tag{2.24}$$

where we used the notation $|_k\,\sigma$ as in (2.14). The function $\Omega_\sigma(\tau)$ is obviously a polynomial of degree $k-2$ which is called a period polynomial of $\sigma \in \Gamma$ (originally called 'period' by Eichler). A function $\Phi(\tau)$ obeying relation (2.23) with a polynomial $\Omega_\sigma(\tau)$ for all $\sigma \in \Gamma$ is called an automorphic integral. Obviously the modular forms of weight $k-2$ are also automorphic integrals [**Kno78**].

Now we rewrite (2.23) as

$$\Omega_\sigma = \Phi\,|_{2-k}\,(\,\sigma - 1\,)$$

where for simplicity the dependence on τ is suppressed. Suppose σ_1, $\sigma_2 \in \Gamma$. Then the equality $\Phi|_{2-k}\,(\sigma_1\sigma_2) = (\Phi|_{2-k}\,\sigma_1)\,|_{2-k}\,\sigma_2$ induces the cocycle relation

$$\Omega_{\sigma_1\,\sigma_2} = \Omega_{\sigma_1}\,|_{2-k}\,\sigma_2 + \Omega_{\sigma_2}\,. \tag{2.25}$$

Hence the period polynomials for the cusp forms of weight k are solutions of these equations and vice versa every family of polynomial solutions of (2.25) defines period polynomials for a cusp form [**Eic57**].

2.4.2. *The period polynomials of* $\Gamma(1)$. Let us briefly recall the special case of the full modular group $\Gamma(1)$ with the two generators Q and T in (2.1). Let $u(z)$ be a holomorphic cusp form. Setting $\Theta = 0$, $\sigma = Q$, $\tau_0 = 0$ in (2.24) and using the condition $u|_k(z) = u(z)$ for the automorphic form $u(z)$ of weight k one gets up to a constant factor the polynomial

$$\Omega_Q(\tau) = \int_0^{i\infty}(\tau - z)^{k-2}u(z)\,dz\,. \tag{2.26}$$

Relations (2.23) for the group $\Gamma(1)$ can be reduced to the two equations for the generators Q and T

$$\Omega_Q = \Phi\,|_{2-k}\,Q - \Phi \quad \text{and} \tag{2.27}$$

$$\Omega_T = \Phi\,|_{2-k}\,T - \Phi\,. \tag{2.28}$$

Since there exists a polynomial P_T of degree smaller than $k-2$ with $\Omega_T = P_T|T - P_T$, one can assume that $\Omega_T = 0$ [**Kno78**], since the polynomials Ω_σ, $\sigma \in \Gamma(1)$ are determined only up to an arbitrary coboundary $P|_\sigma - P$. On the other hand, every element $\sigma \in \Gamma(1)$ can be generated by Q and T, hence an arbitrary period polynomial Ω_σ can be expressed by means of the polynomial $\Omega = \Omega_Q$ by repeated application of the cocycle relation (2.25). The relations for the generators Q and T of $\Gamma(1)$.

$$Q^2 = S^3 = id \quad \text{with} \quad S := QT = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

induce the two cocycle relations

$$\Omega\,|_{2-k}\,(1+Q) = 0 \quad \text{and} \quad \Omega\,|_{2-k}\,(1+S+S^2) = 0 \tag{2.29}$$

for the generating period polynomial $\Omega_Q = \Omega$.

2.4.3. *The period polynomials for the holomorphic cusp forms of* $\Gamma(1)$. Let $u(z)$ be a cusp form of weight k. Then the constant term $a_u(0)$ in the Fourier expansion $u(z) = \sum_{n=0}^{\infty} a_u(n) e^{2\pi i n z}$ vanishes. The period polynomial in (2.26) for $u(z)$ then reads [**Zag91**]:

$$r_u(z) \;=\; \int_0^{i\infty} (t-z)^{k-2} u(t)\, dt \tag{2.30}$$

$$\;=\; -\sum_{n=0}^{k-2} \binom{k-2}{n} \frac{L(u,n+1)}{(2\pi i)^{n+1}} n!\, z^{k-2-n}, \tag{2.31}$$

where $L(u, n+1)$ is the analytic continuation of the L function defined through the series $\sum_{n=1}^{\infty} a_u(n) n^{-s}$. This polynomial is of degree $\leq k-2$.

A simple calculation shows that the function $r_u|\,\sigma := (r_u|_{2-k}\,\sigma)(z)$ can be expressed through the same integral in (2.30) with the path of integration from $\sigma^{-1}(0)$ to $\sigma^{-1}(i\infty)$. Then the two relations (2.29) can be verified easily [**Lan76**]:

$$r_u\,|\,(1+Q) = \int_0^{i\infty} + \int_{i\infty}^{0} = 0\,, \tag{2.32}$$

$$r_u\,|\,(1+S+S^2) = \int_0^{i\infty} + \int_{-1}^{0} + \int_{i\infty}^{-1} = 0\,. \tag{2.33}$$

Since the functions $r_u(z)$ are polynomials of degree at most $k-2$ and fulfill the two relations (2.32) and (2.33), the period polynomials of the cusp forms of weight k lie in the space

$$W_{S_k(\Gamma(1))} := \left\{ p(z) \in \oplus_{n=0}^{k-2} \mathbb{C} z^n \;\middle|\; p\,|\,(1+Q) = p\,|\,(1+S+S^2) = 0 \right\}.$$

Every polynomial p can be decomposed into an even and an odd polynomial p_+ respectively p_-. The Eichler-Shimura-Manin theory [**Lan76**] then tells us that the map $u \mapsto r_u^-$ induces an isomorphism of the space $S_k(\Gamma(1))$ of cusp forms of weight k to the space of odd polynomials in $W_{S_k(\Gamma(1))}$. On the other hand, the map $u \to r_u^+$ gives an isomorphism of the space $S_k(\Gamma(1))$ to the space of even polynomials in $W_{S_k(\Gamma(1))}$ modulo the polynomial $z^{k-2} - 1$.

2.4.4. *The period functions for the holomorphic Eisenstein series of* $\Gamma(1)$. The theory of period polynomials for the holomorphic cusp forms was generalized by Zagier in [**Zag91**] to the holomorphic Eisenstein series $E_k(z) = \sum_{n=0}^{\infty} a_{E_k}(n)\, e^{2\pi i n z}$ of weight k. Instead of (2.30) Zagier considered

$$r_{E_k}(z) = \sum_{n \in \mathbb{Z}} i^{1-n} \binom{k-2}{n} L^*(E_k, n+1)\, z^{k-2-n}\,, \tag{2.34}$$

where L^* for $\Re s \gg 0$ is defined as

$$L^*(u,s) = \int_0^\infty (u(i\,y) - a_u(0)\,)\, y^{s-1} d\,y = (2\pi)^{-s}\Gamma(s)\, L(u,s)\,. \tag{2.35}$$

The function $L^*(u,s)$ has only two singularities at $s=0$ and $s=k$ with residua $-a_{E_k}(0)$ and $(-1)^{k/2}a_{E_k}(0)$ [**Zag91**]. The binomial coefficients $\binom{k-2}{n} = \frac{\Gamma(k-1)}{\Gamma(n+1)\,\Gamma(k-1-n)}$ vanish for all $n \in \mathbb{Z}$ besides the numbers $n = 0, \cdots, k-2$. For $n=-1$ and $n=k-1$ the singularities of $L^*(u,s)$ for $s = n+1$ are cancelled by the zeros of $\binom{k-2}{n}$. Thus comparing with $r_u(z)$ in (2.31) shows that $r_{E_k}(x)$ has two additional terms proportional to $\frac{1}{z}$ and z^{k-1}:

Proposition 2.2. [**Zag91**] *For $k > 2$ let p_k^+ respectively p_k^- be the following even respectively odd rational functions:*

$$p_k^+(z) \;:=\; z^{k-2} - 1\,, \tag{2.36}$$

$$p_k^-(z) \;:=\; \sum_{-1 \le n \le k-1,\; n\ odd} \frac{B_{n+1}}{(n+1)\,!}\,\frac{B_{k-n-1}}{(k-n-1)\,!} z^n\,, \tag{2.37}$$

where B_i denotes the Bernoulli numbers. Then the period function r_{E_k} of the holomorphic Eisenstein series E_k is

$$r_{E_k}(z) = \omega^- p_k^-(z) + \omega^+ p_k^+(z)\,, \tag{2.38}$$

with $\omega^- = -\frac{(k-2)\,!}{2}$ and $\omega^+ = \frac{\zeta(k-1)}{(2\pi i)^{k-1}}\omega^-$.

The period functions respectively period polynomials of the modular forms of weight k of the group $\Gamma(1)$ hence belong to the space [**Zag91**]

$$W_{M_k(\Gamma(1))} := \left\{ \phi(z) \in \oplus_{n=-1}^{k-1} \mathbb{C} z^n \;\middle|\; \phi\,|\,(1+Q) = \phi\,|\,(1+S+S^2) = 0 \right\}. \tag{2.39}$$

In [**Zag91**] and [**CM99**] it is shown that the even respectively odd functions in $W_{M_k(\Gamma(1))}$ are exactly the solutions of the equation

$$\lambda\,(\phi(z) - \phi(z+1)) = z^{k-2}\phi(1+\frac{1}{z})\,, \tag{2.40}$$

with $\lambda = 1$ respectively -1 in the space $\oplus_{n=-1}^{k-1}\mathbb{C}z^n$.

2.4.5. *The period functions for the Maaß wave forms of* $\Gamma(1)$. The theory of period polynomials of modular forms of even positive weight for the group $\Gamma(1)$ was generalized by D. Zagier and J. Lewis to nonholomorphic automorphic forms like the Maaß wave forms [**LZ97**]. Let $u_s(z)$ be a Maaß wave form of the Laplace-Beltrami operator $-\Delta$ with eigenvalue $s(1-s)$, i.e., $-\Delta u_s(z) = s(1-s)u_s(z)$ and assume $u_s(z)$ obeys the conditions in section 2.3.2. Then the function $u_s(z)$ can be decomposed into an even and odd part under the reflection This follows from the invariance of the Laplace-Beltrami operator under this reflection with respect to the imaginary axis. Writing $z = x+iy$, the Taylor expansion of the even respectively the odd Maaß wave form $u_s(z)$ has the form [**LZ97**]

$$u_s^+(x+iy) = \sqrt{y}\sum_{n=1}^\infty a_n K_{s-\frac{1}{2}}(2\pi n y)\, cos(2\pi n x)$$

respectively

$$u_s^-(x+iy) = \sqrt{y}\sum_{n=1}^\infty a_n K_{s-\frac{1}{2}}(2\pi n y)\, sin(2\pi n x)\,,$$

where K_ν denotes the modified Bessel function of order ν. None of these functions, besides the constant function $u_s^+(z) = c$ is explicitly known. In his work on harmonic analysis for the modular group $\Gamma(1)$ J. Lewis established an interesting new relation between the Maaß wave forms and certain holomorphic functions [**Lew97**]:

THEOREM 2.3. *Let Γ be the modular group $\Gamma(1)$ and $0 < \Re s \leq 1$. For every even Maaß wave form with eigenvalue $s(1-s)$ there exists a holomorphic function ψ_s on $\mathbb{C} \setminus (-\infty, 0]$ with $\psi_s(z) = O(1/z)$ for $z \to \infty$ which fulfills the functional equation*

$$\psi_s(z) - \psi_s(z+1) = z^{-2s}\psi_s(1+z^{-1})\,. \tag{2.41}$$

On the other hand, for every solution ψ_s of (2.41) which is holomorphic on $\mathbb{C}\setminus(-\infty,0]$ and $\psi_s(z) = O(1/z)$ for $z \to \infty$, there exists an even Maaß wave form with eigenvalue $s(1-s)$.

An analogous result holds for the odd Maaß wave forms when equation (2.41) is replaced by

$$-\psi_s(z) + \psi_s(z+1) = z^{-2s}\psi_s(1+z^{-1})\,. \tag{2.42}$$

The even Maaß wave forms $u_s(z)$ are related to the solutions $\psi_s(z)$ of (2.41) by [**LZ97**]:

$$\psi_s(z) = z\int_0^\infty y^{\,s}\, u_s(iy)\,(z^2+y^2)^{-s-1}\,dy\,, \quad \Re(z) > 0 \tag{2.43}$$

and the odd Maaß wave forms $u_s(z)$ are related to the solutions $\psi_s(z)$ of (2.42) by

$$\psi_s(z) = \int_0^\infty \frac{\partial}{\partial x}\, u_s(iy)\, y^{\,s}\,(\,z^2+y^2\,)^{-s}\,dy\,, \quad \Re(z) > 0\,. \tag{2.44}$$

Comparing with (2.41) respectively (2.42) shows that (2.40) is just a special case of (2.41) respectively (2.42) with $s = \frac{2-k}{2}$. For this reason Lewis and Zagier called the functions $\psi_s(z)$ the period functions of the Maaß wave forms in analogy to the period polynomials for the holomorphic modular forms. Zagier found also a special family of solution of equation (2.41) for arbitrary $s \neq 1$ [**Zag92b**] given for $\Re s > 1$ as

$$\psi_s(z) = \sum_{m,n\geq 1} (\frac{1}{mz+n})^{2s} + \frac{1}{2}\zeta(2s)(1+(\frac{1}{z})^{2s})\,, \tag{2.45}$$

where ζ denotes the Riemann zeta function. This function can be analytically continued to the entire complex s-plane and defines for $s \in \mathbb{C}\setminus\{1\}$ a holomorphic family of solutions of equation (2.41).

Up to a factor the analytic extension of $\psi_s(z)$ in (2.45) is just the transform (2.43) of the non-holomorphic Eisenstein series (2.19) [**CM98**] for $\Gamma(1)$ and coincides for $s = -N$, $N \in \mathbb{N}$ with the odd part of the period functions (2.37) of the holomorphic Eisenstein series [**CM99**] for this group.

2.4.6. *The period functions and the transfer operator of* $\Gamma(1)$. Now let $f_\beta(z)$ for $\Re\beta > \frac{1}{2}$ be an eigenfunction of the transfer operator [**CM99**]

$$\mathcal{L}_\beta f_\beta(z) = \sum_{n=1}^{\infty}(\frac{1}{z+n})^{2\beta} f_\beta(\frac{1}{z+n})\,.$$

for $\Gamma(1)$ with eigenvalue λ_β

$$\left[\lambda_\beta - \mathcal{L}_\beta\right] f_\beta(z) = 0\,. \tag{2.46}$$

A simple calculation shows that $\mathcal{L}_\beta\, f_\beta(z) - \mathcal{L}_\beta\, f_\beta(z+1) = (\frac{1}{z+1})^{2\beta} f_\beta(\frac{1}{z+1})$ and therefore f_β fulfills the functional equation

$$\lambda_\beta \left[f_\beta(z) - f_\beta(z+1) \right] = (z+1)^{-2\beta}\, f_\beta(\frac{1}{1+z})\,. \tag{2.47}$$

Hence every eigenfunction $f_\beta(z)$ of $\mathcal{L}_\beta$ is a solution of equation (2.47). On the other hand, let $f_\beta(z)$ be a solution of (2.47), i.e.,

$$\lambda_\beta f_\beta(z) = \lambda_\beta f_\beta(z+1) + (z+1)^{-2\beta}\, f_\beta(\frac{1}{1+z})\,,$$

then after N iterations one gets

$$\lambda_\beta f_\beta(z) = \lambda_\beta f_\beta(z+N) + \sum_{n=1}^{N} (z+n)^{-2\beta}\, f_\beta(\frac{1}{n+z})\,. \tag{2.48}$$

Assuming $\lim_{N\to\infty} f(z+N) = 0$ this yields

$$\lambda_\beta f_\beta(z) = \sum_{n=1}^{\infty} (z+n)^{-2\beta}\, f_\beta(\frac{1}{n+z}) = \mathcal{L}_\beta\, f_\beta(z)\,.$$

That is, every solution f_β of (2.47) for $\Re\beta > \frac{1}{2}$ with $\lim_{z\to\infty} f_\beta(z) = 0$ is an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue λ_β.

Inserting $f_\beta(z) = \psi_\beta(z+1)$ into (2.47), the functional equation (2.47) for the eigenfunction f_β of $\mathcal{L}_\beta$ turns out to be

$$\lambda_\beta\left(\psi_\beta(z) - \psi_\beta(z+1)\right) = z^{-2\beta}\psi_\beta(1+z^{-1})\,, \tag{2.49}$$

which for $\lambda_\beta = \pm 1$ is nothing but Lewis functional equation of the period polynomials respectively period functions in (2.41). An analogous result holds true for arbitrary β [**CM99**] for the analytic extension of the transfer operator.

