RFSC 04-01
Revised

The Prime Divisors in the Rationality Condition for Odd Perfect Numbers

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Abstract. It is sufficient to prove that there is an excess of prime factors in the product of repunits with odd prime bases defined by the sum of divisors of the integer $N = (4k+1)^{4m+1}\prod_{i=1}^{\ell}\ q_i^{2\alpha_i}$ to establish that there do not exist any odd integers with equality between $\sigma(N)$ and 2N. The existence of distinct prime divisors in the repunits $\frac{(4k+1)^{4m+2}-1}{4k}$, $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, $i = 1,...,\ell$, in $\sigma(N)$ follows from a theorem on the primitive divisors of the Lucas sequences and the square root of the product of $2(4k+1)$, and the sequence of repunits will not be rational unless the primes are matched. Minimization of the number of prime divisors in $\sigma(N)$ yields an infinite set of repunits of increasing magnitude or prime equations with no integer solutions.

MSC: 11D61, 11K65

Keywords: *prime divisors, rationality condition*

1. Introduction

While even perfect numbers were known to be given by $2^{p-1}(2^p-1)$, for 2^p-1 prime, the universality of this result led to the the problem of characterizing any other possible types of perfect numbers. It was suggested initially by Descartes that it was not likely that odd integers could be perfect numbers [13]. After the work of de Bessy [3], Euler proved that the condition $\frac{\sigma(N)}{N}=2$, where $\sigma(N)=\sum_{\substack{d|N \\ d\ integer}} d$ is the sum-of-divisors function, restricted odd integers to have the form $(4k+1)^{4m+1}q_1^{2\alpha_1}...q_\ell^{2\alpha_\ell}$, with $4k+1,\ q_1,...,q_\ell$ prime [18], and further, that there might exist no set of prime bases such that the perfect number condition was satisfied.

Investigations of the equation for the sum of the reciprocals of the divisors of N has led to lower bounds on the number of distinct prime divisors. This number has increased from four to nine [23][38][48] while a minimum of 75 total prime factors [25][26] has been established. When $3\not|N$, it was shown that there would be a minimum of twelve different prime divisors [24][33][38]. It was demonstrated also that, if $3,\ 5,\ 7\nmid N$, greater than 26 distinct prime factors would be required [8][39]. In decreasing order, the three largest prime divisors were bounded below by 10^8+7 [21][31], 10^4+7 [30] and 10^2+1 [29] respectively, while the least prime divisor had to be less than $\frac{2n+6}{3}$ for n different prime factors [22]. Moreover, either one of the prime powers, $(4k+1)^{4m+1}$ or $q_i^{2\alpha_i}$ for some index i, was found to be larger than 10^{20} [11]. Through the algorithms defined by the sum over reciprocals of the divisors, it has been demonstrated that odd perfect numbers had to be greater than 10^{300} [5], while the lower bound was increased to 10^{1500} through sieve methods for the abundancy and size of the larges component [5]. An upper bound for an odd perfect number 2^{4^k} if there are k distinct prime factors and $2^{4^{4\beta^2+2\beta+3}}$ if $\alpha_i=\beta,\ i=1,...,\ell$ [49].

One of the possible methods of proof of the odd perfect number conjecture is based on the harmonic mean $H(N)=\frac{\tau(N)}{\sum_{d|N}\frac{1}{d}}$, where $\tau(N)$ is the number of integer divisors of N. It has been conjectured that $H(N)$ is not integer when N is odd [43]. This statement also would imply the nonexistence of odd perfect numbers as the perfect number condition is $\sum_{d|N}\frac{1}{d}=2$ and $\tau(N)$ must be even, since N is not a perfect square. The use of the harmonic mean leads again to the study of the sum of the reciprocals of the divisors, and the values of this sum have only been approximated. For example, it has been found that there are odd integer with five different prime factors such that $\left|\frac{\sigma(N)}{N}-2\right|<10^{-12}$ [32].

The uniqueness of the prime decomposition of an integer allows for the comparison between its magnitude and the sum of the divisors. Since the sum of the divisors of an odd integer $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} q_i^{2\alpha_i}$ can be represented by the product $\frac{(4k+1)^{4m+2}-1}{4k} \cdot \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, it is sufficient to determine the properties of the prime factors of the repunits to establish that there exist no odd perfect numbers. The irrationality of $\sqrt{2(4k+1)}\left[\sigma((4k+1)^{4m+1}) \cdot \prod_{i=1}^{\ell} \sigma(q_i^{2\alpha_i})\right]^{\frac{1}{2}}$ would imply that $\sigma(N)$ cannot equal $2N$. It has been proven for a large class of primes $\{4k+1; q_i\}$ and exponents $\{4m+1; 2\alpha_i\}$ that the rationality condition is not satisfied. The irrationality of the square root of the product of $2(4k+1)$ and the sequence of repunits is not valid for all sets of primes and exponents, however, and it is verified in §4 that the rationality condition holds for twelve odd integers. The factorizations of these integers have the property that the repunits have prime divisors which form interlocking rings, whereas, in general, the sequence of prime factors does not close. The presence of a sequence of primes of increasing magnitude prevents any finite odd integer from being a perfect number. To prove that $\frac{\sigma(N)}{N} \neq 2$, it is necessary also to obtain a lower bound for the number of prime divisors in $\sigma(N)$. Since the number of distinct prime factors of $\frac{q_i^n-1}{q_i-1}$ is minimized in the class of repunits with exponents containing the prime divisor p when $n = p$, the exponent is presumed to be prime throughout the discussion. It is demonstrated in Theorem 1 that the minimum number of prime divisors in the product of repunits that could represent the sum of divisors $\sigma(N)$ of an odd perfect number is greater than or equal to the number of prime factors of N, Then, either $\sigma(N)$ has an excess of prime divisors or constraints must be imposed on $\{4k+1; q_i\}$ and $\{4m+1; 2\alpha_i\}$ which have no integer solution. The non-existence of odd perfect numbers also follows from the sequence of prime factors of increasing magnitude in the factorization of $\sigma(N)$, when one of three specified relations is satisfied, and constraints on the basis and exponents otherwise.

2. The Existence of Different Prime Divisors in the Repunit Factors of the Sum of Divisors

To prove the nonexistence of odd perfect numbers, it shall be demonstrated the product of repunits in the expression for $\sigma(N)$ contains an excess of prime divisors of $\sigma(N)$, such that the perfect number condition $\sigma(N) = 2N$ cannot be satisfied. The existence of distinct prime divisors in the quotients $\frac{(4k+1)^{4m+2}-1}{4k}$ and $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, $i = 1, ..., \ell$ follows from a theorem on the prime factors of the quotients $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ where q_i and $2\alpha_i+1$ are odd primes. A restriction to the least possible number of distinct prime factors in a product

$\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ yields three equations that complement conditions considered previously [12].

If the inequality

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \tag{2.1}$$

holds, then either the sets of primitive divisors of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ are not identical or the exponents of the prime power divisors are different, given that $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Let $q_i > q_j$. Since

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_j^{2\alpha_j+1}-1} &= \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\frac{q_i^{2\alpha_j+1}-1}{q_j^{2\alpha_j+1}-1} \\ &= \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\Bigg[\left(\frac{q_i}{q_j}\right)^{2\alpha_j+1}\Bigg(1+\frac{q_i^{2\alpha_i+1}-q_j^{2\alpha_j+1}-1}{q_i^{2\alpha_j+1}q_j^{2\alpha_j+1}}+\frac{1}{q_j^{4\alpha_j+2}} \\ &\qquad -\frac{1}{q_i^{2\alpha_j+1}q_j^{4\alpha_j+2}}+...\Bigg)\Bigg], \end{aligned} \tag{2.2}$$

this equals

$$\begin{gathered} \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\left[\left(\frac{q_i}{q_j}\right)^{2\alpha_j+1}+\epsilon\right] \\ \epsilon \simeq \left(\frac{q_i}{q_j}\right)^{2\alpha_j+1}\left(\frac{q_i^{2\alpha_j+1}-q_j^{2\alpha_j+1}-1}{q_i^{2\alpha_j+1}q_j^{2\alpha_j+1}}+\frac{1}{q_j^{4\alpha_j+2}}+...\right). \end{gathered} \tag{2.3}$$

When $q_j < q_i < \left[q_j^{2\alpha_j+1}\cdot(q_j^{2\alpha_j+1}-1)\right]^{\frac{2\alpha_j+1}{2\alpha_i+1}\left[1+\frac{(2\alpha_j+1)^2}{2\alpha_i+1}\right]^{-1}}$, $\epsilon \cdot \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1} < 1$. The exact value of the remainder term is

$$\epsilon \cdot \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1} = \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\frac{q_i^{2\alpha_i+1}-q_j^{2\alpha_j+1}}{q_j^{2\alpha_j+1}(q_j^{2\alpha_j+1}-1)} \tag{2.4}$$

which yields the alternative conditions $q_i < 2^{-\frac{1}{2\alpha_i+1-2(2\alpha_j+1)}}q_j^{\frac{2\alpha_j+1}{2\alpha_i+1-2(2\alpha_j+1)}}$ and $q_i < \frac{1}{2}q_j^{3\frac{2\alpha_i+1}{2\alpha_j+1}}$ for the remainder term to be less than 1. Since the entire quotient is $\frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\left[\frac{q_j^{4\alpha_j+2}(q_j^{2\alpha_j+1}-1)+q_i^{2\alpha_j+1}-q_j^{2\alpha_j+1}}{q_i^{2\alpha_j+1}q_j^{2\alpha_j+1}(q_j^{2\alpha_j+1}-1)}\right]$, it can be an integer or a fraction with the denominator introducing no new divisors other than factors of $q_i^{2\alpha_i-1}$ only if $q_j^{2\alpha_j+1}-q_j^{2\alpha_j+1}$ contains all of the divisors of $q_j^{2\alpha_j+1}-1$, presuming that $q_j^{2\alpha_j+1}$ is cancelled. Similarly, if $q_j < q_i \left[q_j^{2\alpha_j+1}(q_j^{2\alpha_j+1}-1)\right]^{\frac{2\alpha_j+1}{2\alpha_i+1}\left[1+\frac{(2\alpha_j+1)^2}{2\alpha_i+1}\right]^{-1}}$, the remainder term must be fractional and $\frac{q_j^{2\alpha_j+1}-1}{q_i-1}$ must have a higher power of one of the prime divisors or a distinct prime divisor, unless $q_i^{2\alpha_j+1}-1$ has a factor which is different from the divisors of $q_j^{2\alpha_j+1}-1$, which typically occurs if the odd primes $q_i,\ q_j$ are sufficiently large and $\alpha_i,\ \alpha_j \geq 1$.

Example 2.1. Consider the prime divisors of pairs of repunits and the integrality of the quotients $\frac{q_i^n-1}{q_j^n-1}$ when $q_i,\ q_j$ and n are odd primes. The first example of an integer ratio for odd prime bases and exponents is $\frac{q_i^{2\alpha_i+1}-1}{q_j^{2\alpha_j+1}-1} = \frac{29^3-1}{3^3-1} = 938$, but the quotient $\frac{29^3-1}{29-1} = 13\cdot 67$ introduces a new prime divisor. For $q_i,\ q_j \gg 1$, the ratio $\frac{q_i^{2\alpha_i+1}-1}{q_j^{2\alpha_i+1}-1} \to \left(\frac{q_i}{q_j}\right)^{2\alpha_i+1}$, which cannot be integer for odd primes $q_i,\ q_j$, implying that one of the repunits, with $\alpha_i = \alpha_j$, has a new prime divisor or $q_j^{2\alpha_i+1}-1$ contains a higher power of one of the prime factors. The latter possibility is excluded because the numerator and denominator are raised to the same power in the limit of large $q_i,\ q_j$.

Theorem 2.2. There will exists a minimum of n prime divisors of $\prod_{i=1}^n \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ with odd prime bases and exponents, $q_i \neq q_j$ for $i \neq j$ and $n \in \mathbb{Z}^+$, for an odd perfect number $N = (4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$, $\ell \geq n$, if

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq (2\alpha_i+1)\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$$
$$(2\alpha_j+1)\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq \frac{q_j^{2\alpha_j+1}-1}{q_i-1}$$
$$(2\alpha_j+1)\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq (2\alpha_i+1)\frac{q_j^{2\alpha_i+1}-1}{q_j-1}$$

are valid.

Proof.

There are no equalities of the form $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$ that have been established yet for odd prime bases and exponents. Equality of two quotients in the rationality condition would yield an integer multiple of the square root of the product of the remaining repunits. It would not affect, therefore, the rationality of the square root expression for $\frac{\sigma(s)}{s}$, where $N = (4k+1)^{4m+1}s^2 = (4k+1)^{4m+1}q_1^{2\alpha_1}\ \dots\ q_\ell^{2\alpha_\ell}$.

The known positive integer solutions to the exponential Diophantine equation $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$, $m, n > 1$, include $2^3 - 1 = \frac{6^2-1}{6-1}$, $31 = 2^5 - 1 = \frac{5^3-1}{5-1}$ and $8191 = 2^{13} - 1 = \frac{90^3-1}{90-1}$ [7][20][40]. The occurrence of the base 2 in one of the repunits is found to be necessary for the integers given by repunits with cubic powers, which have been proven to be the only solutions having an exponent equal to 3 [28].

The entire set of solutions to this equation has not been determined. If $a = \frac{y-1}{\delta}$, $b = \frac{x-1}{\delta}$, $c = \frac{y-x}{\delta}$, $\delta = gcd(x-1, y-1)$ and s is the least integer such that $x^s \equiv 1$ $(mod\ by^{n_1})$, the equation $\frac{x^{m_i}-1}{x-1} = \frac{y^{n_i}-1}{y-1}$ can be represented as $(y-1)x^{m_i} - (x-1)y^{n_i} = y - x$, $ax^{m_i} - by^{n_i} = c$, which may be satisfied trivially for $m_1 = n_1 = 1$. It has been proven that, if (x, y, m_1, n_1) is a presumed solution of the Diophantine equation, then $m_2 - m_1$ is a multiple of s when (x, y, m_2, n_2) is another solution with the same bases x, y [7]. Additionally, it has been established that there are at most two solutions for fixed x and y if $y \geq 7$ [7].

As $x < y$, x can be identified with q_j, y with with q_i and δ with $gcd(q_j - 1, q_i - 1)$. Given that the first exponents are $2\alpha_i + 1$ and $2\alpha_j + 1$, the constraint on a second exponent of q_j then would be $m_2 - (2\alpha_j + 1) = \iota\varphi\left(\left(\frac{q_j-1}{\delta}\right) q_i^{2\alpha_i+1}\right)$ where $\iota \in \mathbb{Q}$ may be a fraction with a denominator which is a divisor of $\varphi\left(\frac{q_j-1}{\delta}\right)$ or $q_i - 1$ or q_i, and otherwise, $\iota \in \mathbb{Z}$ when $q_j \nmid (q_i - 1)$. Based on the trivial solution to these equations for an arbitrary pair of odd primes q_i, q_j, the first non-trivial solution for the exponent of q_j would be required to be greater than or equal to than $1 + \iota q_i^{2\alpha_i}(q_i - 1)\varphi\left(\frac{q_j-1}{\delta}\right)$. This exponent introduces new prime divisors which are factors of $\frac{q_j^{1+\iota q_i^{2\alpha_i}(q_i-1)\varphi\left(\left(\frac{q_j-1}{\delta}\right)\right)}-1}{q_j-1}$ and therefore does not minimize the number of prime factors in the product of repunits.

There is a proof of the existence of a maximum of one solution for given x and y [27]. It has been demonstrated that another equation, defined by the equality of the abundance $I(n) = \frac{\sigma(n)}{n}$ for the product of two prime powers, may be solved by prime pairs that would represent sets of solutions to this exponential Diophantine equation [44].

This result may be verified for the repunits with odd prime bases and exponents. Consider the abundances of products of two prime powers, $I(q_i^{2\alpha_i} q_j^{2\alpha_j}) = \frac{1}{q_i^{2\alpha_i} q_j^{2\alpha_j}} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $I(q_i^{2\alpha'_i} q_j^{2\alpha'_j}) = \frac{1}{q_i^{2\alpha'_i} q_j^{2\alpha'_j}} \frac{q_i^{2\alpha'_i+1}-1}{q_i-1} \frac{q_j^{2\alpha'_j+1}-1}{q_j-1}$, where $q_i \neq q_j$. Let $2\alpha'_i > 2\alpha_i$ and

$2\alpha_j > 2\alpha'_j$ and suppose that the abundances are equal or

$$q_i^{2\alpha'_i - 2\alpha_i} \frac{q_i^{2\alpha_i+1} - 1}{q_i - 1} \frac{q_j^{2\alpha_j+1} - 1}{q_j - 1} = q_j^{2\alpha_j - 2\alpha'_j} \frac{q_i^{2\alpha'_i+1} - 1}{q_i - 1} \frac{q_j^{2\alpha'_j+1} - 1}{q_j - 1}. \tag{2.5}$$

It would follow that $q_i^{2\alpha'_i - 2\alpha_i} \Big| \frac{q_j^{2\alpha'_j+1}-1}{q_j-1}$ and $q_j^{2\alpha_j - 2\alpha'_j} \Big| \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$. The repunits do not have imprimitive prime divisors except, possibly, for the exponents that divide $q_i - 1$ or $q_j - 1$. However, if $(2\alpha_i + 1) \Big| (q_i - 1)$, for example, $gcd(2\alpha_i + 1, q_i) = 1$, and divisibility conditions satisfied q_i and q_j as primitive divisors would not be affected. Then the q_i must be a primitive divisor of $\frac{q_j^{2\alpha'_j+1-1}}{q_j-1}$ and q_j would be a primitive divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$. Any imprimitive divisors, such as $2\alpha_i + 1$, for example, would be matched with divisors of the corresponding repunit $\frac{q_j^{2\alpha'_j+1}-1}{q_j-1}$. Then the prime factors in $q_i^{2\alpha'_i - 2\alpha_i} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ would be identified with the prime divisors of $q_j^{2\alpha_j - 2\alpha'_j} \frac{q_j^{2\alpha'_j+1}-1}{q_j-1}$. The exponents also would match because these primes cannot divide the remaining repunits in the equation. A cancellation of these factors in Eq.(2.5) would yield an equality between the the larger repunits $\frac{q_i^{2\alpha'_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. With a maximum of one solution to the exponential Diophantine equation $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$ for each pair of integers (x, y), no other solutions to this equation with the prime bases q_i, q_j could exist. A one-to-one correspondence between the equality of the abundances of products of even positive powers of q_i and q_j and a solution to the equation $\frac{q_i^{2\alpha'_i+1}-1}{q_i-1} = \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ is established.