In the present paper this relation between the eigenfunctions of $\mathcal{L}_\beta$ and the solutions of the Lewis equation (2.49) and hence the period functions of modular forms shall be generalized to an arbitrary subgroup $\Gamma \subseteq \Gamma(1)$ with finite index.

2.4.7. *The period polynomials of Eichler for* $\Gamma_0(2)$. Let us briefly consider Eichler's theory for an arbitrary subgroup $\Gamma \subseteq \Gamma(1)$. For simplicity we restrict ourselves to the group $\Gamma_0(2)$. This group has three generators, namely for example

$$e_1 := T,\quad e_2 := QT^{-2}Q = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \quad\text{and}\quad e_3 := T^{-1}QT^{-2}Q = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\,,$$

which fulfill the relations [**BV97**]

$$e_1\, e_2\, e_3 = id \quad\text{and}\quad e_3^2 = id\,. \tag{2.50}$$

An argument analogous to the case $\Gamma(1)$ shows that one can choose $\Omega_{e_1} = \Omega_T = 0$. We write Ω for Ω_{e_3}. Then Ω_{e_2} can obviously be expressed in terms of Ω.

Let $u(z)$ be a cusp form of weight k for the group $\Gamma_0(2)$. Setting $\Theta = 0$ in (2.24) one gets for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$

$$\Omega_\sigma(\tau) = \int_{\tau_0}^{\sigma\tau_0} (a\tau + b - (c\tau + d)z\,)^{k-2}\, u(z)\, dz, \tag{2.51}$$

The relation $e_3^2 = 1$ leads to the condition

$$\Omega\,|_{2-k}\, e_3 + \Omega = 0 \tag{2.52}$$

for $\Omega = \Omega_{e_3}$, which has the explicit form

$$\Omega\Big(\frac{z+1}{-2z-1}\Big)\,(-2z-1)^{k-2} + \Omega(z) = 0\,. \tag{2.53}$$

From the work of Eichler [**Eic57**] and Knopp [**Kno78**] it follows that the polynomial solutions of this equation generate all period polynomials of $\Gamma_0(2)$. Trivial solutions of this equation are given by the period functions $\Omega_Q(z)$ for the group $\Gamma(1)$: one just expresses $\Omega_{e_3}(z)$ with the cocycle relations (2.25) through Ω_Q which gives

$$\Omega_{e_3}(z) = z^{k-2}\Omega_Q(-\frac{2z+1}{z}) + \Omega_Q(z)\,. \tag{2.54}$$

This Ω_{e_3} obviously fulfills equation (2.53). The solutions of (2.53) shall be compared with the eigenfunctions of the transfer operator for $\Gamma_0(2)$ in section 4.5.3.

3. The functional equations for the eigenfunctions of the transfer operator

3.1. Lewis equations and master equations. Let Γ be a subgroup of $\Gamma(1)$ with finite index and λ_β be an eigenvalue of the analytically continued transfer operator (2.8) with eigenfunction $\underline{f}_\beta(z,\varepsilon)$, i.e.,

$$\lambda_\beta \underline{f}_\beta(z,\varepsilon) - \tilde{\mathcal{L}}_\beta \underline{f}_\beta(z,\varepsilon) = 0\,. \tag{3.1}$$

This eigenequation implies immediately that the eigenfunction $\underline{f}_\beta(z,\varepsilon)$ can be analytically continued from D to $\mathbb{C}^*$, where $\mathbb{C}^*$ denotes the complex plane $\mathbb{C}\backslash(-\infty,-1]$ cut along the negative real axis $(-\infty,-1]$. If $\underline{f}_\beta(z,\varepsilon)$ is a solution of (3.1), the function $\underline{f}_\beta(z,\varepsilon)$ is automatically also a solution of the equation:

$$\begin{aligned} &\lambda_\beta \underline{f}_\beta(z,\varepsilon) - \tilde{\mathcal{L}}_\beta \underline{f}_\beta(z,\varepsilon) \\ &= \chi^\Gamma(QT^\varepsilon Q)\left[\lambda_\beta \underline{f}_\beta(z+1,\varepsilon) - \tilde{\mathcal{L}}_\beta \underline{f}_\beta(z+1,\varepsilon)\right]. \end{aligned} \tag{3.2}$$

The factor $\chi^\Gamma(QT^\varepsilon Q)$ has been introduced for convenience as will be seen later. Conversely, a solution of (3.2) which is holomorphic in D is a solution of (3.1) only if after N iterations of relation (3.2), i.e.,

$$\begin{aligned} &\lambda_\beta \underline{f}_\beta(z,\varepsilon) - \tilde{\mathcal{L}}_\beta \underline{f}_\beta(z,\varepsilon) \\ &= \chi^\Gamma(QT^{N\varepsilon} Q)\left[\lambda_\beta \underline{f}_\beta(z+N,\varepsilon) - \tilde{\mathcal{L}}_\beta \underline{f}_\beta(z+N,\varepsilon)\right], \end{aligned} \tag{3.3}$$

the right-hand side of (3.3) vanishes in the limit $N \to \infty$ uniformly for $z \in D$. That is, a solution of equation (3.2) fulfills

$$\lim_{N\to\infty} \chi^\Gamma(QT^\varepsilon Q)\left[\lambda_\beta \underline{f}_\beta(z+N,\varepsilon) - \tilde{\mathcal{L}}_\beta \underline{f}_\beta(z+N,\varepsilon)\right] = 0 \tag{3.4}$$

uniformly for $z \in D$ iff $\underline{f}_\beta(z,\varepsilon)$ is an eigenfunction of $\tilde{\mathcal{L}}_\beta$. However, in the limit $N \to \infty$ one gets for $\Re\beta > -\frac{\kappa}{2}$ from (2.8)

$$\begin{aligned} &\tilde{\mathcal{L}}_\beta \underline{f}_\beta(z+N,\varepsilon) \sim \tilde{\mathcal{A}}_\beta^{(\kappa)} \underline{f}_\beta(z+N,\varepsilon) \\ &= \sum_{l=0}^{\kappa}\sum_{m=1}^{r} (\frac{1}{r})^{2\beta+l}\chi^\Gamma(QT^{m\varepsilon})\,\frac{\underline{f}_\beta^{(l)}(0,-\varepsilon)}{l\,!}\zeta(2\beta+l,\frac{z+m+N}{r})\,. \end{aligned} \tag{3.5}$$

Hence in the region $\Re\beta > -\frac{\kappa}{2}$ the condition

$$\lim_{z\to\infty}\big[\lambda_\beta \underline{f}_\beta(z,\varepsilon) - \sum_{l=0}^{\kappa}\sum_{m=1}^{r}(\frac{1}{r})^{2\beta+l}\chi^\Gamma(QT^{m\varepsilon})\frac{\underline{f}_\beta^{(l)}(0,-\varepsilon)}{l!}\zeta(2\beta+l,\frac{z+m}{r})\,\big]=0 \tag{3.6}$$

is necessary and sufficient for a solution $\underline{f}_\beta(z,\,\varepsilon)$ of equation (3.2) to be an eigenfunction of the operator $\tilde{\mathcal{L}}_\beta$ with eigenvalue λ_β.

Inserting $\tilde{\mathcal{L}}_\beta$ from (2.8) then into (3.2) one gets for an eigenfunction $\underline{f}_\beta$ of $\tilde{\mathcal{L}}_\beta$ the following equation

$$\lambda_\beta\Big[\underline{f}_\beta(z,\varepsilon)-\chi^\Gamma(QT^\varepsilon Q)\,\underline{f}_\beta(z+1,\varepsilon)\Big]-(\frac{1}{z+1})^{2\beta}\chi^\Gamma(QT^\varepsilon)\underline{f}_\beta(\frac{1}{z+1},-\varepsilon)=0\,. \tag{3.7}$$

From this follows that the transfer operator $\tilde{\mathcal{L}}_\beta$ doesn't have eigenvalue zero.

Equation (3.7) is a functional equation for $\underline{f}_\beta$ with a free parameter λ_β. The spectrum of the transfer operator $\tilde{\mathcal{L}}_\beta$ consists of the set of λ_β's for which a holomorphic solution $\underline{f}_\beta \in \oplus_{i=1\cdots 2\mu}B(D)$ of equation (3.7) obeying the asymptotics (3.6) can be found. Since the operator $\tilde{\mathcal{L}}_\beta$ for $\beta \neq \beta_\kappa = \frac{1-\kappa}{2}$ is nuclear [**CMct**] only for a discrete set of λ_β there exist holomorphic solutions $\underline{f}_\beta$ in (3.7) fulfilling the asymptotic condition in (3.6).

Due to relation (2.13) the eigenvalue $\lambda_\beta = +1$ of $\tilde{\mathcal{L}}_\beta$ is responsible for the zeros of Selberg's zeta function. However, for our later discussion we consider here the more general case $\lambda_\beta = \pm 1$. First, let us define the quantity

$$\begin{aligned} Y(z,\varepsilon) \;&:=\; \lambda_\beta\,\underline{f}_\beta(z,\varepsilon) - \lambda_\beta\chi^\Gamma(QT^\varepsilon Q)\,\underline{f}_\beta(z+1,\varepsilon)\\ &\qquad -(\frac{1}{z+1})^{2\beta}\,\chi^\Gamma(QT^\varepsilon)\,\underline{f}_\beta(\frac{1}{z+1},-\varepsilon)\,, \end{aligned} \tag{3.8}$$

which is nothing but the left-hand side of (3.7). Suppose $\underline{f}_\beta$ is a solution of (3.7), then $Y(z,\varepsilon)$ is identically zero. Inserting (3.8) into the identity

$$\begin{aligned} &(\frac{1}{z+1})^{2\beta}\,\chi^\Gamma(QT^\varepsilon)\,Y(\frac{1}{z+1},-\varepsilon)\\ &-\lambda_\beta\,(\frac{1}{z+2})^{2\beta}\,\chi^\Gamma(QT^\varepsilon QT^{-\varepsilon}QT^{-\varepsilon}Q)\,Y(\frac{-1}{z+2},\varepsilon)+\lambda_\beta\,Y(z,\varepsilon)\equiv 0\,, \end{aligned} \tag{3.9}$$

we get

$$\begin{aligned} &(\frac{1}{z+1})^{2\beta}\,\chi^\Gamma(QT^\varepsilon)\Big[\lambda_\beta\,\underline{f}_\beta(\frac{1}{z+1},-\varepsilon)\\ &\quad-\lambda_\beta\chi^\Gamma(QT^{-\varepsilon}Q)\,\underline{f}_\beta(\frac{z+2}{z+1},-\varepsilon)\;-(\frac{z+1}{z+2})^{2\beta}\,\chi^\Gamma(QT^{-\varepsilon})\,\underline{f}_\beta(\frac{z+1}{z+2},\varepsilon)\Big]\\ &-\lambda_\beta\,(\frac{1}{z+2})^{2\beta}\,\chi^\Gamma(QT^\varepsilon QT^{-\varepsilon}QT^{-\varepsilon}Q)\Big[\lambda_\beta\,\underline{f}_\beta(\frac{-1}{z+2},\varepsilon)\\ &\quad-\lambda_\beta\chi^\Gamma(QT^\varepsilon Q)\,\underline{f}_\beta(\frac{z+1}{z+2},\varepsilon)-(\frac{z+2}{z+1})^{2\beta}\,\chi^\Gamma(QT^\varepsilon)\,\underline{f}_\beta(\frac{z+2}{z+1},-\varepsilon)\Big]\\ &+\lambda_\beta\Big[\lambda_\beta\,\underline{f}_\beta(z,\varepsilon)-\lambda_\beta\chi^\Gamma(QT^\varepsilon Q)\,\underline{f}_\beta(z+1,\varepsilon)\\ &\quad-(\frac{1}{z+1})^{2\beta}\,\chi^\Gamma(QT^\varepsilon)\,\underline{f}_\beta(\frac{1}{z+1},-\varepsilon)\Big]=0\,. \end{aligned}$$

It is easy to see that the terms involing $\underline{f}_\beta(\frac{1}{z+1},-\varepsilon)$, $\underline{f}_\beta(\frac{z+2}{z+1},-\varepsilon)$ respectively $\underline{f}_\beta(\frac{z+1}{z+2},\varepsilon)$ vanish. Using $QT^\varepsilon QT^{-\varepsilon}QT^{-\varepsilon}Q = QT^{2\varepsilon}$ which follows from the identity $(QT)^3 = id$ and setting $\lambda_\beta = \pm 1$ one finally gets the relation

$$\underline{f}_\beta(z,\varepsilon) - \chi^\Gamma(QT^\varepsilon Q)\,\underline{f}_\beta(z+1,\varepsilon) - (\frac{1}{z+2})^{2\beta}\chi^\Gamma(QT^{2\varepsilon})\,\underline{f}_\beta(\frac{-1}{z+2},\varepsilon) = 0\,. \tag{3.10}$$

Compared to equation (3.7) the parameter λ_β does not anymore appear and there is no sign change in the variable ε in (3.10). Every solution $\underline{f}_\beta$ of equation (3.7) solves equation (3.10) automatically. Conversely, it is easy to verify that under the condition

$$\underline{f}_\beta(z-1,\varepsilon) - \lambda_\beta\, z^{-2\beta}\,\chi^\Gamma(T^{-\varepsilon})\,\underline{f}_\beta(\frac{1}{z}-1,-\varepsilon) = 0, \quad \lambda_\beta = \pm 1, \tag{3.11}$$

every solution $\underline{f}_\beta$ of equation (3.10) is also a solution of equation (3.7). To see this one only has to replace z in (3.11) by $1 - \frac{1}{z+2}$, leading to

$$\underline{f}_\beta(\frac{-1}{z+2},\varepsilon) = \lambda_\beta\,(\frac{z+1}{z+2})^{-2\beta}\,\chi^\Gamma(T^{-\varepsilon})\,\underline{f}_\beta(\frac{1}{z+1},-\varepsilon)\,, \tag{3.12}$$

and then to replace $\underline{f}_\beta$ in the last term of (3.10) by (3.12). On the other hand, every solution $\underline{f}_\beta$ of equation (3.7) is also a solution of (3.11), since (3.11) simply follows from the identity

$$\chi^\Gamma(QT^\varepsilon QT^{-\varepsilon}Q)\,Y(z-1,\varepsilon) - \lambda_\beta\,\chi^\Gamma(QT^{2\varepsilon}Q)z^{-2\beta}Y(\frac{1}{z}-1,-\varepsilon) \equiv 0\,. \tag{3.13}$$

For $\lambda_\beta = \pm 1$ this implies

$$\begin{aligned}
&\chi^\Gamma(QT^\varepsilon QT^{-\varepsilon}Q)\,\times \\
&\Big[\lambda_\beta\,\underline{f}_\beta(z-1,\varepsilon) - \lambda_\beta\chi^\Gamma(QT^\varepsilon Q)\,\underline{f}_\beta(z,\varepsilon) - (\frac{1}{z})^{2\beta}\,\chi^\Gamma(QT^\varepsilon)\,\underline{f}_\beta(\frac{1}{z},-\varepsilon)\Big] \\
&-\lambda_\beta\,\chi^\Gamma(QT^{2\varepsilon}Q)z^{-2\beta}\,\times \\
&\Big[\lambda_\beta\,\underline{f}_\beta(\frac{1}{z}-1,-\varepsilon) - \lambda_\beta\chi^\Gamma(QT^{-\varepsilon}Q)\underline{f}_\beta(\frac{1}{z},-\varepsilon) - z^{2\beta}\chi^\Gamma(QT^{-\varepsilon})\underline{f}_\beta(z,\varepsilon)\Big] = 0\,.
\end{aligned}$$

The identity $(QT)^3 = id$ then leads to equation (3.11).