Lemma 2.3. The product of two odd prime powers with a given abundance is unique if the exponents are even and equal to one less than an odd prime. There are no solutions to the equation $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ for two different odd primes q_i and q_j and prime exponents $2\alpha_i + 1$ and $2\alpha_j + 1$.

Proof.

Let

$$I(q_i^{2\alpha_i} q^{2\alpha_j}) = \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_i}(q_i-1)} \frac{q_j^{2\alpha_j+1}-1}{q_j^{2\alpha_j}(q_j-1)}$$
$$I(q_i^{2\alpha_i'} q_j^{2\alpha_j'}) = \frac{q_i^{2\alpha_i'+1}-1}{q_i^{2\alpha_i'}(q_i-1)} \frac{q_j^{2\alpha_j'+1}-1}{q_j^{2\alpha_j'}(q_j-1)} \tag{2.6}$$

If $I(q_i^{2\alpha_i} q_j^{2\alpha_j}) = I(q_i^{2\alpha_i} q_j^{2\alpha_j})$, with $\alpha_i' \geq \alpha_i$ and $\alpha_j \geq \alpha_j'$,

$$\left[q_i^{2\alpha_i'-2\alpha_i} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right] \frac{q_j^{2\alpha_j+1}-1}{q_j-1} = \left[q_j^{2\alpha_j-2\alpha_j'} \frac{q_j^{2\alpha_j'+1}-1}{q-1}\right] \frac{q_i^{2\alpha_i'+1}-1}{q_i-1} \tag{2.7}$$

which can be valid only if

$$q_j^{2\alpha_j-2\alpha_j'} \Bigg| \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \qquad q_i^{2\alpha_i'-2\alpha_i} \Bigg| \frac{q_j^{2\alpha_j'+1}-1}{q_j-1}. \tag{2.8}$$

The primitive divisors of $\frac{q_j^{2\alpha_j+1}-1}{q-1}$ and $\frac{q_i^{2\alpha_i'+1}-1}{q_i-1}$ must divide the other repunit in this pair by Eq.(2.7) when $\alpha_i \neq \alpha_i'$ and $\alpha_j \neq \alpha_j'$ [4]. Similarly, the primitive divisors of $\frac{q_j^{2\alpha_j'+1}-1}{q_j-1}$ and $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ are required by this equation to divide only the other repunit in this pair. An imprimitive divisor of $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ must divide q_j-1 and $2\alpha_j+1$, and an imprimitive divisor of $\frac{q_i^{2\alpha_i'+1}-1}{q_i-1}$ must divide q_i-1 and $2\alpha_i'+1$, there exist divisors $\frac{q_j^r-1}{q-1}$, $r|(2\alpha_j+1)$, or $\frac{q_i^t-1}{q_i-1}$, $t|(2\alpha_i'+1)$ or the prime exponent divides one less than the base. Then q_i and q_j cannot be imprimitive divisors of the first kind simultaneously of repunits with bases q_j and q_i respectively, since the divisibility of q_j-1 by q_i and q_i-1 by q_i would yield a contradiction. Proper divisors of the second kind do not exist when the exponents in the repunits are prime. When $2\alpha_i+1$, $2\alpha_j+1$, $2\alpha_i'+1$ or $2\alpha_j'+1$ is an imprimitive prime factor, with $\alpha_i' > \alpha_i$ and $\alpha_j > \alpha_j'$, the divisibility condition

$$(2\alpha_i+1) \Bigg| \left[q_j^{2\alpha_j-2\alpha_j'} \frac{q_j^{2\alpha_j'+1}-1}{q-1}\right],$$
$$(2\alpha_j+1) \Bigg| \frac{q_i^{2\alpha_i'+1}-1}{q_i-1},$$
$$(2\alpha_i'+1) \Bigg| \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \quad or \tag{2.9}$$
$$(2\alpha_j'+1) \Bigg| \left[q_i^{2\alpha_i'-2\alpha_i} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right]$$

must be satisfied. Therefore, the matching of the divisors with the repunits will be defined by the grouping of prime powers and repunits in Eq.(2.7).

A contradiction arises when q_i and q_j are odd prime primitive divisors of the repunits with bases q_j and q_i with odd prime exponents respectively. Let $q_j^{2\alpha_j-2\alpha'_j} = \frac{q_i^{2\alpha_i+1}-1}{U(q_i-1)}$ and

$$q_i^{2\alpha'_i-2\alpha_i}\Bigg|\Bigg[\left(\frac{q_i^{2\alpha_i+1}-1}{U(q_i-1)}\right)^{\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}}-1\Bigg]. \tag{2.10}$$

Since

$$\begin{aligned}(q_i^{2\alpha_i+1}-1)^{\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}} &- (U(q_i-1))^{\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}} \\ = q_i^{(2\alpha_i+1)\left(\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}\right)}&\left[1-\frac{2\alpha'_j+1}{2\alpha'_j-2\alpha_j}\frac{1}{q_i^{2\alpha_i+1}}+...\right] \\ - U^{\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}}q_i^{\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}}&\left[1-\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}\frac{1}{q_i}+...\right],\end{aligned} \tag{2.11}$$

the series would terminate for integer $\frac{2\alpha'_j+1}{2\alpha_j-2\alpha'_j}$, which cannot occur since $2\alpha'_j+1$ is an odd prime, or the infinite series introduces fractional terms that are not divisible by q_i.

Therefore, Eq.(2.7) is valid only if $\alpha'_i-\alpha_i=0$ and $\alpha_j-\alpha'_j=0$ and $q_i^{2\alpha_i}q_j^{2\alpha_j}=q_i^{2\alpha'_i}q_j^{2\alpha'_j}$. It follows that there is only one product of prime powers $q_i^{2\alpha_i+1}q_j^{2\alpha_j+1}$ with the value $I(q_i^{2\alpha_i}q_j^{2\alpha_j})$, where $q_i,\ q_j>2$ and $\alpha_i,\ \alpha_j\geq 1$ such that $2\alpha_i+1$ and $2\alpha_j+1$ are primes. Given the one-to-one correspondence with solutions to the equation $\frac{x^m-1}{x-1}=\frac{y^n-1}{y-1}$, there will be no examples of an equality of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ for different odd prime bases q_i and q_j and different prime exponents $2\alpha_i+1$ and $2\alpha_j+1$.

$\blacksquare$

The known solutions to the exponential Diophantine equation are characterized by $x=2$. The prime factor 2 may be an imprimitive divisor of a repunit with another base, since $2|q-1$ and n if q is an odd prime and the exponent is even. The equality $I(2^4\cdot 5)=\frac{1}{2^4\cdot 5}(2^5-1)\frac{5^2-1}{5-1}=\frac{1}{2^3 5^2}(2^4-1)\frac{5^3-1}{5-1}=I(2^3\cdot 5^2)$, for example, is equivalent to $5\cdot(2^5-1)\frac{5^2-1}{5-1}=2\cdot(2^4-1)\frac{5^3-1}{5-1}$, and it is evident that 2 is an imprimitive divisor of $\frac{5^2-1}{5-1}$, whereas 5 is a primitive divisor of 2^4-1. After cancellation, the equality of the remaining factors gives $2^5-1=\frac{5^3-1}{5-1}$. Therefore, an exception to the theorem arises when the one of the bases is two and the product of prime powers includes odd exponents.

If two repunits satisfied $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = \frac{q_j^{2\alpha_j+1}-1}{q_j-1} = q'^h$, the product would yield only a single prime power q'^{2h} and reduce the total number of prime factors of $\sigma(N)$. There is one known solution to $\frac{q^p-1}{q-1} = q'^h$, for prime $q,\ q',\ p$ [6], and there are no known solutions to the equation $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1} = q'^h$ for $x > y \geq 2$. The introduction of only one prime divisor through the product of two repunits could occur if

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= q'^{h_i} \\ \frac{q_j^{2\alpha_j+1}-1}{q_j-1} &= q'^{h_j}. \end{aligned} \tag{2.12}$$

There are no imprimitive divisors if $(2\alpha_i+1) \not| \, (q_i-1)$ and $(2\alpha_j+1) \not| \, (q_j-1)$, whereas the primitive divisors of the two repunits could be equal if these have the form $\hat{c}(2\alpha_i + 1)^{b_i}(2\alpha_j+1)^{b_j}$. The existence of primitive divisors of $q_i^n - 1$ and $q_j^n - 1$ for $n \geq 3$ implies that q' must belong to this category, and $q' \not|(q_i - 1)$ and $q' \not|(q_j - 1)$.