Define $\underline{\phi}(z,\varepsilon) := \underline{f}_\beta(z-1,\varepsilon)$, where we suppressed for the moment the dependence on β in the function $\underline{\phi}$. In terms of the function $\underline{\phi}$ the three functional equations (3.7), (3.10) and (3.11) can be written as

$$\begin{aligned}
(I)' \quad &\lambda_\beta\,\big[\,\underline{\phi}(z,\varepsilon) - \chi^\Gamma(QT^\varepsilon Q)\,\underline{\phi}(z+1,\varepsilon)\big] \\
&-z^{-2\beta}\chi^\Gamma(QT^\varepsilon)\,\underline{\phi}(1+\frac{1}{z},-\varepsilon) = 0\,,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
(II)' \quad &\underline{\phi}(z,\varepsilon) - \chi^\Gamma(QT^\varepsilon Q)\,\underline{\phi}(z+1,\varepsilon) \\
&-(z+1)^{-2\beta}\chi^\Gamma(QT^{2\varepsilon})\,\underline{\phi}(\frac{z}{z+1},\varepsilon) = 0\,, \quad \lambda_\beta = \pm 1\,,
\end{aligned} \tag{3.15}$$

$$(III)' \quad \underline{\phi}(z,\varepsilon) - \lambda_\beta\,z^{-2\beta}\,\chi^\Gamma(T^{-\varepsilon})\,\underline{\phi}(\frac{1}{z},-\varepsilon) = 0\,, \quad \lambda_\beta = \pm 1. \tag{3.16}$$

We call equation $(I)'$ the (generalized vector-valued) Lewis equation, because for the trivial one-dimensional representation $\chi^\Gamma = 1$ and $\lambda_\beta = +1$ equation (3.14) when supressing the parameter ε is the same as equation (2.41) of Lewis for $\Gamma(1)$ [**Lew97**]. We call equation $(II)'$ the (generalized vector-valued) master equation, because this equation for the trivial one-dimensional representation $\chi^\Gamma = 1$ when supressing the parameter ε was called master equation by Lewis and Zagier [**LZ97**]. Equation (3.7) respectively (3.10) corresponding to $(I)'$ respectively $(II)'$ are the Lewis respectively the master equation for the function $\underline{f}_\beta$.

REMARK 3.1. The three equations (3.14), (3.15) and (3.16) are equations for vector-valued functions. Every equation consists of μ equations, where μ is the dimension of the representation χ^Γ. The concrete form of these μ equations depends on the choice of the basis. For instance applying the matrix $\chi^\Gamma(Q)$ on the left to (3.14) and renaming $\chi^\Gamma(Q)\underline{\phi}$ by $\underline{\phi}$ one obtains a functional equation of the form:

$$\lambda_\beta\,\left[\,\underline{\phi}(z,\varepsilon) - \chi^\Gamma(T^\varepsilon)\,\underline{\phi}(z+1,\varepsilon)\,\right] - z^{-2\beta}\chi^\Gamma(T^\varepsilon Q)\,\underline{\phi}(1+\frac{1}{z},-\varepsilon) = 0\,.$$

Obviously, this equation is equivalent to (3.14).

If the induced representation χ^Γ in $\tilde{\mathcal{L}}_\beta$ in (3.1) has the property $\chi^\Gamma(T^2) = 1$, then the operator $\tilde{\mathcal{L}}_\beta$ can be decomposed as $\tilde{\mathcal{L}}_\beta = \begin{pmatrix} 0 & \mathcal{L}_\beta \\ \mathcal{L}_\beta & 0 \end{pmatrix}$ for $\Gamma(1)$ in (2.11). In complete analogy to the above discussion for the operator $\tilde{\mathcal{L}}_\beta$ one can derive functional equations for the eigenfunctions of $\mathcal{L}_\beta$ similar to $(I)'$, $(II)'$ and $(III)'$:

$$\text{(3.17)}\,(I)\quad \lambda_\beta\,[\underline{\phi}(z) - \chi^\Gamma(QTQ)\underline{\phi}(z+1)] - z^{-2\beta}\chi^\Gamma(QT)\underline{\phi}(1+\frac{1}{z}) = 0,$$

$$\text{(3.18)}\,(II)\quad \underline{\phi}(z) - \chi^\Gamma(QTQ)\underline{\phi}(z+1) - (z+1)^{-2\beta}\chi^\Gamma(Q)\underline{\phi}(\frac{z}{z+1}) = 0,\quad \lambda_\beta = \pm 1\,,$$

$$\text{(3.19)}\,(III)\ \ \underline{\phi}(z) - \lambda_\beta\, z^{-2\beta}\,\chi^\Gamma(T)\,\underline{\phi}(\frac{1}{z}) = 0\,,\quad \lambda_\beta = \pm 1\,.$$

Compared to equations $(I)'$, $(II)'$ and $(III)'$ the function $\underline{\phi}(z,\varepsilon)$ has been changed simply to $\underline{\phi}(z)$. Criterion (3.6) now reads

$$\begin{aligned} &\lim_{z\to\infty}\,\big[\lambda_\beta \underline{f}_\beta(z) - \\ \text{(3.20)}\qquad &\sum_{l=0}^{\kappa}\sum_{m=1}^{r}(\frac{1}{r})^{2\beta+l}\chi^\Gamma(QT^m)\frac{\underline{f}_\beta^{(l)}(0)}{l!}\zeta(2\beta+l,\frac{z+m}{r})\,\big] = 0\,. \end{aligned}$$

Summarizing, we showed that an eigenfunction of the operator $\tilde{\mathcal{L}}_\beta$ with the argument shifted from z to $z+1$ is always a solution of Lewis equation (3.14) with $\lambda_\beta = +1$. Conversely, a solution of Lewis equation (3.14) with parameter value $\lambda_\beta = +1$ is an eigenfunction of $\tilde{\mathcal{L}}_\beta$ with eigenvalue $\lambda_\beta = +1$ only if this solution satisfies the criterion (3.6). Next we will discuss this criterion for different regions of the variable $\beta \in \mathbb{C}$.

3.2. Asymptotic properties of the eigenfunctions of the transfer operator. Let us consider first the region $0 < \Re\beta \le \frac{1}{2}$ which includes the two vertical lines $\Re\beta = \frac{1}{2}$ respectively $\Re\beta = \frac{1}{4}$ on which the spectral respectively the Riemann (if the Riemann hypothesis is true) zeros of $Z_S(\beta;\,\Gamma(1);\,\chi^\Gamma)$ lie. To obtain the analytic continuation of the transfer operator to this region one has to set $\kappa = 0$ in (3.6). Then criterion (3.6) reads

$$\text{(3.21)}\qquad \lim_{z\to\infty}\,\big[\lambda_\beta \underline{f}_\beta(z,\varepsilon) - \sum_{m=1}^{r}(\frac{1}{r})^{2\beta}\chi^\Gamma(QT^{m\varepsilon})\underline{f}_\beta(0,-\varepsilon)\zeta(2\beta,\frac{z+m}{r})\,\big] = 0\,.$$

For $\beta \neq \frac{1}{2}$ the function $\zeta(2\beta, \frac{z+m}{r})$ doesn't have any singularity. For large z with $|arg\, z| < \pi$ the Hurwitz zeta function behaves like [**MOS66**] p.25:

$$\begin{aligned} \zeta(s,z) \quad &= \quad \frac{1}{\Gamma(s)}\Big[z^{1-s}\Gamma(s-1) + \frac{1}{2}\Gamma(s)z^{-s} + \sum_{n=1}^{m-1} \frac{B_{2n}}{(2n)\,!}\Gamma(s+2n-1)z^{-2n-s+1}\Big] \\ &\quad\quad +O(z^{-2m-s-1})\,, \end{aligned} \tag{3.22}$$

where $\Gamma(s)$ is the gamma function and $B_n(z)$ is the n-th Bernoulli polynomial of degree n. That is, in the rigion $0 < \Re s \leq 1$ all the terms in (3.22) vanish in the limit $z \to \infty$, only the leading term $z^{1-s}\frac{\Gamma(s-1)}{\Gamma(s)}$ survives. Hence, in the region $0 < \Re\beta \leq \frac{1}{2}$ criterion (3.21) can be simplified to

$$\begin{aligned} &\lim_{z\to\infty} \big[\lambda_\beta \underline{f}_\beta(z,\varepsilon) - \\ &\frac{\underline{f}_\beta(0,-\varepsilon)}{r}\frac{\Gamma(2\beta-1)}{\Gamma(2\beta)}\sum_{m=1}^{r} \chi^\Gamma(QT^{m\varepsilon})(z+m)^{1-2\beta}\big] = 0\,. \end{aligned} \tag{3.23}$$

To apply criterion (3.6) at the points $\beta = \beta_\kappa = \frac{1-\kappa}{2}$, $\kappa \in \mathbb{N}_0$, which in general are poles of the meromorphic operator $\tilde{\mathcal{L}}_\beta$ in (2.8), we recall first the properties of the eigenvalues and the eigenfunctions of the operators $\tilde{\mathcal{L}}_\beta$ and $\tilde{\mathcal{A}}_\beta^{(\kappa)}$ in the limit $\beta \to \beta_\kappa$: Since in general also $\tilde{\mathcal{A}}_\beta^{(\kappa)}$ is singular at $\beta = \beta_\kappa$, both $\tilde{\mathcal{L}}_\beta$ and $\tilde{\mathcal{A}}_\beta^{(\kappa)}$ can be singular and not defined at the critical points $\beta = \beta_\kappa$. As in the case $\Gamma(1)$ [**CM98**] the operators $\tilde{\mathcal{L}}_\beta$ and $\tilde{\mathcal{A}}_\beta^{(\kappa)}$ have besides the eigenvalues λ_β which are regular in the limit $\beta \to \beta_\kappa$ also eigenvalues $\hat{\lambda}_\beta$ becoming singular for $\beta \to \beta_\kappa$. The eigenfunctions of both λ_β and $\hat{\lambda}_\beta$ can however be rescaled by a multiplicative factor $(\beta - \beta_\kappa)^\delta$ with some $\delta \in \mathbb{Q}$ in such a way that the rescaled eigenfunctions remain regular in the limit $\beta \to \beta_\kappa$. Now if λ_β is a regular eigenvalue of $\tilde{\mathcal{L}}_\beta$ in (2.8) with regular eigenfunction $\underline{f}_\beta(z)$ for $\beta \to \beta_\kappa$, then the only singular term in the sum (2.9) respectively in (3.6), namely the term $l = \kappa$ must vanish. That is, either $\sum_{m=1}^{r} \chi^\Gamma(QT^{m\varepsilon})$ must be zero or $\underline{f}_\beta^{(\kappa)}(0,-\varepsilon)$ in (2.9) respectively (3.6) must vanish fast enough for $\beta \to \beta_\kappa$ to cancel the singularity in the Hurwitz function which for $\beta \to \beta_\kappa$ behaves like

$$\zeta(2\beta + \kappa, \frac{z+m}{r}) \sim \frac{1}{2}\frac{1}{\beta - \beta_\kappa}\,.$$

The remaining terms $l = 0, \cdots, \kappa - 1$ in (3.6) are polynomials of degree $\kappa - l$, because the zeta functions $\zeta(2\beta_\kappa + l, z)$, $l = 0, \cdots, \kappa - 1$ are for these β-values polynomials of degree $-(2\beta_\kappa + l) + 1 = \kappa - l$, since [**MOS66**]

$$\zeta(-n,z) = \frac{B_{n+1}(z)}{n+1},\ n \in \mathbb{N}_0 \tag{3.24}$$

with B_n the Bernoulli polynomial of degree $n+1$ Therefore, if λ_β and $\underline{f}_\beta$ fulfill the Lewis equation (3.7), are regular for $\beta \to \beta_\kappa$ and obey the asymptotics

$$\begin{aligned} &\lim_{z\to\infty} \Big[\lambda_{\beta_\kappa} \underline{f}_{\beta_\kappa}(z,\varepsilon) - \\ &\sum_{l=0}^{\kappa}\sum_{m=1}^{r} (\frac{1}{r})^{2\beta_\kappa + l}\chi^\Gamma(QT^{m\varepsilon})\frac{\underline{f}_{\beta_\kappa}^{(l)}(0,-\varepsilon)}{l\,!}\frac{B_{k-l}(\frac{z+m}{r})}{k-l}\Big] = 0\,, \end{aligned} \tag{3.25}$$

then they define a regular eigenvalue with eigenfunction $\underline{f}_\beta$ for $\tilde{\mathcal{L}}_\beta$ for $\beta = \beta_\kappa$. We call them simply regular eigenvalue and eigenfunction for $\mathcal{L}_{\beta_\kappa}$, although $\mathcal{L}_{\beta_\kappa}$ is not defined in the strict sense. Similar to the arguments in [**CM99**] one shows that, a vector-valued solution of Lewis equation (3.14) respectively (3.17) for $\beta = \beta_\kappa$ with components in $\oplus_{n=0}^{\kappa}\mathbb{C}z^n \oplus \frac{\mathbb{C}}{z}$ is always an eigenfunction of the transfer operator $\tilde{\mathcal{L}}_{\beta_\kappa}$ respectively $\mathcal{L}_{\beta_\kappa}$.

For $\Gamma(1)$ there are two kinds of eigenfunctions for $\mathcal{L}_{\beta_\kappa}$ [**CM99**]. One class lies in the polynomial space $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$. An example of such a polynomial eigenfunction is

$$f_{-5}(z) = 10z + 45z^2 - 240z^3 - 1025z^4 - 1458z^5 - 915z^6 - 240z^7 + 10z^9 + z^{10}$$

for $\kappa = 11$ (corresponding to $\beta = -5$). The function $\phi(z) = f_{-5}(z-1)$ is just the even part of the period polynomial of the holomorphic cusp form of $\Gamma(1)$ of weight 12. The other class consists of functions in $\oplus_{n=0}^{\kappa}\mathbb{C}z^n \oplus \frac{\mathbb{C}}{z+1}$. An example for such a solution is

$$f_{-1}(z) = \frac{1}{z+1} - 4 - 2z + 3z^2 + z^3$$

for $\kappa = 3$ (corresponding to $\beta = -1$), with the property $f(z) \sim_{z\to\infty} z^3$ and $f^{(3)}(0) = 0$. The function $\phi(z) = f_{-1}(z-1)$ is the odd part of the period function of the holomorphic Eisenstein series of weight 4 for the group $\Gamma(1)$.

We will next study solutions of the Lewis equation (3.14) for different subgroups Γ of $\Gamma(1)$ which are also eigenfunctions for the corresponding transfer operators.

4. Eigenfunctions of the transfer operator for subgroups of $\Gamma(1)$

Let Γ be one of the subgroups $\{\Gamma(1), \Gamma_2, \Gamma_0(2), \Gamma^0(2), \Gamma_\vartheta, \Gamma(2)\}$ of $\Gamma(1)$. Consider then the unitary irreducible representations of $\Gamma(1)$ defined through the following representations of the generators Q and T:

$$\chi_1(Q) = 1, \qquad \chi_1(T) = 1, \tag{4.1}$$

$$\chi_{-1}(Q) = -1, \qquad \chi_{-1}(T) = -1, \tag{4.2}$$

$$\chi_2(Q) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \qquad \chi_2(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.3}$$

The induced representations χ^Γ of $\Gamma(1)$ induced from the trivial representation of the subgroup Γ can then be decomposed as follows [**CMct**]:

$$\begin{aligned}
\chi^{\Gamma(1)} &= \chi_1\,, \\
\chi^{\Gamma_2} &= \chi_1 \oplus \chi_{-1}\,, \\
\chi^{\Gamma'} &= \chi_1 \oplus \chi_2\,, \quad \text{for } \Gamma' \text{ one of the subgroups } \{\Gamma_0(2), \Gamma^0(2), \Gamma_\vartheta\}, \\
\chi^{\Gamma(2)} &= \chi_1 \oplus \chi_{-1} \oplus \chi_2 \oplus \chi_2\,.
\end{aligned} \tag{4.4}$$

Since any of these representations χ^Γ has the property $\chi^\Gamma(T^2) = 1$, we can restrict the discussion of the operator $\tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi^\Gamma}$ to the operator $\mathcal{L}_\beta^{\Gamma(1),\chi^\Gamma}$ since $\tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi^\Gamma} =$

$\begin{pmatrix} 0 & \mathcal{L}_\beta^{\Gamma(1),\,\chi^\Gamma} \\ \mathcal{L}_\beta^{\Gamma(1),\,\chi^\Gamma} & 0 \end{pmatrix}$ (see section 2.2.2). With

$$\mathcal{L}_\beta^{\chi_1} f_\beta(z) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2\beta} f_\beta(\frac{1}{z+n}), \tag{4.5}$$

$$\mathcal{L}_\beta^{\chi_{-1}} f_\beta(z) = \sum_{n=1}^{\infty} (-1)^{n+1} (\frac{1}{z+n})^{2\beta} f_\beta(\frac{1}{z+n}), \tag{4.6}$$

$$\mathcal{L}_\beta^{\chi_2} \underline{f}_\beta(z) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2\beta} \chi_2(QT^n) \underline{f}_\beta(\frac{1}{z+n}) \tag{4.7}$$

the tranfer operators for the different subgroups Γ can be written as follows:

$$\mathcal{L}_\beta^{\Gamma(1),\,\chi^{\Gamma(1)}} f(z) = \mathcal{L}_\beta^{\chi_1} f(z), \tag{4.8}$$

$$\mathcal{L}_\beta^{\Gamma(1),\,\chi^{\Gamma_2}} \underline{f}(z) = \begin{pmatrix} \mathcal{L}_\beta^{\chi_1} & 0 \\ 0 & \mathcal{L}_\beta^{\chi_{-1}} \end{pmatrix} \underline{f}(z), \tag{4.9}$$

$$\mathcal{L}_\beta^{\Gamma(1),\,\chi^{\Gamma}} \underline{f}(z) = \begin{pmatrix} \mathcal{L}_\beta^{\chi_1} & 0 \\ 0 & \mathcal{L}_\beta^{\chi_2} \end{pmatrix} \underline{f}(z), \tag{4.10}$$

$$\mathcal{L}_\beta^{\Gamma(1),\,\chi^{\Gamma(2)}} \underline{f}(z) = \begin{pmatrix} \mathcal{L}_\beta^{\chi_1} & & & 0 \\ & \mathcal{L}_\beta^{\chi_{-1}} & & \\ & & \mathcal{L}_\beta^{\chi_2} & \\ 0 & & & \mathcal{L}_\beta^{\chi_2} \end{pmatrix} \underline{f}(z). \tag{4.11}$$

We can look for solutions $\underline{\varphi}(z)$ of Lewis equation (3.17) for the two parameter values $\lambda_\beta = \pm 1$ where χ^Γ has been replaced by χ_i for $i = 1, -1, 2$. Then the functions $\underline{f}_\beta(z) = \underline{\varphi}(z+1)$ fulfilling criterion (3.20) (with χ^Γ replaced by χ_i) are eigenfunctions of the operator $\tilde{\mathcal{L}}_\beta^{\chi_i}$. Let us briefly recall for the following discussion the results for the transfer operator $\mathcal{L}_\beta^{\chi_1}$.

4.1. The eigenfunctions of the operator $\mathcal{L}_\beta^{\chi_1}$. The transfer operator (4.5)

$$\mathcal{L}_\beta^{\chi_1} f_\beta(z) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2\beta} f_\beta(\frac{1}{z+n})$$

is nothing else but the transfer operator of the Gauß transformation [**May91a**]. The corresponding functional equations in (3.17), (3.18) and (3.19) are then

$$(I) \quad \lambda_\beta \, [\, \phi(z) - \phi(z+1)\,] - z^{-2\beta}\, \phi(1+\frac{1}{z}) = 0, \tag{4.12}$$

$$(II) \quad \phi(z) - \phi(z+1) - (z+1)^{-2\beta}\, \phi(\frac{z}{z+1}) = 0, \quad \lambda_\beta = \pm 1, \tag{4.13}$$

$$(III) \quad \phi(z) - \lambda_\beta\, z^{-2\beta}\, \phi(\frac{1}{z}) = 0, \quad \lambda_\beta = \pm 1. \tag{4.14}$$

The functions $f_\beta(z)$ and $\phi(z+1)$ are scalar functions. Suppose $\phi(z)$ is a solution of (4.12). Then the function $f_\beta(z) = \phi(z+1)$ is an eigenfunction of the operator $\mathcal{L}_\beta^{\chi_1}$ if it satisfies condition (3.20) with $r = 1$ and $\chi^\Gamma = 1$ which hence reads

$$\lim_{z\to\infty} \left[\lambda_\beta f_\beta(z) - \sum_{l=0}^{\kappa} \frac{f_\beta^{(l)}(0)}{l\,!} \zeta(2\beta + l, z+1) \,\right] = 0. \tag{4.15}$$

Defining $F(z) := z^\beta \phi(z)$ equation (4.14) can be rewritten as

$$F(z) = \lambda_\beta \, F(\frac{1}{z}) \,, \quad \lambda_\beta = \pm 1 \,. \tag{4.16}$$

That is, under the transformation $z \leftrightarrow \frac{1}{z}$ the function $F(z)$ is symmetric for $\lambda_\beta = +1$ respectively antisymmetric for $\lambda_\beta = -1$. If $F(z)$ is a solution of (4.16), then $\phi(z) = z^{-\beta} F(z)$ is a solution of (4.14).

It is well known [**Hej83**] that Selberg's zeta function $Z_S(\beta; \Gamma(1); 1)$ for the trivial representation of the modular group $\Gamma(1)$ has trivial zero's at $\beta = -N$, $N \in \mathbb{N}$, spectral zero's on $\Re\beta = \frac{1}{2}$ and Riemann zero's on $\Re\beta = \frac{1}{4}$ (if the Riemann conjecture is true). Since [**CMct**]:

$$Z_S(\beta; \Gamma(1); 1) = \det(1 - \tilde{\mathcal{L}}_\beta^{\chi_1}) = \det(1 + \mathcal{L}_\beta^{\chi_1}) \, \det(1 - \mathcal{L}_\beta^{\chi_1}) \,,$$

all the zeros β of the function $Z_S(\beta; \Gamma(1); 1)$ can be explained by the presence of an eigenvalue $\lambda_\beta = \pm 1$ for $\mathcal{L}_\beta^{\chi_1}$ at the corresponding β value. For such β's the eigenfunctions of $\mathcal{L}_\beta^{\chi_1}$ with $\lambda_\beta = \pm 1$ are explicitly related to the holomorphic modular forms and the Maaß wave forms of $\Gamma(1)$ as follows [**CM99**], [**CM98**]:

PROPOSITION 4.1. *Let $\beta_\kappa = \frac{1-\kappa}{2}$ and $\kappa \in \{-1\} \cup \mathbb{N}_0$. Then one has*

(i) $\mathcal{L}_1$ ($\kappa = -1$) has the leading eigenvalue $\lambda_1 = +1$ with eigenfunction $f_1(z) = \frac{1}{z+1}$. The function $\phi(z) = f_1(z-1) = \frac{1}{z}$ is the Lewis transform (2.43) of the constant eigenfunction of $-\Delta$ with eigenvalue $\rho = 0$. ($\frac{1}{\log 2}\frac{1}{z+1}$ is the normalized Gauß measure on the unit intervall $[0,1]$).

(ii) $\mathcal{L}_{\frac{1}{2}}$ ($\kappa = 0$) has eigenvalue $\lambda_{\frac{1}{2}} = -1$ with eigenfunction $f_{\frac{1}{2}}(z) = \frac{1}{z+1} - 1$.

$\mathcal{L}_0$ ($\kappa = 1$) has eigenvalue $\lambda_0 = +1$ with eigenfunction $f_0(z) = \frac{1}{z+1} - 2 + z$.

(iii) For $\kappa \geq 2$ the operator $\mathcal{L}_{\beta_\kappa}$ has eigenvalue $\lambda_{\beta_\kappa} = -1$ with eigenfunction

$$f_{\beta_\kappa}(z) = (z+1)^{\kappa-1} - 1 \,, \tag{4.17}$$

where $\phi(z) = f_{\beta_\kappa}(z-1) = p^+_{\kappa+1}(z)$ is the even part of the period polynomial $p^+_k(z)$ in (2.36) for the holomorphic Eisenstein series of weight $k = \kappa + 1$.

For $\kappa = 3, 5, 7, \cdots$ the operator $\mathcal{L}_{\beta_\kappa}$ has eigenvalue $\lambda_{\beta_\kappa} = +1$ with eigenfunction

$$f_{\beta_\kappa}(z) = \sum_{-1 \leq n \leq \kappa, \, n \text{ odd}} \frac{B_{n+1}}{(n+1)\,!} \frac{B_{\kappa-n}}{(\kappa-n)\,!} \, (z+1)^n \,, \tag{4.18}$$

where $\phi(z) = f_{\beta_\kappa}(z-1) = p^-_{\kappa+1}(z)$ is the odd part of the period polynomial $p^-_k(z)$ in (2.37) for the holomorphic Eisenstein series of weight $k = \kappa + 1$.

(iv) $\mathcal{L}_{\beta_\kappa}$ has eigenvalues $\lambda_{\beta_\kappa} = -1$ respectively $\lambda_{\beta_\kappa} = +1$ with polynomial eigenfunctions $f_+(z)$ respectively $f_-(z)$ in the space $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$ for those κ, for which $\phi_+(z) = f_+(z-1)$ respectively $\phi_-(z) = f_-(z-1)$ is the even respectively odd part of the period polynomial of a holomorphic cusp form of $\Gamma(1)$ of weight $k = \kappa + 1$ (see section 2.3.1). The dimension of these eigenfunctions for fixed κ is equal to the dimension (2.15) of the cusp forms of weight $k = \kappa + 1$.

(v) Let $\psi_\beta(z)$ be the analytic continuation of the function given for $\Re\beta > 1$ as

$$\psi_\beta(z) = \sum_{m,n \geq 1} (\frac{1}{mz+n})^{2\beta} + \frac{1}{2}\zeta(2\beta)(1 + (\frac{1}{z})^{2\beta}) \,.$$

Then $f_\beta(z) = \psi_\beta(z+1)$ is an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue $\lambda_\beta = +1$ for those values of β for which $\zeta(2\beta) = 0$.

4.2. Eigenfunctions of the operator $\mathcal{L}_\beta^{\chi_{-1}}$. The transfer oprator for the representation χ_{-1} in (4.6)

$$\mathcal{L}_\beta^{\chi_{-1}} f_\beta(z) = \sum_{n=1}^{\infty} (-1)^{n+1} (\frac{1}{z+n})^{2\beta} f_\beta(\frac{1}{z+n})$$

differs from $\mathcal{L}_\beta^{\chi_1}$ in (4.5) only in the additional factor $(-1)^{n+1}$ under the sum. The functional equations (3.17), (3.18) and (3.19) read now

$$\begin{aligned} &(I) && \lambda_\beta \,[\, \phi(z) + \phi(z+1)\,] - z^{-2\beta}\, \phi(1+\frac{1}{z}) = 0\,, \\ &(II) && \phi(z) + \phi(z+1) + (z+1)^{-2\beta}\, \phi(\frac{z}{z+1}) = 0\,, \\ &(III) && \phi(z) + \lambda_\beta\, z^{-2\beta}\phi(\frac{1}{z}) = 0\,, \quad \lambda_\beta = \pm 1\,. \end{aligned} \tag{4.19}$$

Criterion (3.20) can be simplified in the region $\Re\beta > -\frac{\kappa}{2}$ with $\kappa \in \mathbb{N}_0$ and for $r = 2$ to

(4.20)
$$\lim_{z\to\infty} \Big[\lambda_\beta f_\beta(z) - \sum_{l=0}^{\kappa} 2^{-2\beta-l}\, \frac{f_\beta^{(l)}(0)}{l\,!}\Big(\zeta(2\beta+l, \frac{z+1}{2}) - \zeta(2\beta+l, \frac{z+2}{2})\Big)\Big] = 0\,.$$

Let $\phi(z)$ be a solution of (4.19). Then $f_\beta(z) = \phi(z+1)$ is an eigenfunction of $\mathcal{L}_\beta^{\chi_{-1}}$ if $f_\beta(z)$ fulfills this criterion (4.20).

Since the singularities in the two zeta functions for $l = \kappa$ in (4.20) cancel each other for $\beta \to \beta_\kappa$, the operator $\mathcal{L}_\beta^{\chi_{-1}}$ is holomorphic at $\beta = \beta_\kappa$ and hence for all β in $\mathbb{C}$ [**CMct**]. The operator $\mathcal{L}_\beta^{\chi_{-1}}$ is nuclear in the entire complex β-plane with pure point spectrum [**CMct**]. This just reflects the fact that the representation χ_{-1} is regular (see section 2.3.2): the Laplace-Beltrami operator $-\Delta$ in (2.17) with this representation has only a discrete spectrum [**BV97**].

For $\beta = \frac{1-\kappa}{2}$ with $\kappa \in \mathbb{N}_0$ Lewis equation (4.19) has again solutions $\phi(z)$ in the space $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$ of polynomials. It is straightforward to see that the corresponding polynomials $f_\beta(z-1)$ automatically fulfill the asymptotics in (4.20) and hence are eigenfunctions of the transfer operator $\mathcal{L}_\beta^{\chi_{-1}}$. These polynomial solutions can be determined exactly on the computer.

The number of polynomial solutions of the Lewis equation (4.19) as determined numerically for different $\beta = \beta_\kappa$ is shown in table 2 of section 2.

4.3. Eigenfunctions of the operator $\mathcal{L}_\beta^{\chi_2}$. With the two matrices

$$\chi_2(QT^n) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} & n \text{ even} \\ \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} & n \text{ odd} \end{cases} \tag{4.21}$$

the transfer operator (4.7) for the representation χ_2 can be written as

$$\begin{aligned}
\mathcal{L}_\beta^{\chi_2} \underline{f}_\beta(z) &= \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2\beta} \chi_2(QT^n) \underline{f}_\beta(\frac{1}{z+n}) \\
&= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \sum_{n=1}^{\infty} \Big(\frac{1}{z+(2n-1)}\Big)^{2\beta} \underline{f}_\beta\Big(\frac{1}{z+(2n-1)}\Big) \\
&\quad + \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \sum_{n=1}^{\infty} (\frac{1}{z+2n})^{2\beta} \underline{f}_\beta(\frac{1}{z+2n}) \,.
\end{aligned} \tag{4.22}$$

The corresponding functional equations (3.17), (3.18) and (3.19) for χ_2 then read

$$(I)\ \lambda_\beta \left[\underline{\phi}(z) - \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \underline{\phi}(z+1) \right] - z^{-2\beta} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \underline{\phi}(1+\frac{1}{z}) = 0 \,, \tag{4.23}$$

$$(II)\ \underline{\phi}(z) - \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \underline{\phi}(z+1) - (z+1)^{-2\beta} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \underline{\phi}(\frac{z}{z+1}),\ \lambda_\beta = \pm 1,$$

$$(III)\ \underline{\phi}(z) - z^{-2\beta} \lambda_\beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underline{\phi}(\frac{1}{z}) = 0 \,, \quad \lambda_\beta = \pm 1 \,. \tag{4.24}$$

Criterion (3.20) for χ_2 in the region $\Re\beta > -\frac{\kappa}{2}$, $\kappa \in \mathbb{N}_0$ has the form

$$\lim_{z\to\infty} \left[\lambda_\beta \underline{f}_\beta(z) - \sum_{l=0}^{\kappa} \sum_{m=1}^{2} (\frac{1}{2})^{2\beta+l} \chi_2(QT^m) \frac{\underline{f}_\beta^{(l)}(0)}{l\,!} \zeta(2\beta+l, \frac{z+m}{2}) \right] = 0 \,. \tag{4.25}$$

Let $\underline{\phi} = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix}$ then be a solution of the Lewis equation (4.23). The ϕ_i fulfill the equations

$$\lambda_\beta \left[\phi_1(z) + \phi_1(z+1) + \phi_2(z+1) \right] - z^{-2\beta} \phi_2(1+\frac{1}{z}) = 0 \,, \tag{4.26}$$

$$\lambda_\beta \left[\phi_2(z) - \phi_2(z+1) \right] + z^{-2\beta} \phi_1(1+\frac{1}{z}) + z^{-2\beta} \phi_2(1+\frac{1}{z}) = 0, \tag{4.27}$$

We denote the left-hand sides of (4.26) respectively (4.27) by $Y_1(z)$ respectively $Y_2(z)$. If $\underline{\phi}$ is a solution of (4.23), then $Y_1(z)$ and $Y_2(z)$ are identical zero and the identity

$$-\frac{z^{-2\beta}}{\lambda_\beta} Y_1(1+\frac{1}{z}) + Y_2(z) + (z+1)^{-2\beta} Y_2(\frac{z}{z+1}) \equiv 0$$

implies

$$\begin{aligned}
&-z^{-2\beta} \left[\phi_1(\frac{z+1}{z}) + \phi_1(\frac{2z+1}{z}) + \phi_2(\frac{2z+1}{z}) \right] + \frac{1}{\lambda_\beta} (z+1)^{-2\beta} \phi_2(\frac{2z+1}{z+1}) \\
&+ \lambda_\beta \left[\phi_2(z) - \phi_2(z+1) \right] + z^{-2\beta} \phi_1(\frac{z+1}{z}) + z^{-2\beta} \phi_2(\frac{z+1}{z}) \\
&+ (z+1)^{-2\beta} \lambda_\beta \left[\phi_2(\frac{z}{z+1}) - \phi_2(\frac{2z+1}{z+1}) \right] + z^{-2\beta} \phi_1(\frac{2z+1}{z}) \\
&+ z^{-2\beta} \phi_2(\frac{2z+1}{z}) = 0 \,.
\end{aligned}$$

A simple calculation then gives

$$\begin{aligned} &\lambda_\beta\,[\phi_2(z)-\phi_2(z+1)]+\lambda_\beta\,(z+1)^{-2\beta}\,\phi_2(\frac{z}{z+1})+z^{-2\beta}\phi_2(\frac{z+1}{z}) \\ &+(z+1)^{-2\beta}\phi_2(\frac{2z+1}{z+1})\,(\frac{1}{\lambda_\beta}-\lambda_\beta)=0\,, \end{aligned} \tag{4.28}$$

where the function $\phi_1(z)$ has been eliminated. Since we are interested only in solutions for $\lambda_\beta = \pm 1$, the last term in (4.28) vanishes and one gets an equation for ϕ_2:

$$\lambda_\beta\,[\phi_2(z)-\phi_2(z+1)]+\lambda_\beta\,(z+1)^{-2\beta}\,\phi_2(\frac{z}{z+1})+z^{-2\beta}\phi_2(\frac{z+1}{z})=0\,. \tag{4.29}$$

Similarly, adding $Y_1(z)$ and $Y_2(z)$ leads to the equation

$$\lambda_\beta\,[\phi_1(z)+\phi_1(z+1)+\phi_2(z)]+z^{-2\beta}\,\phi_1(1+\frac{1}{z})=0\,. \tag{4.30}$$

From this one can express the function $\phi_2(z)$ in terms of ϕ_1. Inserting this $\phi_2(z)$ into one of the relations (4.29), (4.26) or (4.27) one gets an equation for ϕ_1. That is, a solution $\underline{\phi}(z)$ of the Lewis equation (4.23) determines uniquely a solution $\phi_2(z)$ of equation (4.29). Conversely a solution $\phi_2(z)$ of (4.29) together with ϕ_1, determined from relation (4.27), gives a solution $\underline{\phi}(z) = \begin{pmatrix}\phi_1(z)\\ \phi_2(z)\end{pmatrix}$ of the Lewis equation (4.23). Hence equation (4.29) and criterion (4.25) determine the eigenfunctions of the transfer operator $\mathcal{L}_\beta^{\chi_2}$ with eigenvalue $\lambda_\beta = \pm 1$.