All solutions to the equation $\frac{x^n-1}{x-1} = y^m$, except $(x, y, m, n) = (18, 7, 3, 3)$, is characterized by the existence of a prime $p|x$ such that $m|(p-1)$ [6]. Then h_i must divide $q_i - 1$ and h_j must divide $q_j - 1$ for any integer solution to either relation in Eq.(2.12). This condition is satisfied when the exponents equal 2 and the bases are odd primes, and there are no solutions other than $\frac{3^5-1}{3-1} = 11^2$ [36][40], while it is nontrivial for $h_i,\ h_j \geq 3$.

Lemma 2.4. The equations

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= q'^{h_i} \\ \frac{q_j^{2\alpha_j+1}-1}{q_j-1} &= q'^{h_j}. \end{aligned}$$

cannot be solved by odd primes q_i, q_j and q', $2\alpha_i'+1$ and $2\alpha_j'+1$ with $h_i,\ h_j \geq 3$.

Proof. Let $\tau_i h_i = \tau_j h_j = l.c.m.(q_i - 1, q_j - 1)$. Then it follows from the equalities with the powers of the prime q' that

$$\left(\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right)^{\tau_i} = \left(\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right)^{\tau_j} \tag{2.13}$$

and

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &\equiv 1 \qquad (mod\ q_i) \\ \left(\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right)^{\tau_j} &\equiv 1 \qquad (mod\ q_i). \end{aligned} \tag{2.14}$$

It is known that $ord_q(a) = q-1$ if $a \not\equiv 1\ (mod\ q)$ and $gcd(a, q-1) = 1$. Since $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \equiv 2\alpha_i + 1 \equiv 0\ (mod\ 2\alpha_i)$ and $gcd\left(\frac{q_j^{2\alpha_j+1}-1}{q_j-1}, q_i - 1\right) = gcd(q'^{h_j}, 2\alpha_j + 1) = 1$ or a contradiction arises because there must be a primitive divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ of the form $c_i(2\alpha_i+1)+1$. Furthermore, $\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \equiv 1\ (mod\ q_i)$ only if $q_j^{2\alpha_i} \equiv 1\ (mod\ q_i)$ or $2\alpha_j|(q_i-1)$.

Let $q_i - 1 = \eta_{ij}(2\alpha_j)$. Then

$$\frac{q_j^{\frac{q_i-1}{\eta_{ij}}+1} - 1}{q_j - 1} = q'^{\frac{q_j-1}{\kappa_j}}. \tag{2.15}$$

Consider the integrality of $\left(\frac{q_j^{\frac{q_i-1}{\eta_{ij}}+1}-1}{q_j-1}\right)^{\frac{\kappa_j}{q_j-1}}$. The factor $(q_j-1)^{\frac{1}{q_j-1}}$ is irrational for $q_j \geq 3$, although $(q_j-1)^{\frac{\kappa_j}{q_j-1}}$ may be rational if $(q_j-1)^{\frac{1}{q_j-1}} = w^{\frac{1}{\kappa_j}}$ for $w \in \mathbb{Q}$. The expansion of the numerator is

$$\left(q_j^{\frac{q_i-1}{\eta_{ij}}+1} - 1\right)^{\frac{\kappa_j}{q_j-1}}$$

$$= q_j^{\frac{\kappa_j}{\eta_{ij}}\frac{q_i-1}{q_j-1}+\frac{\kappa_j\cdot}{q_j-1}}\left(1 - \frac{\kappa_j}{q_j-1}\frac{1}{q_j^{\frac{q_i-1}{\eta_{ij}}+1}} + \frac{1}{2}\frac{\kappa_j}{q_j-1}\left(1-\frac{\kappa_j}{q_j-1}\right)\frac{1}{\left(q_j^{\frac{q_i-1}{\eta_{ij}}+1}\right)^2} + ...\right), \tag{2.16}$$

which is a product of an irrational number $q_j^{\frac{2\alpha_j+1}{h_j}}$, if $h_j \not|(2\alpha_j+1)$ and a series of rational terms. Given that $2\alpha_j+1$ is prime, h_j must equal an integer multiple of $2\alpha_j+1$ for $q_j^{\frac{2\alpha_j+1}{h_j}}$ to be an integer, and then the inequality

$$\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \neq q'^{c(2\alpha_j+1)} \tag{2.17}$$

following from

$$\left(\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right)^{\frac{1}{2\alpha_j+1}} = q_j^{\frac{2\alpha_j}{2\alpha_j+1}}\left(\frac{1-\frac{1}{q_j^{2\alpha_j+1}}}{1-\frac{1}{q_j}}\right)^{\frac{1}{2\alpha_j+1}} < q_j \tag{2.18}$$

$\left(\frac{1-\frac{1}{q_j^{2\alpha_j+1}}}{1-\frac{1}{q_j}}\right) < q_j^{\frac{1}{2\alpha_j+1}}$, yields the contradiction. The product in Eq.(2.16) is irrational because $\left(1-\frac{1}{q_j^{\frac{q_i-1}{\eta_{ij}}+1}}\right)^{\frac{\kappa_j}{q_j-1}} < 1$ and the factor will not compensate for the irrationality of $q_j^{\frac{(2\alpha_j+1}{h_j}-\lfloor\frac{2\alpha_j+1}{h_j}\rfloor}$. Then, if $2\alpha_i \neq (q_i-1)$, $\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \not\equiv 1\ (mod\ q_i)$ and τ_j must equal q_i-1. Similarly, τ_i must equal q_j-1 and

$$\left(\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right)^{q_j-1} = \left(\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right)^{q_i-1}. \tag{2.19}$$

Then

$$q'^{(h_i-h_j)} = \left(\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right)^{\frac{q_i-q_j}{q_i-1}} = \left(\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right)^{\frac{q_i-q_j}{q_j-1}}, \tag{2.20}$$

which could be valid only if $(q_j-1)|(q_i-1)$.

With $q_i-1 = c'(q_j-1)$,

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} = \frac{[c'(q_j-1)+1]^{2\alpha_i+1}-1}{c'(q_j-1)} &\equiv 2\alpha_i+1 \qquad (mod\ q_j-1) \\ \frac{q_j^{2\alpha_j+1}-1}{q_j-1} &\equiv 2\alpha_j+1 \qquad (mod\ q_j-1). \end{aligned} \tag{2.21}$$

It is evident that

$$\begin{aligned} q'^{h_i} &\equiv \frac{[-c'+1]^{2\alpha_i+1}-1}{(-c')} = \left[[c'-1]^{2\alpha_i+1}+1\right]\cdot c'^{-1} \qquad (mod\ q_j) \\ q'^{h_j} &\equiv 1 \qquad (mod\ q_j) \end{aligned} \tag{2.22}$$

as $2\alpha_i+1$ is odd. If h_i is a multiple of h_j,

$$\begin{aligned} [c'-1]^{2\alpha_i+1}+1 &\equiv c' \qquad (mod\ q_j) \\ [c'-1]^{2\alpha_i+1} &\equiv c'-1 \qquad (mod\ q_j). \end{aligned} \tag{2.23}$$

The last congruence requires $(q_i-1)|2\alpha_i$.