Let us next discuss some explicit solutions of these equations.

4.3.1. *Old solutions of Lewis' equation for the representation χ_2.* The first kind of solutions of the Lewis equation (4.23) for χ_2 are explicitely related to the solutions of the Lewis equation (4.12) for χ_1. Hence we call such solutions old solutions in analogy to the theory of old forms of Atkin-Lehner (see section 2.3.3).

PROPOSITION 4.2. *Let $\psi_\beta(z)$ be the analytic extension of Zagier's function in (2.45), which for $\Re\beta > 1$ reads*

$$\psi_\beta(z)=\sum_{m,n\geq 1}(mz+n)^{-2\beta}+\frac{1}{2}\zeta(2\beta)(1+z^{-2\beta})\,, \tag{4.31}$$

which for all β-values different from 1 is a solution of the Lewis equation (4.12) for χ_1 with $\lambda_\beta = 1$. Then

$$\underline{\psi}_\beta(z)=\begin{pmatrix}\phi_1(z)\\ \phi_2(z)\end{pmatrix}=\begin{pmatrix}\frac{1}{3}(1+2^{1-2\beta}) & 0 & -2^{-2\beta}\\ \frac{1}{3}(1+2^{1-2\beta}) & -1 & 0\end{pmatrix}\begin{pmatrix}\psi_\beta(z)\\ \psi_\beta(2z)\\ \psi_\beta(\frac{z}{2})\end{pmatrix} \tag{4.32}$$

for β-values with $\beta \neq 1$ is a solution of the Lewis equation (4.23) for the representation χ_2 with $\lambda_\beta = 1$.

The proof of this proposition follows from Lemma 4.7 in section 4.5.1. We will argue next that $\underline{\psi}_\beta(z+1)$ is indeed an eigenfunction of $\mathcal{L}_\beta^{\chi_2}$ for β's with $\zeta(2\beta) = 0$.

COROLLARY 4.3. Let $\underline{\psi}_\beta(z)$ be defined as in proposition 4.2. Then $\underline{\psi}_\beta(z+1)$ defines an eigenfunction of the transfer operator $\mathcal{L}_\beta^{\chi_2}$ for all β's with $\zeta(2\beta) = 0$.

PROOF. The vector-valued function $\underline{\psi}_\beta(z) \in B(D)\oplus B(D)$ in (4.32) is obviously meromorphic in $\beta \in \mathbb{C}$, because $\psi_\beta(z)$ is meromorphic in the complex β-plane [**CM99**]. Since the function $\underline{\psi}_\beta(z)$ by Proposition 4.2 is for $\beta \neq 1$ a solution of the Lewis equation (4.23) with $\lambda_\beta = +1$, according to (3.7) the function $\underline{f}_\beta(z) = \underline{\psi}_\beta(z+1)$ is a solution of the following equation:

$$\underline{f}_\beta(z) = \chi_2(QTQ)\,\underline{f}_\beta(z+1) + (\frac{1}{z+1})^{2\beta}\,\chi_2(QT)\,\underline{f}_\beta(\frac{1}{z+1})\,, \tag{4.33}$$

where we have set $\lambda_\beta = +1$. Iterating this equation N times one finds

$$\underline{f}_\beta(z) = \chi_2(QT^NQ)\,\underline{f}_\beta(z+N) + \sum_{n=1}^{N}(\frac{1}{z+n})^{2\beta}\,\chi_2(QT^n)\,\underline{f}_\beta(\frac{1}{z+n})\,,$$

where we used $\chi_2(Q^2) = 1$. Inserting the matrices (4.21) gives

$$\underline{f}_\beta(z) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,\underline{f}_\beta(z+N) + \sum_{n=1}^{N}(\frac{1}{z+n})^{2\beta}\,\chi_2(QT^n)\,\underline{f}_\beta(\frac{1}{z+n}) \tag{4.34}$$

for even N and

$$\underline{f}_\beta(z) = \begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}\,\underline{f}_\beta(z+N) + \sum_{n=1}^{N}(\frac{1}{z+n})^{2\beta}\,\chi_2(QT^n)\,\underline{f}_\beta(\frac{1}{z+n}) \tag{4.35}$$

for odd N. However due to $\lim_{N\to\infty}\psi_\beta(z+1+N) = \frac{\zeta(2\beta)}{2}$ for $\Re\beta > 1$ we find

$$\lim_{N\to\infty}\underline{f}_\beta(z+N) = \lim_{N\to\infty}\underline{\psi}_\beta(z+1+N) = \begin{pmatrix}d_\beta\\ -2d_\beta\end{pmatrix} \quad\text{with}\quad d_\beta = \frac{1-2^{-2\beta}}{6}\zeta(2\beta)\,.$$

That is, in the limit $N \to \infty$ relations (4.34) and (4.35) reduce to

$$\underline{f}_\beta(z) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}d_\beta\\ -2d_\beta\end{pmatrix} + \mathcal{L}_\beta^{\chi_2}\,\underline{f}_\beta(z) \quad\text{for even } N \tag{4.36}$$

$$\text{and}\qquad \underline{f}_\beta(z) = \begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}d_\beta\\ -2d_\beta\end{pmatrix} + \mathcal{L}_\beta^{\chi_2}\,\underline{f}_\beta(z) \quad\text{for odd } N, \tag{4.37}$$

which obviously are identical to

$$\underline{f}_\beta(z) = \begin{pmatrix}d_\beta\\ -2d_\beta\end{pmatrix} + \mathcal{L}_\beta^{\chi_2}\,\underline{f}_\beta(z)\,. \tag{4.38}$$

Since the function $\underline{f}_\beta(z) = \underline{\psi}_\beta(z+1)$ on the left-hand side of (4.38) is meromorphic for all $\beta \in \mathbb{C}$, the right-hand side must also be moromorphic in $\mathbb{C}$. For those $\beta \in \mathbb{C}$ with $\zeta(2\beta) = 0$, i.e., $d_\beta = 0$, the function $\underline{f}_\beta(z)$ is hence an eigenfunction of $\mathcal{L}_\beta^{\chi_2}$ with eigenvalue $\lambda_\beta = +1$. □

It is well known that Riemann's zeta function $\zeta(2\beta)$ has trivial zeros at $2\beta = -2, -4, -6, \cdots$ which correspond to $\beta = \beta_\kappa = \frac{1-\kappa}{2}$ with $\kappa = 3, 5, 7, \cdots$. For such β values the function $\psi_{\beta_\kappa}(z)$ coincides up to a factor with the odd part of the period polynomial $p^-_{\kappa+1}(z)$ in (2.37) for the holomorphic Eisenstein series of weight $k = \kappa + 1$ [**CM99**]. The Lewis equation (4.23) for $\lambda_\beta = +1$ has therefore the solution

$$\underline{p}^-_{\kappa+1}(z) = \begin{pmatrix}\frac{1}{3}(1+2^\kappa) & 0 & -2^{\kappa-1}\\ \frac{1}{3}(1+2^\kappa) & -1 & 0\end{pmatrix}\begin{pmatrix}p^-_{\kappa+1}(z)\\ p^-_{\kappa+1}(2z)\\ p^-_{\kappa+1}(\frac{z}{2})\end{pmatrix} \tag{4.39}$$

which, up to a factor, is the same as the function $\underline{\psi}_\beta(z)$ at $\beta = \beta_\kappa$, which as we have shown is an eigenfunction if $\mathcal{L}_\beta^{\chi_2}$ for all β with $\zeta(2\beta) = 0$.

Another polynomial solution of the Lewis equation (4.23) for $\beta = \beta_\kappa$, $\kappa = 3, 5, 7, \cdots$ for the parameter value $\lambda_\beta = -1$ is

$$\underline{p}^+_{\kappa+1}(z) = \begin{pmatrix} \frac{1}{3}(1+2^\kappa) & 0 & -2^{\kappa-1} \\ \frac{1}{3}(1+2^\kappa) & -1 & 0 \end{pmatrix} \begin{pmatrix} p^+_{\kappa+1}(z) \\ p^+_{\kappa+1}(2z) \\ p^+_{\kappa+1}(\frac{z}{2}) \end{pmatrix}, \tag{4.40}$$

where $p^+_{\kappa+1}(z) = z^{\kappa-1} - 1$ is the even part of the period polynomial (2.36) of the holomorphic Eisenstein series of weight $k = \kappa + 1$. To verify this, one just inserts $p^+_{\kappa+1}(z) = z^{\kappa-1} - 1$ into (4.40) and obtains

$$\underline{p}^+_{\kappa+1}(z) = c_\kappa \begin{pmatrix} 2z^{\kappa-1} + 1 \\ -z^{\kappa-1} - 2 \end{pmatrix}, \tag{4.41}$$

with $c_\kappa = \frac{2^{\kappa-1}-1}{3}$. As one can check immediately, the two components $\phi_1(z) = c_\kappa(2z^{\kappa-1} + 1)$ and $\phi_2(z) = c_\kappa(-z^{\kappa-1} - 2)$ in $\underline{p}^+_{\kappa+1}(z) = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix}$ fulfill the two equations (4.26) and (4.27) which are equivalent to the Lewis equation (4.23). Therefore $\underline{p}^+_{\kappa+1}(z)$ is a solution of the Lewis equation (4.23) and the function $\underline{f}_{\beta_\kappa}(z) = \underline{p}^+_{\kappa+1}(z+1)$ is an eigenfunction of the operator $\mathcal{L}_\beta^{\chi_2}$ with eigenvalue $\lambda_\beta = -1$.

PROPOSITION 4.4. *Let $\underline{\psi}_\beta(z)$ be defined as in Proposition 4.2. Then the function $\underline{\psi}_\beta(z)$ is an eigenfunction of $\mathcal{L}_\beta^{\chi_2}$ for $\beta = 0$ with eigenvalue $\lambda_\beta = 1$.*

For the proof of this Proposition we refer to section 4.5.1.

Explicit examples for $\underline{p}^-_{\kappa+1}(z)$ and $\underline{p}^+_{\kappa+1}(z)$ for $\kappa = 3$ (corresponding to weight $k = 4$ and $\beta = -1$) are

$$\underline{p}^-_4(z) = \begin{pmatrix} -\frac{5}{z} - 5z + \frac{5}{2}z^3 \\ \frac{5}{2}\frac{1}{z} - 5z - 5z^3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -4 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} p^-_4(z) \\ p^-_4(2z) \\ p^-_4(\frac{z}{2}) \end{pmatrix},$$

$$\underline{p}^+_4(z) = \begin{pmatrix} 1 + 2z^2 \\ -2 - z^2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -4 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} p^+_4(z) \\ p^+_4(2z) \\ p^+_4(\frac{z}{2}) \end{pmatrix},$$

with the odd part of the period polynomial $p^-_4(z)$ respectively the even part of the period polynomial $p^+_4(z)$ of the Eisenstein series of weight 4 for $\Gamma(1)$ given as:

$$p^-_4(z) = \frac{1}{z} - 5z + z^3 \quad \text{respectively} \quad p^+_4(z) = z^2 - 1\,.$$

4.3.2. *Two classes of polynomial solutions for Lewis equation for χ_2.* Besides the solutions mentioned above, which are closely related to the nonholomorphic Eisenstein series respectively holomorphic Eisenstein series for β with $\zeta(2\beta) = 0$ for $\Gamma(1)$ the Lewis equation (4.23) has further polynomial solutions $\underline{\phi}(z) = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix}$ for certain $\beta = \beta_\kappa$ with $\kappa \in \mathbb{N}_0$ and $\phi_1(z),\ \phi_2(z) \in \oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$ which obviously correspond to period polynomials of the holomorphic cusp forms of weight $k = \kappa+1$ for the group $\Gamma(1)$ (see section 2.4.3) respectively the corresponding polynomial eigenfunctions of the operator $\mathcal{L}_\beta^{\chi_1}$.

The two components of these solutions $\underline{\varphi}$ are linear combinations of the even respectively odd parts of the period polynomials $\varphi(z)$ of the group $\Gamma(1)$ with argument z, $2z$ and $\frac{z}{2}$ as shown in table 1. We call these solutions and the corresponding eigenfunctions old solutions respectively old eigenfunctions, because they can be constructed from the solutions and the eigenfunctions for the representation χ_1 of $\Gamma(1)$.

A second class of solutions $\underline{\phi}(z)$ has the property that the component $\phi_2(z)$ is proportional to the component $\phi_1(2z)$ plus the even part of the period polynomial of the holomorphic Eisenstein series in the case $\lambda_\beta = -1$ as shown in table 1. We call these solutions and eigenfunctions new solutions and new eigenfunctions, because they are not related to the solutions and the eigenfunctions for the representation χ_1 of $\Gamma(1)$. These solutions have been determined numerically

TABLE 1. Two classes of solutions of Lewis equation (4.23) for χ_2 with components in $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$. $\varphi_o(z)$ respectively $\varphi_e(z)$ denote the odd respectively even part of a period polynomial of $\Gamma(1)$ and $p^+_{\kappa+1}$ is the even part of the period function of the holomorphic Eisenstein series of weight $\kappa+1$.