Then $2\alpha_i+1 = \tilde{c}_i(q_j-1)+1$ and the primitive prime divisor is

$$q' = \hat{c}_i[\tilde{c}_i(q_j-1)+1]+1 = \hat{c}_i\tilde{c}_i(q_j-1)+\hat{c}_i+1 \tag{2.24}$$

for some constant $\hat{c}_i$. Since

$$\begin{aligned} \frac{q_i^{\tilde{c}_i(q_j-1)+1}-1}{q_i-1} &\equiv 1 \qquad\qquad (mod\ q_j-1) \\ [\hat{c}_i\tilde{c}_i(q_j-1)+\hat{c}_i+1]^{h_i} &\equiv (\hat{c}_i+1)^{h_i} \qquad\qquad (mod\ q_j-1), \end{aligned} \tag{2.25}$$

it follows that $\varphi(q_j-1)|h_i$, where φ is the totient function. Let $h_i = \mu_i\varphi(q_j-1)$. From

$$\frac{q_i^{\tilde{c}_i(q_j-1)+1}-1}{q_i-1} = [\hat{c}_i\tilde{c}_i(q_j-1)+\hat{c}_i+1]^{\mu_i\varphi(q_j-1)} \tag{2.26}$$

the constraint

$$(1+q_i+...+q_i^{2\alpha_i})^{\frac{1}{\varphi(q_j-1)}} = q'^{\mu_i} \tag{2.27}$$

is derived. However,

$$\begin{aligned} &q_i^{\frac{2\alpha_i}{\varphi(q_j-1)}}\left(1+\frac{1}{q_i}+...+\frac{1}{q_i^{2\alpha_i}}\right)^{\frac{1}{\varphi(q_j-1)}} \\ &= q_i^{\frac{2\alpha_i}{\varphi(q_j-1)}}\left[1+\sum_{n=1}^{\infty}\frac{\prod_{m=1}^{n}(1-(m-1)\varphi(q_j-1))(q_i^{2\alpha_i}-1)^n}{n!q_i^{2n\alpha_i}\varphi(q_j-1)^n(q_i-1)^n}\right]. \end{aligned} \tag{2.28}$$

For sufficiently large q_i, this series approximately equals

$$q_i^{\frac{2\alpha_i}{\varphi(q_j-1)}}\left[1+\frac{(q_i^{2\alpha_i}-1)}{q_i^{2\alpha_i}\varphi(q_j-1)(q_i-1)}+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n\varphi(q_j-1)(q_j-1)^n}\right]. \tag{2.29}$$

If $2\alpha_i = q_j-1$, then $\varphi(q_j-1) \not| 2\alpha_i$ and $q^{\frac{2\alpha_i}{\varphi(q_j-1)}}$ is irrational. If $\varphi(q_j-1)|2\alpha_i$, the powers of q_i will not cancel the denominators in the sum while the summation does not yield an integer and there will be no odd prime solution to Eq.(2.19). It follows that both repunits cannot be powers of the same odd prime.

■

The possibility of equating the product of two repunits in $\sigma(N)$ with a factor q'^h is circumvented.

The primitive prime divisors of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ are $a(2\alpha_i+1)^b+1$ while the primitive divisors of $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ are $a'(2\alpha_j+1)^{b'}+1$ [4][47]. Since the exponents in the repunits are primes, this equality is valid only if $a = c(2\alpha_j+1)^{b'}$ and $a' = c(2\alpha_i+1)^b$. For the k^{th} common primitive prime divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, the coefficients would be $a_k = c_k(2\alpha_j+1)^{b'_k}$ and

$a'_k = c_k(2\alpha_i+1)^{b_k}$ yielding the corresponding factor $a_k(2\alpha_i+1)^{b_k}+1 = c_k(2\alpha_i+1)^{b_k}(2\alpha_j+1)^{b'_k}+1$. If no new primitive divisors are introduced by each repunit, it would follow that

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = (product\ of\ imprimitive\ divisors)\cdot\prod_k[c_k(2\alpha_i+1)^{b_k}(2\alpha_j+1)^{b'_k}+1]^{h_k}$$
$$\frac{q_j^{2\alpha_j+1}-1}{q_j-1} = (product\ of\ imprimitive\ divisors)\cdot\prod_k[c_k(2\alpha_i+1)^{b_k}(2\alpha_j+1)^{b'_k}+1]^{h'_k}. \tag{2.30}$$

Let $q_i - 1 = p_1^{h_{i1}}...p_s^{h_{is}}$ and $q_j - 1 = p_1^{h_{j1}}...p_s^{h_{js}}$. Then

$$\frac{(p_1^{h_{i1}}...p_s^{h_{is}}+1)^{2\alpha_i+1}-1}{p_1^{h_{1s}}...p_s^{h_{is}}} = 2\alpha_i+1+\alpha_i(2\alpha_i+1)(p_1^{h_{i1}}...p_s^{h_{is}})+...+(p_1^{h_{i1}}...p_s^{h_{is}})^{2\alpha_i}$$
$$\frac{(p_1^{h_{j1}}...p_s^{h_{js}}+1)^{2\alpha_j+1}-1}{p_1^{h_{1s}}...p_s^{h_{js}}} = 2\alpha_j+1+\alpha_j(2\alpha_j+1)(p_1^{h_{j1}}...p_s^{h_{is}})+...+(p_1^{h_{j1}}...p_s^{h_{js}})^{2\alpha_j}. \tag{2.31}$$

Let P be a primitive divisor of the $q_i^{2\alpha_i+1}-1$. Consider the congruence $c_0+c_1x+...+c_{2\alpha_i}x^{2\alpha_i}\equiv 0\ (mod\ P)$, $c_t>0$, $t=0,1,2,...,2\alpha_i$. There is a unique solution x_0 to the congruence relation $f(x)\equiv c_0+c_1x+...+c_nx^n\equiv 0\ (mod\ P)$ within $[x_0-\epsilon, x_0+\epsilon]$ [35], where

$$\epsilon f'(x_0)+\frac{\epsilon^2}{2!}f''(x_0)+...\ <\ P. \tag{2.32}$$

Setting $c_t = \binom{2\alpha_i+1}{t+1}$,

$$f'(q_i-1) = \binom{2\alpha_i+1}{2}+2\binom{2\alpha_i+1}{3}(q_i-1)+...+2\alpha_i(q_i-1)^{2\alpha_i-1}, \tag{2.33}$$

the constraint (2.33) implies a bound on ϵ of $\left(1-\frac{2}{2\alpha_i+1}\right)\frac{q_i-1}{2\alpha_i}$. Consequently, if

$$|q_i-q_j| < \left(1-\frac{2}{2\alpha_i+1}\right)\frac{q_i-1}{2\alpha_i}, \tag{2.34}$$

the prime P is not a common factor of both repunits. The number of repunits with primitive divisor P will be bounded by the number of integer solutions to the congruence relation.

If the same powers of the common primitive prime divisors occurred in the repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_i+1}-1}{q_j-1}$, an inequality between the two integers necessarily would imply the existence of a distinct primitive factor in one of the repunits. Then, the congruences

$$\begin{aligned} &1+q_i+q_i^2+...+q_i^{2\alpha_i}\equiv 0 \qquad (mod\ P^r) \qquad r\geq 2\\ &1+q_j+q_j^2+...+q_j^{2\alpha_i}\equiv 0 \qquad (mod\ P)\end{aligned} \tag{2.35}$$

could be considered. By Hensel's lemma, solutions to these constraints would satisfy $q_i = q_j + \hat{c}_1 P + \hat{c}_2 P^2 + ... + \hat{c}_{r-1} P^{r-1}$, and therefore, $q_i - q_j$ must be greater than P.

If P is a prime divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, and $\alpha_i = \alpha_j$,

$$P|(q_i - q_j)(q_i^{2\alpha_i} + (2\alpha_i + 1)q_i^{2\alpha_i - 1}q_j + ... + (2\alpha_i + 1)q_i q_j^{2\alpha_i - 1} + q_j^{2\alpha_i}). \tag{2.36}$$

Since $gcd(q_i - q_j, q_i^{2\alpha_i} + (2\alpha_i + 1)q_i^{2\alpha_i - 1}q_j + ... + (2\alpha_i + 1)q_i q_j^{2\alpha_i - 1} + q_j^{2\alpha_i}) = 1$, either $P|(q_i - q_j)$ or $P|(q_i^{2\alpha_i} + (2\alpha_i + 1)q_i^{2\alpha_i - 1}q_j + ... + (2\alpha_i + 1)q_i q_j^{2\alpha_i - 1} + q_j^{2\alpha_i})$. When $P|(q_i - q_j)$, it may satisfy the divisibility condition for the congruence relation. However, there also would be a primitive divisor which is a factor of $q_i^{2\alpha_i} + q_i^{2\alpha_i - 1}q_j + ... + q_j^{2\alpha_i - 1}q_i + q_j^{2\alpha_i}$, and not $q_i - q_j$. While there may exist primes which divide the difference between the repunits and not each quotient, the common prime factors of $\frac{q_i^{2\alpha_i+1}1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ must be divisors of $q_i - q_j$ for the congruences to hold modulo different powers. The common prime factors $\{P'\}$ which do not divide $q_i - q_j$ must be equal to the primes that exactly divide $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$.

The distinct exponents of the primitive prime divisors dividing $\frac{q_j^{2\alpha_i+1}-1}{q_j-1}$ must be less than those of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ since the P-adic expansion for the reverse inequality would produce a contradiction with $q_i > q_j$. A precise coincidence of the set of primitive prime divisors then would produce the divisibility condition $\frac{q_j^{2\alpha_i+1}-1}{q_j-1} \Big| \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, which is not valid generically for arbitrarily large pairs (q_i, q_j), $q_i \neq q_j$. It follows that there should be a prime divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ distinct from the factors of $\frac{q_j^{2\alpha_i+1}-1}{q_j-1}$. Setting $q_i = 5$ and $q_j = 3$, for instance, it can be proven by induction that one of the repunits $\frac{5^{2\alpha_i+1}-1}{4}$ and $\frac{3^{2\alpha_j+1}-1}{2}$ has a distinct prime divisor, since $\frac{5^3-1}{5-1} = 31$, $\frac{5^5-1}{4} = 11 \cdot 71$, $\frac{3^3-1}{2} = 13$, $\frac{3^5-1}{2} = 11 \cdot 11$, and only primitive divisors of the integers occur in these sequences when $2\alpha_i + 1$ is prime.