Class	λ_β	solutions in linear combinations
old	$+1$ (*odd*)	$\underline{\varphi}_o = \begin{pmatrix} c_1 & 0 & c_2 \\ c_1 & c_3 & 0 \end{pmatrix} \begin{pmatrix} \varphi_o(z) \\ \varphi_o(2z) \\ \varphi_o(\frac{z}{2}) \end{pmatrix}$
	-1 (*even*)	$\underline{\varphi}_e = \begin{pmatrix} c_1 & 0 & c_2 \\ c_1 & c_3 & 0 \end{pmatrix} \begin{pmatrix} \varphi_e(z) \\ \varphi_e(2z) \\ \varphi_e(\frac{z}{2}) \end{pmatrix} + \begin{pmatrix} d_1 & 0 & d_2 \\ d_1 & d_3 & 0 \end{pmatrix} \begin{pmatrix} p^+_{\kappa+1}(z) \\ p^+_{\kappa+1}(2z) \\ p^+_{\kappa+1}(\frac{z}{2}) \end{pmatrix}$
new	$+1$ (*odd*)	$\underline{\phi}_o = \begin{pmatrix} c_4\,\phi_o(z) \\ \phi_o(2z) \end{pmatrix}$
	-1 (*even*)	$\underline{\phi}_e = \begin{pmatrix} c_4\,\phi_e(z) \\ \phi_e(2z) \end{pmatrix} + \begin{pmatrix} d_5\,p^+_{\kappa+1}(z) \\ p^+_{\kappa+1}(2z) \end{pmatrix}$

Explicit examples for old solutions $\underline{\phi}(z)$ for $\kappa = 11$ (corresponding to weight $k = 12$ and $\beta = -5$) are

$$
\begin{aligned}
\underline{\varphi}_o(z) &= \begin{pmatrix} 52z - 85z^3 + 42z^5 - 10z^7 + z^9 \\ z - 10z^3 + 42z^5 - 85z^7 + 52z^9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 128 \\ 1 & \frac{1}{8} & 0 \end{pmatrix} \begin{pmatrix} \varphi_o(z) \\ \varphi_o(2z) \\ \varphi_o(\frac{z}{2}) \end{pmatrix}, \\
\underline{\varphi}_e(z) &= \begin{pmatrix} (t-3) + 22z^2 - 18z^4 + 6z^6 - z^8 + (2t+3)z^{10} \\ (-2t-3) + z^2 - 6z^4 + 18z^6 - 22z^8 + (3-t)z^{10} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 128 \\ 1 & \frac{1}{8} & 0 \end{pmatrix} \begin{pmatrix} \varphi_e(z) \\ \varphi_e(2z) \\ \varphi_e(\frac{z}{2}) \end{pmatrix} + \begin{pmatrix} 3 + \frac{683}{341}t & 0 & -\frac{1024}{341}t \\ 3 + \frac{683}{341}t & -\frac{1}{341}t & 0 \end{pmatrix} \begin{pmatrix} p^+_{12}(z) \\ p^+_{12}(2z) \\ p^+_{12}(\frac{z}{2}) \end{pmatrix},
\end{aligned}
$$

with $t \in \mathbb{C}$ a free parameter and

$$\varphi_o(z) = \frac{4}{5}z - 5z^3 + \frac{42}{5}z^5 - 5z^7 + \frac{4}{5}z^9 \text{ respectively } \varphi_e(z) = \frac{2}{3}z^2 - 2z^4 + 2z^6 - \frac{2}{3}z^8$$

the odd respectively even part of the period polynomial of the holomorphic cusp form for $\Gamma(1)$ of weight 12 and $p_{12}^+(z) = z^{10} - 1$ the even part of the period function of the holomorphic Eisenstein series for $\Gamma(1)$ of weight 12.

Explicit examples of new solutions for $\kappa = 7$ (corresponding to weight $k = 8$ and $\beta = -3$) are

$$\text{(4.42)} \qquad \underline{\phi}_o(z) = \begin{pmatrix} 4z - 5z^3 + z^5 \\ z - 5z^3 + 4z^5 \end{pmatrix} = \begin{pmatrix} 8\phi_o(\frac{z}{2}) \\ \phi_o(z) \end{pmatrix},$$

$$\text{(4.43)} \qquad \underline{\phi}_e(z) = \begin{pmatrix} (t-1) + 6z^2 - 3z^4 + (2t+1)z^6 \\ (-2t-1) + 3z^2 - 6z^4 + (-t+1)z^6 \end{pmatrix}$$

$$= \begin{pmatrix} 8\phi_e(\frac{z}{2}) \\ \phi_e(z) \end{pmatrix} + \begin{pmatrix} 1 + \frac{43}{21}t & 0 & -\frac{64}{21}t \\ 1 + \frac{43}{21}t & -\frac{1}{21}t & 0 \end{pmatrix} \begin{pmatrix} p_8^+(z) \\ p_8^+(2z) \\ p_8^+(\frac{z}{2}) \end{pmatrix},$$

with $t \in \mathbb{C}$ is a free parameter and

$$\phi_o(z) = z - 5z^3 + 4z^5 \text{ respectively } \phi_e(z) = 3z^2 - 6z^4.$$

The polynomial $p_8^+(z) = z^6 - 1$ is the even part of the period function of the holomorphic Eisenstein series for $\Gamma(1)$ of weight 8. The components of the solutions in (4.42) respectively (4.43) lie in the polynomial space $\oplus_{n=0}^{6}\mathbb{C}z^n$. The functions $\phi_o(z)$ and $\phi_e(z)$ for $\kappa = 7$ have obviously nothing to do with solutions of the Lewis equation (4.12) for the representation χ_1 of $\Gamma(1)$, because for $\kappa = 7$ equation (4.12) with $\lambda_\beta = \pm 1$ doesn't have any solution in the space $\oplus_{n=0}^{6}\mathbb{C}z^n$ since there doesn't exist any cusp form for $\Gamma(1)$ of weight 8.

4.4. New and old solutions and new and old forms. The solutions of the Lewis equations for χ_1 and χ_{-1} in (4.12) and (4.19) in the space $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$ of polynomials and the one for χ_2 in (4.23) with components in $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$ can be determined easily numerically. The numbers of solutions found this way are summarized in table 2. Since 'odd' ($\lambda_\beta = +1$) and 'even' ($\lambda_\beta = -1$) solutions always arise in pairs, each pair will be counted as one solution. Numbers with the symbol $*$ in the row for χ_2 characterize the number of old solutions, the other number counts the new solutions.

According to the decompositions in (4.4)

$$\begin{aligned} \chi^{\Gamma(1)} &= \chi_1, \\ \chi^{\Gamma_2} &= \chi_1 \oplus \chi_{-1}, \\ \chi^{\Gamma'} &= \chi_1 \oplus \chi_2, && \Gamma' \in \{\Gamma_0(2), \Gamma^0(2), \Gamma_\vartheta\}, \\ \chi^{\Gamma(2)} &= \chi_1 \oplus \chi_{-1} \oplus \chi_2 \oplus \chi_2 \end{aligned}$$

the number of solutions of the Lewis equation (3.17) with $\lambda_\beta = \pm 1$ for the groups $\Gamma(1)$, Γ_2, $\Gamma_0(2)$, $\Gamma^0(2)$, Γ_ϑ and $\Gamma(2)$ are determined by the sum of solutions for χ_1, χ_{-1} and χ_2. They are also given in table 2, where the numbers with the symbol $*$ count the old solutions in the following sense:

The solutions of the Lewis equation (3.17) for χ^{Γ_2} are determined by the solutions of the Lewis equations for χ_1 and χ_{-1}. The first ones are called 'old' solutions since they are solutions of Lewis equation for the representation $\chi^{\Gamma(1)}$ of the larger group $\Gamma(1)$. The ones for χ_{-1} are 'new' solutions. The same notation is used for the other representations. For $\Gamma(2)$ the induced representation $\chi^{\Gamma(2)}$ can be decomposed into the representations χ_1, χ_{-1} and twice χ_2. The 'old' solutions for $\chi^{\Gamma(2)}$ then consist of the solutions for the representations χ_1 and for χ_2, since these representations appear also in the induced representations $\chi^{\Gamma(1)}$ respectively $\chi^{\Gamma_0(2)}$ for the larger groups $\Gamma(1)$ respectively $\Gamma_0(2)$ with $\Gamma(2) \subset \Gamma_0(2) \subset \Gamma(1)$.

TABLE 2. The number of solutions of the Lewis equation for different representations χ with components in the space $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$, where $\Gamma' \in \{\Gamma_0(2), \Gamma^0(2), \Gamma_\vartheta\}$

β	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11
κ	3	5	7	9	11	13	15	17	19	21	23
χ_1	0	0	0	0	1	0	1	1	1	1	2
χ_{-1}	0	1	0	1	1	1	1	2	1	2	2
χ_2	0	0	1	1	1*	2	1+1*	1+1*	1+1*	2+1*	1+2*
$\chi^{\Gamma(1)}$	0	0	0	0	1	0	1	1	1	1	2
χ^{Γ_2}	0	1	0	1	1+1*	1	1+1*	2+1*	1+1*	2+1*	2+2*
$\chi^{\Gamma'}$	0	0	1	1	2*	2	1+2*	1+2*	2+2*	2+2*	1+4*
$\chi^{\Gamma(2)}$	0	1	2*	1+2*	1+3*	1+4*	1+5*	2+5*	1+7*	2+7*	2+8*

Comparing this table with the table on p.296 of [**Miy89**] the numbers for the 'old' respectively the 'new' solutions for the representations $\chi^{\Gamma_0(2)}$ and $\chi^{\Gamma(2)}$ coincide exactly with the numbers given there for the old respectively the new cusp forms for the groups $\Gamma_0(2)$ and $\Gamma(2)$ (indeed $\Gamma(2)$ is conjugate to $\Gamma_0(4)$ [**BV97**] p.27). For the modular forms of the groups $\Gamma(1)$, $\Gamma_0(2)$, $\Gamma^0(2)$ and $\Gamma(2)$ the following result namely holds [**BV97**]:

LEMMA 4.5.
(i) If $g(z)$ is a modular form for $\Gamma(1)$, then

$$\begin{aligned} &\textit{the group } \Gamma_0(2) && \textit{has the old forms } g(z) \textit{ and } g(2z)\,, \\ &\textit{the group } \Gamma^0(2) && \textit{has the old forms } g(z) \textit{ and } g(\tfrac{z}{2})\,, \\ &\textit{the group } \Gamma(2) && \textit{has the old forms } g(z), g(2z) \textit{ and } g(\tfrac{z}{2})\,. \end{aligned} \tag{4.44}$$

(ii) If $h(z)$ is a modular form of $\Gamma_0(2)$ respectively $\Gamma^0(2)$, then

$$\begin{aligned} &\textit{the group } \Gamma^0(2) && \textit{has the modular form } h(\tfrac{z}{2})\,, \\ &\textit{the group } \Gamma(2) && \textit{has the modular forms } h(z) \textit{ and } h(\tfrac{z}{2}) \end{aligned} \tag{4.45}$$

respectively

$$\begin{aligned} &\textit{the group } \Gamma_0(2) && \textit{has the modular form } h(2z)\,, \\ &\textit{the group } \Gamma(2) && \textit{has the modular forms } h(z) \textit{ and } h(2z)\,. \end{aligned} \tag{4.46}$$

According to this lemma the space $M(\Gamma)^{old}$ of old modular forms for Γ is spanned by the following forms:

$$M(\Gamma_0(2))^{old} = \{c_1\, g_1(z) + c_2\, g_2(2z) \,|\, c_i \in \mathbb{C},\ g_i(z) \in M(\Gamma(1))\}\,, \tag{4.47}$$

$$M(\Gamma^0(2))^{old} = \{c_1\, g_1(z) + c_2\, g_2(\frac{z}{2}) \,|\, c_i \in \mathbb{C},\ g_i(z) \in M(\Gamma(1))\}\,, \tag{4.48}$$

$$\begin{aligned} M(\Gamma(2))^{old} = &\{c_1\, g_1(z) + c_2\, g_2(2z) + c_3\, g_3(\frac{z}{2}) \,|\, c_i \in \mathbb{C},\ g_i(z) \in M(\Gamma(1))\} \\ &\oplus\{d_1\, h_1(z) + d_2\, h_2(\frac{z}{2}) \,|\, d_i \in \mathbb{C},\ h_i(z) \in M(\Gamma_0(2))\}\,. \end{aligned} \tag{4.49}$$

The last subspace of $M(\Gamma(2))^{old}$ can be replaced by

$$\{d_1\, h_1(z) + d_2\, h_2(2z) \,|\, d_i \in \mathbb{C},\ h_i(z) \in M(\Gamma^0(2))\}\,.$$

The dimensions of these different spaces are hence related as follows:

$$\begin{aligned} dim(M(\Gamma_0(2))^{old}) &= dim(M(\Gamma^0(2))^{old}) = 2\, dim(M(\Gamma(1)))\,, && (4.50) \\ dim(M(\Gamma_0(2))) &= dim(M(\Gamma^0(2))^{old}) + dim(M(\Gamma(1))^{new})\,, \\ dim(M(\Gamma(2))) &= 3\, dim(M(\Gamma(1))) + 2\, dim(M(\Gamma_0(2))^{old}) \\ &\quad + dim(M(\Gamma(2))^{new})\,. && (4.51) \end{aligned}$$

Obviously the numbers of solutions given in table 2 for the different groups fulfill relations (4.50) and (4.51). The form of old solutions for the Lewis equation for χ_2 in table 1 and in (4.32) is quite similar to the one of the old modular forms in (4.47), (4.48) and (4.49). The representation χ_2 appears in the induced representations $\chi^{\Gamma_0(2)}$, $\chi^{\Gamma_0(2)}$ and $\chi^{\Gamma(2)}$. This suggests that the transfer operator approach respects in a surprizing way the theory of old and new forms for such congruence subgroups which for instance in Eichlers period polynomials is not so obviously seen.

To study this in more detail, we restrict our discussion now to the group $\Gamma_0(2)$ and consider its induced representation $\chi^{\Gamma_0(2)}$ and the corresponding Lewis equation respectively transfer operator.

4.5. Solutions of the Lewis equation for $\Gamma_0(2)$. For the group $\Gamma_0(2)$ Lewis' equation in (3.17) reads

$$\lambda_\beta\, \Big[\underline{\phi}(z) - \chi^{\Gamma_0(2)}(QTQ)\, \underline{\phi}(z+1)\Big] - z^{-2\beta}\chi^{\Gamma_0(2)}(QT)\, \underline{\phi}(1+\frac{1}{z}) = 0\,, \tag{4.52}$$

with $\chi^{\Gamma_0(2)}$ the induced representation defined by [**CMct**]:

$$\chi^{\Gamma_0(2)}(Q) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \chi^{\Gamma_0(2)}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.53}$$

Let $\underline{\phi}(z) = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \\ \phi_3(z) \end{pmatrix}$ be a solution of equation (4.52). With

$$\chi^{\Gamma_0(2)}(QTQ) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \chi^{\Gamma_0(2)}(QT) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

equation (4.52) is then equivalent to the following three equations:

$$\lambda_\beta\,[\,\phi_1(z)-\phi_3(z+1)\,]-z^{-2\beta}\,\phi_3(1+\frac{1}{z})=0\,, \tag{4.54}$$

$$\lambda_\beta\,[\,\phi_2(z)-\phi_2(z+1)\,]-z^{-2\beta}\,\phi_1(1+\frac{1}{z})=0\,, \tag{4.55}$$

$$\lambda_\beta\,[\,\phi_3(z)-\phi_1(z+1)\,]-z^{-2\beta}\,\phi_2(1+\frac{1}{z})=0\,. \tag{4.56}$$

Summing up these three equations leads to

$$\lambda_\beta\,\Big[\,\hat\phi(z)-\hat\phi(z+1)\Big]-z^{-2\beta}\,\hat\phi(1+\frac{1}{z})=0 \tag{4.57}$$

with $\hat\phi(z)=\phi_1(z)+\phi_2(z)+\phi_3(z)$. This however is just Lewis' equation (4.12) for the representation χ_1. That is, if $\underline{\phi}$ is a solution of Lewis' equation for $\chi^{\Gamma_0(2)}$, then $\hat\phi(z)$ must be a solution of the Lewis equation (4.12) for χ_1, which can also be trivial. The three equations above can be rewritten as

$$\phi_1(z) \;=\; \phi_3(z+1)+\frac{1}{\lambda_\beta}\,z^{-2\beta}\phi_3(1+\frac{1}{z})\,, \tag{4.58}$$

$$\phi_1(z) \;=\; \lambda_\beta\Big[\,\phi_2(\frac{1}{z-1})-\phi_2(1+\frac{1}{z-1})\Big]\,(z-1)^{-2\beta}\,, \tag{4.59}$$

$$\phi_3(z) \;=\; \phi_1(z+1)+\frac{1}{\lambda_\beta}\,z^{-2\beta}\phi_2(1+\frac{1}{z})\,. \tag{4.60}$$

Inserting (4.59) into (4.60) with $\lambda_\beta=\pm1$ implies

$$\begin{aligned}\phi_3(z) &= \lambda_\beta\Big[\,\phi_2(\frac{1}{z})-\phi_2(1+\frac{1}{z})\Big]\,z^{-2\beta}+\frac{1}{\lambda_\beta}\,z^{-2\beta}\phi_2(1+\frac{1}{z})\\ &= \lambda_\beta\,z^{-2\beta}\,\phi_2(\frac{1}{z})\,.\end{aligned} \tag{4.61}$$

Inserting $\phi_1(z)$ from (4.58) into relation (4.59) furthermore gives

$$\phi_3(z+1)+\frac{1}{\lambda_\beta}\,z^{-2\beta}\phi_3(1+\frac{1}{z})=\lambda_\beta\Big[\,\phi_2(\frac{1}{z-1})-\phi_2(1+\frac{1}{z-1})\Big]\,(z-1)^{-2\beta}\,.$$

Due to expression (4.61) the function ϕ_3 can be replaced by $\phi_2(z)$, which leads finally to the following equation for $\phi_2(z)$:

$$\lambda_\beta\Big[\phi_2(z)-\phi_2(z+1)\Big]=(2z+1)^{-2\beta}\Big[\lambda_\beta\,\phi_2(\frac{z}{2z+1})+\phi_2(\frac{z+1}{2z+1})\Big]\,. \tag{4.62}$$

Hence, if $\underline{\phi}(z)$ is a solution of the Lewis equation for $\chi^{\Gamma_0(2)}$, then $\phi_2(z)$ must fulfill equation (4.62). Conversely, any solution $\phi_2(z)$ of this equation defines a solution $\underline{\phi}$ of Lewis' equation (4.52) for $\lambda_\beta=\pm1$ by means of the relations (4.59) and (4.61) as

$$\underline{\phi}(z)=\begin{pmatrix}\phi_1(z)\\ \phi_2(z)\\ \phi_3(z)\end{pmatrix}=\begin{pmatrix}\lambda_\beta\Big[\,\phi_2(\frac{1}{z-1})-\phi_2(1+\frac{1}{z-1})\Big]\,(z-1)^{-2\beta}\\ \phi_2(z)\\ \lambda_\beta\,z^{-2\beta}\,\phi_2(\frac{1}{z})\end{pmatrix}\,, \tag{4.63}$$

This shows, that the solutions $\phi_2(z)$ of equation (4.62) and the solutions $\underline{\phi}(z)$ of Lewis' equation (4.52) are in $1-1$ correspondence.