Furthermore, there would be a new prime divisor of $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ if $\alpha_j > \alpha_i$. This prime divisor is congruent to 1 modulo $2\alpha_j + 1$ and would differ from the other prime factor distinguishing the two repunits with the exponent $2\alpha_i + 1$ that is congruent to 1 modulo $2\alpha_i + 1$ unless both equal the same integer having the form $a(2\alpha_i + 1)(2\alpha_j + 1) + 1$. Under the first condition, $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ has a minimum of one new prime divisor other han the set of prime factors of each repunit. If the second condition is valid, it remains to be determined if any new prime divisors arise in the product.

When the prime exponents are different, the existence of primitive divisors of the Lucas numbers $\frac{\alpha^n-\beta^n}{\alpha-\beta}$ for $n>12$, when $\alpha,\ \beta$ are real, is a consequence of the factorization into cyclotomic polynomials $F_n(\alpha,\beta)=\prod_{\substack{1\le k<\frac{n}{2}\\ gcd(k,n)=1}}\left(\alpha-2\ cos\left(\frac{2\pi k}{n}\right)\beta\right)$ [4][49]. The Lucas sequence has different primitive divisors as n changes because it is not possible to match all of the factors with $cos\left(\frac{2\pi k}{n}\right)$.

If $\alpha_i\ne\alpha_j$,

$$
\begin{aligned}
q_i^{2\alpha_i+1}-1 &= \prod_{k=0}^{2\alpha_i}\left(q_i-\omega_{2\alpha_i+1}^k\right)\\
q_j^{2\alpha_j+1}-1 &= \prod_{k'=0}^{2\alpha_j}\left(q_j-\omega_{2\alpha_j+1}^{k'}\right)\\
\omega_n &= e^{\frac{2\pi i}{n}}.
\end{aligned}
\tag{2.37}
$$

Quotients of products of linear factors in $q_i^{2\alpha_i+1}-1$ and $q_j^{2\alpha_j+1}-1$ generally will not involve cancellation of integer divisors because products of powers of the different units of unity cannot be equal unless the exponents are a multiples of the primes $2\alpha_i+1$ or $2\alpha_j+1$. The sumsets which give rise to equal exponents can be enumerated by determining the integers $s_1,...,s_t$ such that the sums are either congruent to 0 modulo $2\alpha_i+1$ and $2\alpha_j+1$. It is apparent that the entire integer sets $\{1,2,...,2\alpha_i\}$ and $\{1,2,...,2\alpha_j\}$ cannot be used for unequal repunits. The number of sequences satisfying the congruence relations for each repunit must equal $\sum_{\sum t_k=2\alpha_i+1}(2^{t_k}-1)$ and $\sum_{\sum t_l'=2\alpha_j+1}(2^{t_l'}-1)$, with sequences of integers which sum to a non-zero value giving rise to cancellation of complex numbers in the product being included.

Consider two products $\prod_{m=1}^{t_k}(q_i-\omega_{2\alpha_i+1}^{s_m})$ and $\prod_{n=1}^{t'_k}(q_j-\omega_{2\alpha_j+1}^{s_n'})$. Upper and lower bounds for these products are

$$
\begin{aligned}
(q_i-1)^{t_k} &< \prod_{m=1}^{t_k}(q_i-\omega_{2\alpha_i+1}^{s_m})<(q_i+1)^{t_k}\\
(q_j-1)^{t_l'} &< \prod_{n=1}^{t_l'}(q_j-\omega_{2\alpha_j+1}^{s_n'})<(q_j+1)^{t_l'}.
\end{aligned}
\tag{2.38}
$$

Since the minimum difference between primes is 2, one of the inequalities $(q_i+1)^{t_k}\le(q_j-1)^{t_l'}$, $(q_j+1)^{t_l'}\le(q_i-1)^{t_k}$, $(q_i+1)^{t_k}>(q_j-1)^{t_l'}$ or $(q_j+1)^{t_l'}>(q_i-1)^{t^k}$ will be satisfied when $t_l'\ne t_k$. The first two inequalities imply that the factors of the repunits defined by the products $\prod_{m=1}^{t_k}(q_i-\omega_{2\alpha_i+1}^{s_m})$ and $\prod_{n=1}^{t_k'}(q_j-\omega_{2\alpha_j+1}^{s_n'})$ are distinct.

If either of the next two inequalities hold, the factors possibly could be equal when $(q_i-1)^{t_k}=(q_j-1)^{t'_i}$ or $(q_i+1)^{t_k}=(q_j+1)^{t'_l}$. In the first case, $t'_l=n_{ij}t_k$, $q_i-1=(q_j-1)^{n_{ij}}$, $n_{ij}>2$. This relation holds for all t_k and t'_l such that $\sum_k t_k=n_{ij}\sum_l t'_l$. However, $2\alpha_i+1$ and $2\alpha_j+1$ are prime, so that this equation is not valid. When $q_i+1=(q_j+1)^{n_{ij}}$, $n_{ij}\geq 2$, the relation $\sum_k t_k=n_{ij}\sum_l t'_l$ again leads to a contradiction.

The identification of the products in Eq.(2.37) for each corresponding k,l is necessary, since any additional product would give rise either to a new prime factor after appropriate rescaling, or a different power of the same number, which would would prevent a cancellation of an extra prime, described later in the proof. The inequalities $(q_j-1)^{t'_l}<(q_i-1)^{t_k}<(q_j+1)^{t'_l}$ imply that

$$t'_l\,\frac{ln(q_j-1)}{ln(q_i-1)}<t_k<t'_l\frac{ln(q_j+1)}{ln(q_i-1)}. \tag{2.39}$$

If these inequalities hold for two pairs of exponents (t_{k_1},t'_{l_1}), (t_{k_2},t'_{l_2}), the interval $\left[t'_{l_2}\frac{ln(q_j-1)}{ln(q_i-1)},t'_{l_2}\frac{ln(q_j+1)}{ln(q_i-1)}\right]$ contains $\frac{t'_{l_2}}{t'_{l_1}}t_{k_1}$. The fractions $\frac{t_{k_m}}{t_{k_n}}$ cannot be equal to $\frac{t'_{l_m}}{t'_{l_n}}$ for all m,n as this equality would imply that $\frac{t_k}{t'_l}=\frac{2\alpha_i+1}{2\alpha_j+1}$, which is not possible since $\frac{2\alpha_i+1}{2\alpha_j+1}$ is an irreducible prime fraction when $\alpha_i\neq\alpha_j$.

Then

$$\left|\frac{t_{k_1}}{t'_{l_1}}-\frac{t_{k_2}}{t'_{l_2}}\right|\geq\frac{1}{t'_{l_1}t'_{l_2}}. \tag{2.40}$$

and $t_{k_2}\notin\left[t'_{l_2}\frac{ln(q_j-1)}{ln(q_i-1)},t'_{l_2}\frac{ln(q_j+1)}{ln(q_i-1)}\right]$ if

$$ln\,\left(\frac{q_j+1}{q_j-1}\right)<\frac{ln(q_i-1)}{t'_{l_1}t'_{l_2}}. \tag{2.41}$$

As $t'_l\leq 2\alpha_j+1$, this inequality is valid when

$$q_j ln(q_i-1)>\frac{9}{4}(2\alpha_j+1)^2. \tag{2.42}$$

Therefore, the products of linear factors with $t_k<2\alpha_i+1$ and $t'_\ell<2\alpha_j+1$ will not be equal given the Eqs.(2.41) and (2.42), When these products cannot be equated, there does not exist sets of linear terms in the decomposition of $q_i^{2\alpha_i+1}-1$ and $q_j^{2\alpha_j+1}-1$ that give the same prime divisor $a(2\alpha_i+1)(2\alpha_j+1)+1$ within multiplication by a prime power.

Since the arithmetic primitive factor of $\frac{q^n-1}{q-1}$, $n\neq 6$ is $\Phi_n(q)=\prod_{\substack{1\leq k\leq n\\ gcd(k,n)=1}}(q_i-e^{\frac{2\pi ik}{n}})$ or $\frac{\Phi_n(q)}{p}$, where p is the largest prime factor of both $\frac{n}{gcd(n,3)}$ and $\Phi_n(q)$ [1][2][4][47][50], any prime, which is a primitive divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, divides $\Phi_{2\alpha_i+1}(q_i)$. It follows that these primes can be obtained by appropriate multiplication of the products $\prod_{m=1}^{t_k}(q_i - \omega_{2\alpha_i+1}^{s_m})$ and $\prod_{n=1}^{t'_k}(q_j - \omega_{2\alpha_j+1}^{s'_n})$. Although these products may not be the primitive divisors, they are real, and multiplication of $\prod_{m=1}^{t_1}(q_i - \omega_{2\alpha_i+1}^{s_m})$ by $\kappa_1 \in \mathbb{R}$ can be compensated by multiplication of $\prod_{m'=1}^{t_2}(q_i - \omega_{2\alpha_j+1}^{s'_n})$ by κ_1^{-1} to obtain the integer factors. By Lemma 2, the occurrence of the minimal set of prime divisors in the two repunits $\frac{q_i^{2i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ introduces two prime factors in the products of the linear terms.

Since the prime divisors of the repunits can be obtained by rescaling of products of the form (2.38), t'_{l_1} and t'_{l_2} can be set equal, provided that the factor of a power of a prime is included to match the intervals. Suppose, for example, that $t'_{\ell_1} = t'_{\ell_2} = 2$. Then, $P^h(q_j-1)^2 < (q_i-1)^{t_{k_1}} < P^h(q_j+1)^2$ and $t_{k_1} \in \left[\frac{2\ ln(q_j-1)+h\ ln\ P}{ln(q_i-1)}, \frac{2\ ln(q_j+1)+h\ ln\ P}{ln(q_i-1)}\right]$. Again, for some k_1, k_2, $\frac{t_{k_1}}{t'_{l_1}} \neq \frac{t_{k_2}}{t'_{l_2}}$ if $\alpha_i \neq \alpha_j$, and $|t_{k_1} - t_{k_2}| \geq 1$. The inequalities $P^h(q_j - 1)^2 < (q_i-1)^{t_{k_2}} < P^h(q_j+1)^2$ arise because each of the terms $y_l = (q_j - e^{\frac{2\pi i k_l}{2\alpha_j+1}})(q_j - e^{\frac{2\pi i k_l}{2\alpha_j+1}})$, which is matched with a corresponding product of linear factors $\prod_{s_m=1}^{t_k}(q_i - \omega_{2\alpha_i+1}^{s_m})$, has the same upper and lower bound. However, $t_{k_2} \notin \left[\frac{2\ ln(q_j-1)+h\ lnP}{ln(q_i-1)}, \frac{2\ ln(q_j+1)+h\ lnP}{ln(q_i-1)}\right]$ if $\frac{2\ ln\left(\frac{q_j+1}{q_j-1}\right)}{ln(q_i-1)} < 1$ which implies $q_i > 5$. The equality is also valid because $t_{k_2} = \frac{2ln(q_j+1)+h\ lnP}{ln(q_i-1)}$ only if $P^h = \frac{(q_i-1)^{t_{k_2}}}{(q_j+1)^2}$ and $q_i - 1 = (q_j+1)^{h'}$, $h' \geq 1$. Then $t'_l = h' t_k$, and if $h' > 1$, summation over k, l gives $2\alpha_j + 1 = h'(2\alpha_i+1)$ which is not possible as $2\alpha_i + 1$ and $2\alpha_j + 1$ are prime. If $h' = 1$, $t_k = t'_l$ and $2\alpha_i + 1$ must be set equal to $2\alpha_j + 1$, and a new prime divisor will occur. When the limits t_k and $t_{\ell'}$ are set equal to 2 or a bounded integer, which is feasible since there would exist only $4m + 2 + \sum_{i=1}^{\ell}(2\alpha_i+1)$ prime factors for the odd integer $(4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$ to be a perfect number, the products $\prod_{m=1}^{t_k}(q_i - \omega_{2\alpha_i+1}^{s_m})$ and $\prod_{n=1}^{t'k}(q_j - \omega_{2\alpha_j+1}^{s'_n})$ would equal a real number that must be rescaled by to give an integer. The rescaling will be given by a prime power in the following.

A minimum set of prime divisors will be derived for a product of a given set of repunits. Since $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ are not powers of the same prime by Lemma 2, there will a minimum of two distinct prime divisors of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Suppose that

1. $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = q_1'^{h_i} x_i \qquad \frac{q_j^{2\alpha_j+1}-1}{q_j-1} = q_2'^{h_j} x_j.$

Now consider a third repunit $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$. Either

1a. $\frac{q_k^{2\alpha_k+1}-1}{q_k-1} = q_3'^{h_k} x_a$

1b. $\frac{q_k^{2\alpha_k+1}-1}{q_k-1} = q_1'^{h_{1k}} q_3'^{h_k} x_b$

1c. $\frac{q_k^{2\alpha_k+1}-1}{q_k-1} = q_2'^{h_{jk}} q_3'^{h_k} x_c.$

1d. $\frac{q_k^{2\alpha_k+1}-1}{q_k-1} = q_1'^{h_{ik}} q_2'^{h_{jk}} x_d$

where $x_i,\ x_j,\ x_a,\ x_b,\ x_c$ and x_d represent additional factors. Under the first three conditions, there are three distinct prime divisors of the product of the three repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \frac{q_k^{2\alpha_k+1}-1}{q_k-1}$.

Given that the prime divisors are primitive divisors in Condition (1d), q_1' must have the form $a(2\alpha_i+1)(2\alpha_k+1)+1$, while q_2' would equal $b(2\alpha_j+1)(2\alpha_k+1)+1$. If $ln(q_i-1) \geq \frac{9}{4}$ or $q_i \geq 11$, then Eq.(2.42) will be satisfied if $q_\ell > (2\alpha_\ell+1)^2$. Provided that the odd prime bases are greater than or equal to 11, and q_1' is labelled q_ℓ, the repunit $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ will have a prime divisor that is not a factor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ or $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ if $\alpha_\ell \leq \alpha_i,\ \alpha_k$, and

$$\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} = q_2'^{h_{2\ell}} q_3'^{h_{3\ell}} x_\ell. \tag{2.43}$$

Therefore, three prime divisors occur in the product

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}. \tag{2.44}$$

If $\alpha_\ell > \alpha_i$ or $\alpha_\ell > \alpha_k$, a primitive prime divisor must be introduced that is different from the repunits with the basis q_ℓ and exponents less than or equal to $\alpha_i,\ \alpha_k$. It can equal q_2' only if $h_{2\ell} = 0$ for an exponent less than or equal to $\alpha_i,\ \alpha_k$. Since the repunits $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ both cannot be powers of the same prime q_2' by Lemma 2, a third prime factor must occur. The new factor may be labelled q_3', and again this product of three repunits is divisible by three distinct primes.

Similarly, if q_m is set equal to q_2',

$$\frac{q_m^{2\alpha_m+1}-1}{q_m-1} = q_1'^{h_{1m}} q_4'^{h_{3m}} x_m. \tag{2.44}$$

when $\alpha_m \leq \alpha_j,\ \alpha_k$. and the product

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_m^{2\alpha_m+1}-1}{q_m-1} \tag{2.45}$$

would have a minimum of three distinct prime divisors. If $\alpha_m > \alpha_j$ or α_k, a new prime divisor q_4' will arise and there would be a minimum of three different prime factors of this product.

If $q_3' \neq q_4'$, there would be four different primes divisors of

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_\ell^{2\alpha_j+1}-1}{q_\ell-1}\frac{q_m^{2\alpha_m+1}-1}{q_m-1}. \tag{2.46}$$

When $q_3' = q_4'$. it must have the form $c(2\alpha_\ell+1)(2\alpha_m+1)+1$ to be a primitive prime divisor. Setting $q_r = q_3' = q_4'$, $\frac{q_r^{2\alpha_r+1}-1}{q_r-1}$ has a prime divisor distinct from the factors of $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ or $\frac{q_m^{2\alpha_m+1}-1}{q_m-1}$ if $\alpha_r \leq \alpha_\ell, \alpha_m$. If $\alpha_r > \alpha_\ell$ or $\alpha_r > \alpha_m$, a new primitive prime divisor is introduced unless it coincides with q_1' or q_2'. Then another prime factor must occur because the pairs of repunits cannot be powers of the same prime. It follows that

$$\frac{q_i^{2\alpha_j+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}\frac{q_r^{2\alpha_r+1}-1}{q_r-1} \tag{2.47}$$

or

$$\frac{q_i^{2\alpha_j+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_m^{2\alpha_m+1}-1}{q_m-1}\frac{q_r^{2\alpha_r+1}-1}{q_r-1} \tag{2.48}$$

would be divisible by a minimum of four distinct prime factors.

Now consider the condition

2. $\quad \frac{q_i^{2\alpha_i+1}-1}{q_i-1} = q_1'^{h_i} \qquad \frac{q_j^{2\alpha_j+1}-1}{q_j-1} = q_1'^{h_{1j}} q_2'^{h_{2j}}$.