Two special solutions of the form (4.63) can be obtained by choosing $\phi_2(z)=\phi(z)$ respectively $\phi_2(z)=\phi(2z)$, where $\phi(z)$ is an arbitrary solution of Lewis' equation (4.12) for the representation χ_1 and parameter value β. Indeed one shows

PROPOSITION 4.6. *Let $\phi(z)$ be a solution of the Lewis equation (4.12) for the representation χ_1 and parameter β with $\lambda_\beta = \pm 1$. Then the Lewis equation (4.52) for the representation $\chi^{\Gamma_0(2)}$ and the same β with $\lambda_\beta = \pm 1$ has the solutions*

$$\underline{\phi}^{(1)}(z) = \begin{pmatrix} \phi(z) \\ \phi(z) \\ \phi(z) \end{pmatrix} \quad \textit{and} \quad \underline{\phi}^{(2)}(z) = \begin{pmatrix} 2^{-2\beta}\left[\phi(\frac{z-1}{2}) - z^{-2\beta}\phi(\frac{z-1}{2z})\right] \\ \phi(2z) \\ 2^{-2\beta}\phi(\frac{z}{2}) \end{pmatrix}. \tag{4.64}$$

PROOF. One has to show $\phi(z)$ and $\phi(2z)$ to be solutions of equation (4.62). Then $\underline{\phi}^{(1)}(z)$ and $\underline{\phi}^{(2)}(z)$ in (4.64) correspond to the general solution (4.63) for $\phi_2(z) = \phi(z)$ respectively $\phi_2(z) = \phi(2z)$.

In the case $\phi_2(z) = \phi(z)$ we define

$$Y_1(z) := \lambda_\beta\,[\phi(z) - \phi(z+1)] - z^{-2\beta}\,\phi(1+\frac{1}{z})\,, \tag{4.65}$$

$$Y_2(z) := \phi(z) - \lambda_\beta\, z^{-2\beta}\,\phi(\frac{1}{z})\,, \tag{4.66}$$

where $Y_1(z)$ respectively $Y_2(z)$ are just the left-hand sides of the functional equations (4.12) respectively (4.14). For $\phi(z)$ a solution of Lewis' equation (4.12) with $\lambda_\beta = \pm 1$, $Y_1(z) = Y_2(z) \equiv 0$. The trivial identity

$$Y_1(z) + \lambda_\beta\, z^{-2\beta}\, Y_1(1+\frac{1}{z}) + z^{-2\beta}\, Y_2(2+\frac{1}{z}) + \lambda_\beta\, Y_2(1+\frac{z}{z+1})\,(z+1)^{-2\beta} \equiv 0$$

then implies

$$\begin{aligned} &\lambda_\beta\,[\phi(z) - \phi(z+1)] - z^{-2\beta}\,\phi(\frac{z+1}{z}) \\ &+z^{-2\beta}\left[\phi(\frac{z+1}{z}) - \phi(\frac{2z+1}{z})\right] - \lambda_\beta\,(z+1)^{-2\beta}\,\phi(\frac{2z+1}{z+1}) \\ &+z^{-2\beta}\,\phi(\frac{2z+1}{z}) - \lambda_\beta\,(2z+1)^{-2\beta}\,\phi(\frac{z}{2z+1}) \\ &+\lambda_\beta\,(z+1)^{-2\beta}\,\phi(\frac{2z+1}{z+1}) - (2z+1)^{-2\beta}\,\phi(\frac{z+1}{2z+1}) = 0\,. \end{aligned}$$

The terms proportional $\phi(\frac{z+1}{z})$, $\phi(\frac{2z+1}{z})$ and $\phi(\frac{2z+1}{z+1})$ cancel leading to equation (4.62). Hence $\phi(z)$ is a solution of this equation.

In the case $\phi_2(z) = \phi(2z)$ with $\phi(z)$ a solution of the Lewis equation for the representation χ_1 with $\lambda_\beta = \pm 1$, we define the quantities $Y_3(z)$ respectively $Y_4(z)$ as

$$Y_3(z) := \lambda_\beta\left[\phi_2(\frac{z}{2}) - \phi_2(\frac{z+1}{2})\right] - z^{-2\beta}\,\phi_2(\frac{1}{2}+\frac{1}{2z})\,, \tag{4.67}$$

$$Y_4(z) := \phi_2(\frac{z}{2}) - \lambda_\beta\, z^{-2\beta}\,\phi_2(\frac{1}{2z})\,. \tag{4.68}$$

Since $\phi(z)$ is a solution of the Lewis equation for the representation χ_1, $Y_3(z)$ and $Y_4(z)$ vanish identically in z. The trivial identity

$$Y_3(2z) + Y_3(2z+1) + (2z)^{-2\beta}\, Y_4(\frac{2z+1}{2z}) \equiv 0 \tag{4.69}$$

then implies

$$\lambda_\beta \left[\phi_2(z) - \phi_2(\frac{2z+1}{2}) \right] - (2z)^{-2\beta}\, \phi_2(\frac{1}{2} + \frac{1}{4z})$$
$$\lambda_\beta \left[\phi_2(\frac{2z+1}{2}) - \phi_2(z+1) \right] - (2z+1)^{-2\beta}\, \phi_2(\frac{1}{2} + \frac{1}{4z+2})$$
$$+(2z)^{-2\beta}\, \phi_2(\frac{2z+1}{4z}) - \lambda_\beta\, (2z+1)^{-2\beta}\, \phi_2(\frac{z}{2z+1}) = 0\,,$$

which leads to equation (4.62). That is, $\phi_2(z) = \phi(2z)$ is also a solution of equation (4.62).

Replacing now in the general formula (4.63) $\phi_2(z)$ by $\phi(z)$ respectively $\phi(2z)$, one gets finally the two solutions:

$$\underline{\phi}^{(1)}(z) = \begin{pmatrix} \lambda_\beta \left[\phi(\frac{1}{z-1}) - \phi(1 + \frac{1}{z-1}) \right] (z-1)^{-2\beta} \\ \phi(z) \\ \lambda_\beta\, z^{-2\beta}\, \phi(\frac{1}{z}) \end{pmatrix} = \begin{pmatrix} \phi(z) \\ \phi(z) \\ \phi(z) \end{pmatrix}, \tag{4.70}$$

$$\begin{aligned} \underline{\phi}^{(2)}(z) &= \begin{pmatrix} \lambda_\beta \left[\phi(\frac{2}{z-1}) - \phi(2 + \frac{2}{z-1}) \right] (z-1)^{-2\beta} \\ \phi(2z) \\ \lambda_\beta\, z^{-2\beta} \phi(\frac{2}{z}) \end{pmatrix} \\ &= \begin{pmatrix} 2^{-2\beta} \left[\phi(\frac{z-1}{2}) - z^{-2\beta} \phi(\frac{z-1}{2z}) \right] \\ \phi(2z) \\ 2^{-2\beta} \phi(\frac{z}{2}) \end{pmatrix}, \end{aligned} \tag{4.71}$$

where we used relations (4.12) and (4.14) to get (4.70) and (4.14) to get (4.71). □

The two solutions in (4.64) are called 'old' solutions of the Lewis equation for $\chi^{\Gamma_0(2)}$ since they are determined by solutions for $\chi^{\Gamma(1)}$. According to Proposition 4.6 the number of such old solutions of the Lewis equation for $\chi^{\Gamma_0(2)}$ is at least twice the number of the solutions for $\chi^{\Gamma(1)}$. This agrees with the numerical results in table 2, where we found that the number of old solutions of the Lewis equation for the representation $\chi^{\Gamma_0(2)} = \chi_1 \oplus \chi_2$, is at least for $\kappa \le 23$, always twice the number of solutions for $\chi^{\Gamma(1)} = \chi_1$. This reflects also the fact that the dimension of the space of old forms for $\Gamma_0(2)$ is twice the dimension of the space of modular forms for $\Gamma(1)$. We obviously expect, that the number of old solutions of the Lewis equation for $\chi^{\Gamma_0(2)}$ for all κ is exactly twice the number of solutions of the Lewis equation for $\chi^{\Gamma(1)}$. Since all polynomial solutions of Lewis' equation are also eigenfunctions of the corresponding transfer operator, an analogous result holds for the 'old' eigenfunctions of the transfer operator for $\chi^{\Gamma_0(2)}$.

4.5.1. *Old solutions of the Lewis equation for the representation* χ_2. Knowing the explicit form of two kinds of old solutions (4.64) of the Lewis equation for $\chi^{\Gamma_0(2)}$, one can ask how these solutions are related to the old solutions of the Lewis equation for the representation χ_2 as determined numerically in table 2. To answer this we rewrite the Lewis equation (4.52) for the representation $\chi^{\Gamma_0(2)}$ as

$$\begin{aligned} &\lambda_\beta \left[M\, \underline{\phi}(z) - M\chi^{\Gamma_0(2)}(QTQ)M^{-1}\, M\underline{\phi}(z+1) \right] \\ &-z^{-2\beta} M\chi^{\Gamma_0(2)}(QT)M^{-1}\, M\underline{\phi}(1 + \frac{1}{z}) = 0\,, \end{aligned} \tag{4.72}$$

where M denotes the matrix

$$M = \begin{pmatrix} \frac{1}{27} & \frac{1}{27} & \frac{1}{27} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \text{ and } M^{-1} = \begin{pmatrix} 9 & 1 & 1 \\ 9 & 0 & -1 \\ 9 & -1 & 0 \end{pmatrix} \text{ its inverse.}$$

The two matrices $M\chi^{\Gamma_0(2)}(QTQ)M^{-1}$ and $M\chi^{\Gamma_0(2)}(QT)M^{-1}$ in (4.72) then have the form

$$\begin{pmatrix} \chi_1(QTQ) & 0 \\ 0 & \chi_2(QTQ) \end{pmatrix} \text{ and } \begin{pmatrix} \chi_1(QT) & 0 \\ 0 & \chi_2(QT) \end{pmatrix}, \tag{4.73}$$

with χ_1 and χ_2 the representations given in (4.1) and (4.3). Equation (4.72) splits then into the two equations

$$\lambda_\beta \left[\hat{\phi}_1(z) - \chi_1(QTQ)\,\hat{\phi}_1(z+1)\right] - z^{-2\beta}\chi_1(QT)\hat{\phi}_1(1+\frac{1}{z}) = 0, \tag{4.74}$$

$$\lambda_\beta \left[\hat{\underline{\phi}}_3(z) - \chi_2(QTQ)\,\hat{\underline{\phi}}_3(z+1)\right] - z^{-2\beta}\chi_2(QT)\hat{\underline{\phi}}_3(1+\frac{1}{z}) = 0, \tag{4.75}$$

where $\hat{\phi}_1(z)$ respectively $\hat{\underline{\phi}}_3(z)$ denote the first respectively second component of $M\underline{\phi}(z) = \begin{pmatrix} \hat{\phi}_1(z) \\ \hat{\underline{\phi}}_3(z) \end{pmatrix}$. The equations (4.74) respectively (4.75) coincide however with the Lewis equations for the representations χ_1 respectively χ_2 as defined in (4.1) respectively (4.3).

For the solution $\underline{\phi}^{(1)}(z) = \begin{pmatrix} \phi(z) \\ \phi(z) \\ \phi(z) \end{pmatrix}$ in (4.64) one then finds

$$\begin{pmatrix} \hat{\phi}_1(z) \\ \hat{\underline{\phi}}_3(z) \end{pmatrix} = M\underline{\phi}^{(1)}(z) = \begin{pmatrix} \frac{1}{27} & \frac{1}{27} & \frac{1}{27} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \phi(z) \\ \phi(z) \\ \phi(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\phi(z) \\ 0 \\ 0 \end{pmatrix},$$

and hence $\hat{\phi}_1(z) = \frac{1}{3}\phi(z)$ and $\hat{\underline{\phi}}_3(z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Obviously $\hat{\phi}_1(z)$ respectively $\hat{\underline{\phi}}_3(z)$ satisfy the Lewis equations for the representations χ_1 respectively χ_2.

For the solution $\underline{\phi}^{(2)}(z)$ in (4.64) on the other hand one gets

$$\begin{pmatrix} \hat{\phi}_1(z) \\ \hat{\underline{\phi}}_3(z) \end{pmatrix} = M\underline{\phi}^{(2)}(z) = \begin{pmatrix} \frac{1}{27} & \frac{1}{27} & \frac{1}{27} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2^{-2\beta}\left[\phi(\frac{z-1}{2}) - z^{-2\beta}\phi(\frac{z-1}{2z})\right] \\ \phi(2z) \\ 2^{-2\beta}\phi(\frac{z}{2}) \end{pmatrix}$$

and hence

$$\hat{\phi}_1(z) = \frac{1}{27}\Big[2^{-2\beta}[\phi(\frac{z-1}{2}) - z^{-2\beta}\phi(\frac{z-1}{2z})] + \phi(2z) + 2^{-2\beta}\phi(\frac{z}{2})\Big], \tag{4.76}$$

according to (4.57) is a solution of the Lewis equation for the representation χ_1. The second component $\hat{\underline{\phi}}_3(z)$, which is a solution of Lewis equation for the representation χ_2, can be rewritten as

$$\hat{\underline{\phi}}_3(z) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \underline{\phi}^{(2)}(z) = \Big[\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} N^{-1}\Big]\Big[N\,\underline{\phi}^{(2)}(z)\Big], \tag{4.77}$$

with

$$N = \begin{pmatrix} \frac{1}{3\delta} & \frac{1}{3\delta} & \frac{1}{3\delta} \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2\beta} \end{pmatrix} \text{ respectively } N^{-1} = \begin{pmatrix} 3\delta & -1 & -2^{-2\beta} \\ 0 & 1 & 0 \\ 0 & 0 & 2^{-2\beta} \end{pmatrix}$$

for arbitrary $\delta \neq 0$. After a simple calculation one finds

$$\hat{\underline{\phi}}_3(z) = \begin{pmatrix} \delta & 0 & -2^{-2\beta} \\ \delta & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{9}{\delta}\hat{\phi}_1(z) \\ \phi(2z) \\ \phi(\frac{z}{2}) \end{pmatrix}, \tag{4.78}$$

with $\hat{\phi}_1(z)$ given in (4.76). The form of these old solutions (4.78) of the Lewis equation for the representation χ_2 explains completely the form of the old solutions given in table 1.

As shown in (4.57) the functions $\hat{\phi}_1(z)$ in (4.76) as well as $\phi(z)$ from which it is constructed are solutions of the Lewis equation for the representation χ_1. In general $\hat{\phi}_1(z)$ is not necessarily proportional to $\phi(z)$ and lies only in the vector space spanned by the solutions of the Lewis equation for χ_1 for a fixed β. Hence, if for some β-value with $\Re\beta = \frac{1}{2}$ the functions $\hat{\phi}_1(z)$ and $\phi(z)$ are not proportional to each other and satisfy criterion (4.15), then the transfer operator $\mathcal{L}_\beta$ has a degenerate eigenvalue $\lambda_\beta = \pm 1$ and also the Laplace-Beltrami operator $-\triangle$ has a degenerate eigenvalue $\beta(1-\beta)$ and vice versa. This could be a possibility to test if the operator $-\triangle$ on the modular surface for $\Gamma(1)$ has degenerate eigenvalues. For β-values with $\zeta(2\beta) = 0$ and $\Re\beta > 0$ the solutions of the Lewis equation for χ_1 fulfilling criterion (4.15) are related to the nonholomorphic Eisenstein series. Their multiplicity is one and hence $\hat{\phi}_1(z)$ must be proportional to $\phi(z)$.

Indeed for $\phi(z) = \psi_\beta(z)$ defined in (4.31) one finds with $\delta = \frac{1}{3}(1 + 2^{1-2\beta})$ for (4.78):

$$\hat{\underline{\phi}}_3(z) = \begin{pmatrix} \frac{1}{3}(1+2^{1-2\beta}) & 0 & -2^{-2\beta} \\ \frac{1}{3}(1+2^{1-2\beta}) & -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\psi}_\beta(z) \\ \psi_\beta(2z) \\ \psi_\beta(\frac{z}{2}) \end{pmatrix}, \tag{4.79}$$

with

$$\hat{\psi}_\beta(z) = \frac{2^{-2\beta}\left[\psi_\beta(\frac{z-1}{2}) - z^{-2\beta}\psi_\beta(\frac{z-1}{2z})\right] + \psi_\beta(2z) + 2^{-2\beta}\psi_\beta(\frac{z}{2})}{1+2^{1-2\beta}}. \tag{4.80}$$

Then one shows:

LEMMA 4.7. *$\hat{\psi}_\beta(z) = \psi_\beta(z)$.*

PROOF. Using definition (4.31)

$$\psi_\beta(z) = \sum_{m,n\geq 1} (mz+n)^{-2\beta} + \frac{1}{2}\zeta(2\beta)(1+z^{-2\beta}), \quad \Re\beta > 1$$

one gets for $\Re\beta > 1$

$$
\begin{aligned}
&2^{-2\beta}\Big[\psi_\beta(\frac{z-1}{2}) - z^{-2\beta}\psi_\beta(\frac{z-1}{2z})\Big] \\
&= \sum_{m,n\geq 1} (m(z-1)+2n)^{-2\beta} + \frac{1}{2}\zeta(2\beta)(2^{-2\beta} + (z-1)^{-2\beta}) \\
&\quad - \sum_{m,n\geq 1} (m(z-1)+2nz)^{-2\beta} - \frac{1}{2}\zeta(2\beta)((2z)^{-2\beta} + (z-1)^{-2\beta}) \\
&= \sum_{m,n\geq 1} (mz+(2n-m))^{-2\beta} - \sum_{m,n\geq 1} ((m+2n)z-m)^{-2\beta} \\
&\quad + \frac{1}{2}\zeta(2\beta)(2^{-2\beta} - (2z)^{-2\beta})\,. \qquad (4.81)
\end{aligned}
$$

Writing

$$
\begin{aligned}
&\sum_{m,n\geq 1} (mz+(2n-m))^{-2\beta} \\
&= \Big[\sum_{m,n\geq 1,\,2n>m} + \sum_{m,n\geq 1,\,2n=m} + \sum_{m,n\geq 1,\,2n<m}\Big] (mz+(2n-m))^{-2\beta} \\
&= \Big[\sum_{m_o,n_0\geq 1} (m_o z+n_0)^{-2\beta} + \sum_{m_e,n_e\geq 1} (m_e z+n_e)^{-2\beta}\Big] + \sum_{n\geq 1}(2nz)^{-2\beta} \\
&\quad + \sum_{n,\,m'=m-2n\geq 1} ((m'+2n)z-m')^{-2\beta}\,,
\end{aligned}
$$

where m_o, n_0 respectively m_e, n_e run over the odd respectively the even natural numbers, (4.81) can be simplified to

$$
\begin{aligned}
&2^{-2\beta}\Big[\psi_\beta(\frac{z-1}{2}) - z^{-2\beta}\psi_\beta(\frac{z-1}{2z})\Big] \\
&= \sum_{m_o,n_0\geq 1} (m_o z+n_0)^{-2\beta} + \sum_{m_e,n_e\geq 1} (m_e z+n_e)^{-2\beta} \\
&\quad + \frac{1}{2}\zeta(2\beta)\Big[2^{-2\beta} + (2z)^{-2\beta}\Big]\,. \qquad (4.82)
\end{aligned}
$$

Furthermore one finds

$$
\begin{aligned}
\psi_\beta(2z) &= \sum_{m,n\geq 1} (2mz+n)^{-2\beta} + \frac{1}{2}\zeta(2\beta)(1+(2z)^{-2\beta}) \\
&= \sum_{m_e,n\geq 1} (m_e z+n)^{-2\beta} + \frac{1}{2}\zeta(2\beta)[1+(2z)^{-2\beta}] \qquad (4.83)
\end{aligned}
$$

respectively

$$
\begin{aligned}
2^{-2\beta}\psi_\beta(\frac{z}{2}) &= \sum_{m,n\geq 1} (mz+2n)^{-2\beta} + \frac{1}{2}\zeta(2\beta)(2^{-2\beta} + z^{-2\beta}) \\
&= \sum_{m,n_e\geq 1} (mz+n_e)^{-2\beta} + \frac{1}{2}\zeta(2\beta)[2^{-2\beta} + z^{-2\beta}]\,. \qquad (4.84)
\end{aligned}
$$

Adding (4.82), (4.83) and (4.84) the numerator of the function $\hat{\psi}_\beta(z)$ in (4.80) is equal to

$$\begin{aligned}
&\sum_{m_o,n_0\geq 1}(m_o z+n_0)^{-2\beta}\sum_{m_e,n_e\geq 1}(m_e z+n_e)^{-2\beta}\\
&\sum_{m_e,n_0\geq 1}(m_e z+n_0)^{-2\beta}\sum_{m_e,n_e\geq 1}(m_e z+n_e)^{-2\beta}\\
&\sum_{m_o,n_e\geq 1}(m_o z+n_e)^{-2\beta}\sum_{m_e,n_e\geq 1}(m_e z+n_e)^{-2\beta}\\
&+\frac{1}{2}\zeta(2\beta)[(1+z^{-2\beta})\,(1+2^{1-2\beta})]\\
&=(1+2^{1-2\beta})\,\psi_\beta(z)\,,
\end{aligned}$$

where we used $\sum_{m_e,n_e\geq 1}(m_e z+n_e)^{-2\beta}=2^{-2\beta}\sum_{m,n\geq 1}(mz+n)^{-2\beta}=2^{-2\beta}\psi_\beta(z)$ for $\Re\beta>1$. This implies immediately $\hat{\psi}_\beta(z)=\psi_\beta(z)$ for $\Re\beta>1$ and by analytic continuation for arbitrary β. □

This Lemma together with the definition of $\hat{\underline{\phi}}_3(z)$ in (4.79) proves also Proposition 4.2 and Proposition 4.4 in section 4.3.1.

In **[CM99]** we showed that the function $\varphi_0(z)=\frac{1}{z}+z-3$ is a solution of Lewis equation (4.12) for $\Gamma(1)$. The function $\psi_\beta(z)$ on the other hand takes for $\beta=0$ the form $\psi_0(z)=\frac{1}{12}\,(\frac{1}{z}+z)$. Inserting φ_0 for ϕ in (4.78) hence leads to a solution $\hat{\underline{\phi}}_3$ of Lewis equation for χ_2. By accident the same solution $\hat{\underline{\phi}}_3$ is obtained when inserting instead of $\varphi_0(z)$ the function $\psi_0(z)$ since the constant term in $\varphi_0(z)$ just cancels when inserted into (4.78).

4.5.2. *New solutions of the Lewis equation for* $\Gamma_0(2)$. Besides the old solutions given in (4.64) the Lewis equation (4.52) for the representation $\chi^{\Gamma_0(2)}$ has other solutions, which obviously are not related to solutions of the Lewis equation for the representation χ_1. These solutions obviously must be related to the new forms of the group $\Gamma_0(2)$. For $\beta=\frac{1-\kappa}{2}$, $\kappa=3,\,5,\,\cdots 21$ we found numerically, that these new solutions have the structure as shown in table 3.

TABLE 3. New solutions of the Lewis equation (4.52) with components in $\oplus_{n=0}^{\kappa-1}\mathbb{C}z^n$, where $c\in\mathbb{Q}$, $e=\pm 1$ and $p^+_{\kappa+1}$ denotes the even part of the period polynomial of the holomorphic Eisenstein series of weight $\kappa+1$.

λ_β	Structure of the solutions
$+1(odd)$	$\underline{\phi}_o=\begin{pmatrix}-\phi_o(z)-e\phi_o(\frac{z}{2})\,2^{\frac{\kappa-1}{2}}\\ \phi_o(z)\\ e\phi_o(\frac{z}{2})\,2^{\frac{\kappa-1}{2}}\end{pmatrix}$
$-1(even)$	$\underline{\phi}_e=\begin{pmatrix}-\phi_e(z)-e\phi_e(\frac{z}{2})\,2^{\frac{\kappa-1}{2}}+c\,p^+_{\kappa+1}(z)\\ \phi_e(z)\\ e\phi_e(\frac{z}{2})\,2^{\frac{\kappa-1}{2}}\end{pmatrix}$

Explicit examples of such solutions $\underline{\phi}(z)$ for $\kappa = 7$ (corresponding to weight $k = 8$ and $\beta = -3$) are

$$\begin{aligned}
\underline{\phi}_o(z) &= \begin{pmatrix} -\phi_o(z) - \phi_o(\frac{z}{2})\, 2^{\frac{\kappa-1}{2}} \\ \phi_o(z) \\ \phi_o(\frac{z}{2})\, 2^{\frac{\kappa-1}{2}} \end{pmatrix} = \begin{pmatrix} 5z - 10z^3 + 5z^5 \\ -z + 5z^3 - 4z^5 \\ -4z + 5z^3 - z^5 \end{pmatrix} \\
\underline{\phi}_e(z) &= \begin{pmatrix} -\phi_e(z) - \phi_e(\frac{z}{2})\, 2^{\frac{\kappa-1}{2}} + p_8^+(z) \\ \phi_e(z) \\ \phi_e(\frac{z}{2})\, 2^{\frac{\kappa-1}{2}} \end{pmatrix} = \begin{pmatrix} 1 - 3z^2 + 3z^4 - z^6 \\ z^2 - 2z^4 \\ 2z^2 - z^4 \end{pmatrix}.
\end{aligned}$$

We expect the polynomials

$$\phi_o = -z + 5z^3 - 4z^5 \quad \text{respectively} \quad \phi_e = z^2 - 2z^4$$

to be related to a new cusp form of weight 8 for the group $\Gamma_0(2)$ since there are no such forms for $\Gamma(1)$ for this weight. The polynomial $p_8^+(z) = z^6 - 1$ is the even part of the period polynomial of the holomorphic Eisenstein series of weight 8 for $\Gamma(1)$.

For $\underline{\phi}(z) = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \\ \phi_3(z) \end{pmatrix}$ a general solution of the Lewis equation (4.52) for the representation $\chi^{\Gamma_0(2)}$, we have shown that $\hat{\phi}(z) = \phi_1(z) + \phi_2(z) + \phi_3(z)$ must be a solution of the Lewis equation for the representation χ_1 of $\Gamma(1)$. Our numerical results show that for the odd new solutions $\underline{\phi}_o(z)$ one always has $\hat{\phi}(z) \equiv 0$ for $\beta = -N$, $N \in \mathbb{N}$. For the even new solutions $\underline{\phi}_e(z)$ however one finds $\hat{\phi}(z) \not\equiv 0$ proportional to the even part of the period polynomial of the Eisenstein form.

4.5.3. *Solutions of the Lewis equations and period functions of* $\Gamma_0(2)$. The old solutions of Lewis equation for $\Gamma_0(2)$ have the form

$$\underline{\phi}^{(1)}(z) = \begin{pmatrix} \phi(z) \\ \phi(z) \\ \phi(z) \end{pmatrix} \quad \text{and} \quad \underline{\phi}^{(2)}(z) = \begin{pmatrix} 2^{-2\beta}\left[\phi(\frac{z-1}{2}) - z^{-2\beta}\phi(\frac{z-1}{2z})\right] \\ \phi(2z) \\ 2^{-2\beta}\phi(\frac{z}{2}) \end{pmatrix} \tag{4.85}$$

According to Lemma 4.5 in section 4.5 we know on the other hand that for $g(z)$ a modular form for $\Gamma(1)$ $g(z)$ and $g(2z)$ are old forms for $\Gamma_0(2)$ respectively $g(z)$ and $g(\frac{z}{2})$ are old forms for $\Gamma^0(2)$. Consider then the Eichler period polynomial for the modular form $u(z)$ of weight k given in (2.30)

$$r(z) = \int_0^{i\infty} (t-z)^{k-2} u(t)\, dt$$

respectively the period function (2.43) for an even Maaß wave form $u_s(z)$ with eigenvalue $\lambda = s(1-s)$

$$\psi_s(z) = z \int_0^{\infty} y^{\,s}\, u_s(iy)\, (z^2 + y^2)^{-s-1}\, dy\,, \quad \Re(z) > 0\,.$$

A trivial calculation then shows, that

$$r(2z) = 2^{k-1} \int_0^{i\infty} (t-z)^{k-2} u(2t)\, dt \quad \text{and} \quad r(\frac{z}{2}) = 2^{1-k} \int_0^{i\infty} (t-z)^{k-2} u(\frac{t}{2})\, dt$$

and similarly $\psi_s(2z)$ respectively $\psi_s(\frac{z}{2})$ is related through the above integral to $u(2z)$ and $u(\frac{z}{2})$.

These relations obviously are also reflected in the form of the old solutions of the Lewis equation for $\Gamma_0(2)$ respectively in the old eigenfunctions of the transfer operator $\tilde{\mathcal{L}}_\beta^{\chi^{\Gamma_0(2)}}$. The third component of both the old solutions in (4.85) and the new solutions in Table 3 seem to be directly related to the old and new forms for the group $\Gamma_0(2)$. One could then also expect that the first component of these solutions is related to forms of the group Γ_ϑ which is also conjugate to the two groups $\Gamma_0(2)$ and $\Gamma^0(2)$.

There arises immediately the question how the above polynomial solutions are related to Eichlers period polynomials for the group $\Gamma_0(2)$ respectively $\Gamma^0(2)$. Besides for the polynomial

$$\Omega_{e_3}(z) = z^{k-2}\Omega_Q(-\frac{2z+1}{z}) + \Omega_Q(z)$$

in (2.54) with $k = \kappa + 1$ and Ω_Q the period polynomial of $\Gamma(1)$ we could not find an explicit relation between the polynomial solutions of Lewis equation (4.58) for $\Gamma_0(2)$ and the solutions of Eichler's cocycle relation (2.53) for the period polynomials for the modular cusp forms of the group $\Gamma_0(2)$. Numerical calculations however show that the number of polynomial solutions of the cocycle relation (2.53) and the number of polynomial solutions of equation (4.62) for the parameter values $\beta = \beta_\kappa = \frac{1-\kappa}{2}$ coincide at least for the κ values we considered. From this we expect that there exists a close relation between the two equations.

An obvious remarkable fact concerning our approach via Lewis equation (4.61) or the transfer operator is, that contrary to Eichler's cocycle relations (2.53) the theory of old and new forms for $\Gamma_0(2)$ can be recognized immediately also in the solutions of this equation respectively the eigenfunctions of the transfer operator $\tilde{\mathcal{L}}_\beta^{\chi^{\Gamma_0(2)}}$. Interestingly enough the old solutions of Lewis's equation respectively the old eigenfunctions of the transfer operator are obtained just like the automorphic forms of $\Gamma_0(2)$ from those of the larger group $\Gamma(1)$. The same seems to hold true for the group $\Gamma(2)$. Unfortunately, we did not succeed up to now to relate also the new solutions of Lewis equation for $\Gamma_0(2)$ or $\Gamma(2)$ to the corresponding new automorphic forms for these groups. The old solutions respectively eigenfunctions however seem to be related to the old automorphic forms through the same transformations Lewis found for $\Gamma(1)$. This is true at least for the groups $\Gamma_0(2)$ and $\Gamma(2)$.

5. Conclusion

From the transfer operator $\tilde{\mathcal{L}}_\beta^\Gamma$ for subgroups of finite index of the modular group $\Gamma(1) = PSL(2, \mathbb{Z})$ one can derive a functional equation for the eigenfunctions of $\tilde{\mathcal{L}}_\beta^\Gamma$ which generalize the functional equation of Lewis for $\Gamma(1)$. For the group $\Gamma_0(2)$ we discussed in more detail this equation and found explicit solutions. They can be characterized as old ones and new ones in complete analogy to the well known theory of old and new automorphic forms for subgroups of $\Gamma(1)$. The old polynomial solutions seem to be related to the old cusp forms by Eichler's integral transformation for $\Gamma(1)$, the same holds true for the old nonpolynomial solutions which seem to be related to the old Maaß wave forms through Lewis transformations for $\Gamma(1)$. From this one should expect that this holds true also for the new solutions and the new automorphic forms.

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