It follows that that the differentiation between the prime divisors of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ depends on a factor of the form $a(2\alpha_i+1)(2\alpha_j+1)+1$, given that exponents do not divide either repunit. Since it must be a prime, it will the basis for a new repunit $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$, where $q_k = a(2\alpha_i+1)(2\alpha_j+1)+1$ for the integer N to be an odd perfect number. The prime satisfies the inequality Eq.(2.42) with respect to the repunit with the lesser exponent if $\alpha_k \leq \alpha_i, \alpha_j$. Then there will be a prime factor of $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ that does not divide $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ or $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and there will be a minimum of three different prime divisors of the product of the three repunits. When $\alpha_k > \alpha_i$ or $\alpha_k > \alpha_j$, there is a new primitive divisor that is a third prime factor unless it coincides with q_2', since it cannot be divisible by $q_1' = q_k$. Given that

$$\frac{q_k^{2\alpha_k+1}-1}{q_k-1} = q_2'^{h_{2k}} x_k, \tag{2.49}$$

q_2' would have the form $b(2\alpha_j+1)(2\alpha_k+1)+1$. Labelling this prime to be q_ℓ, $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ would have a new prime factor not dividing $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ if $\alpha_\ell \leq \alpha_j,\ \alpha_k$. If $\alpha_\ell > \alpha_j$ or $\alpha_\ell > \alpha_k$, there would be a new primitive prime divisor unless it coincided with q_1'. Since the repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ both cannot be powers of the prime q_1', it would follow then that

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \tag{2.50}$$

is divisible by three distinct prime divisors.

If $\alpha_k,\ \alpha_\ell \leq \alpha_i,\ \alpha_j$, the new prime divisors in $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ may be labelled q_3' and q_4' respectively, and there would be a minimum of four different prime divisors of

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_k^{2\alpha_k+1}-1}{q_k-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \tag{2.51}$$

if $q_3' \neq q_4'$. When $q_3' = q_4'$,

$$\begin{aligned} \frac{q_k^{2\alpha_k+1}-1}{q_k-1} &= q_3'^{h_{3k}} x_k \\ \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} &= q_3'^{h_{4\ell}} x_\ell. \end{aligned} \tag{2.52}$$

Either $x_k \neq 1$ or $x_\ell \neq 1$ or $x_k \neq 1,\ x_\ell \neq 1$. A fourth prime factor q_5' is introduced, and the product of the four reupunits is divisible by four different primes when $q_i, ..., q_\ell \geq 11$.

For $\alpha_k,\ \alpha_\ell > \alpha_i$ or $\alpha_k,\ \alpha_\ell > \alpha_j$, it is feasible for the repunits to have the form

$$\begin{aligned} \frac{q_k^{2\alpha_k+1}-1}{q_k-1} &= q_2'^{h_{2k}} x_k \\ \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} &= q_1'^{h_{1\ell}} x_\ell. \end{aligned} \tag{2.53}$$

if the one of the primitive divisors coincides with q_2' and q_1' respectively. Then $q_1' = a'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_\ell+1)+1$ and $q_2' = b(2\alpha_j+1)(2\alpha_k+1)+1$. When $2\alpha_i+1,\ 2\alpha_j+1 < 2\alpha_k+1 \leq \sqrt{a'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_k+1)+1}$, there is a prime factor q_3' of x_k that does not divide $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ or $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$. By Lemma 2, there must be a prime factor of x_ℓ that does not equal q_1', q_2' or q_3'. Then the product $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_k^{2\alpha_k+1}-1}{q_k-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ is divisible by a minimum of four different primes.

If $\alpha_k > \alpha_i$ or $\alpha_k > \alpha_j$ and $2\alpha_k + 1 > \sqrt{a'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_\ell+1)+1}$, there is a primitive divisor occurring in $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$, different from q_2' and the prime factors of x_k, that may be labelled q_3'. It is possible that $h_{2k} = 0$ in Eq.(2.53) and the prime q_2' rather than q_3' is introduced into the factorization of $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$. Then the proofs of the existence of a fourth prime divisor will be interchanged and conclusions remain valid. To minimize the number of primes, suppose that x_ℓ is divisible only by q_3'. Then

$$\begin{aligned}\frac{q_k^{2\alpha_k+1}-1}{q_k-1} &= q_3'^{h_{3k}} x_k' \\ \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} &= q_1'^{h_{1\ell}} q_3'^{h_{3\ell}}.\end{aligned} \tag{2.54}$$

It follows that $q_3' = c(2\alpha_k+1)(2\alpha_\ell+1)+1 \equiv q_r$. If $\alpha_r \leq \alpha_k,\ \alpha_\ell$, there exists a prime factor of $\frac{q_r^{2\alpha_r+1}-1}{q_r-1}$ that does not divide $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ or $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ and may be labelled q_4'. If $q_4' \neq q_2'$, there is a minimum of four distinct prime divisors of

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_k^{2\alpha_k+1}-1}{q_k-1}\frac{q_r^{2\alpha_r+1}-1}{q_r-1} \tag{2.55}$$

or

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}\frac{q_r^{2\alpha_r+1}-1}{q_r-1} \tag{2.56}$$

for $q_i,\ q_j,\ q_k,\ q_\ell,\ q_r \geq 11$.

If q_4' coincides with q_2', the number of primes will be minimized when $\frac{q_r^{2\alpha_r+1}-1}{q_r-1} = q_2'^{h_{2r}}$. Then $q_2' = b'(2\alpha_j+1)(2\alpha_k+1)(2\alpha_r+1)+1$. A fourth prime factor arises when $\alpha_\ell < \alpha_j,\ \alpha_k$, Given that $2\alpha_j+1,\ 2\alpha_k+1 < \sqrt{b'(2\alpha_j+1)(2\alpha_k+1)(2\alpha_r+1)+1}$, $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ will have a prime divisor not dividing $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ or $\frac{q_r^{2\alpha_r+1}-1}{q_r-1}$. Therefore, the least number of prime factors would occur when

$$\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} = q_1'^{h_{1\ell}} q_3'^{h_{3\ell}} q_4'^{h_{4\ell}} \tag{2.57}$$

Then the product of the four repunits with bases $q_i,\ q_j,\ q_k,\ q_\ell \geq 11$ would have a minimum of four different prime divisors. If $2\alpha_\ell + 1 > \sqrt{b'(2\alpha_j+1)(2\alpha_k+1)(2\alpha_\ell+1)+1}$,

$$\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} = q_5'^{h_{5\ell}} x_\ell' \tag{2.58}$$

where q_5' does not equal q_1', q_2' or q_3'. Again, there are four primes in the factorization of the product of the four repunits.

Suppose that

3. $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = q_1'^{h_{1i}} q_2'^{h_{2i}}$ $\qquad\qquad$ $\frac{q_j^{2\alpha_j+1}-1}{q_j-1} = q_1'^{h_{1j}} q_2'^{h_{2j}}.$

Let $q_k = q_1' = a(2\alpha_i+1)(2\alpha_j+1)+1$ and $q_\ell = q_2' = b(2\alpha_i+1)(2\alpha_j+1)+1$. If $\alpha_k \le \alpha_i,\ \alpha_j$, there is a prime factor of $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ that does not divide $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ or $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and it may be labelled q_3'. Similarly, there is prime divisor q_4' that is not a factor of both repunits in Condition 3. Consequently, given these inequalities for the exponents, $\frac{q_i^{2\alpha_i+1}-1}{q-i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ and $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ have a minimum of three different prime factors. Furthermore, there is a minimum of four distinct prime divisors of

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_k^{2\alpha_k+1}-1}{q_k-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \tag{2.59}$$

when $q_3' \neq q_4'$. If $q_3' = q_4'$, both $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ cannot be powers of the same prime by Lemma 2, and a fourth prime divisor is introduced into the product of the four repunits.

If $\alpha_k > \alpha_i$ or $\alpha_k > \alpha_j$, there would be a new primitive divisor different from q_3' that could be q_2', but it cannot equal q_1'. Suppose that

$$\frac{q_k^{2\alpha_k+1}-1}{q_k-1} = q_2'^{h_{2k}} x_k. \tag{2.60}$$

Similarly, if $\alpha_\ell > \alpha_i$ or $\alpha_\ell > \alpha_j$, the primitive divisor could coincide with q_1' such that

$$\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} = q_1'^{h_{1\ell}} x_\ell. \tag{2.61}$$

Then $q_k = q_1' = a'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_\ell+1)+1$ and $q_\ell = q_2' = b'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_k+1)+1$. If $2\alpha_i+1, 2\alpha_j+1 < 2\alpha_\ell+1 < \sqrt{b'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_\ell+1)+1}$, $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ must have a prime factor that does not divide $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ or $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $x_k = q_3''$. When $2\alpha_\ell+1 > \sqrt{b'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_\ell+1)+1}$, it follows that there is a new primitive divisor different from q_2' or q_3'', which may be labelled q_5''. Similarly, there would be a prime factor q_4'' of $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ that does not divide $\frac{q_i^{2\alpha_j+1}-1}{q_j-1}$ or $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ if $2\alpha_i+1,\ 2\alpha_j+1 < 2\alpha_\ell+1 < \sqrt{a'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_k+1)+1}$. When $2\alpha_\ell+1 >$

$\sqrt{a'(2\alpha_i+1)(2\alpha_j+1)(2\alpha_k+1)+1}$, the primitive divisor would be different from q_1', q_4'' and may be labelled q_6'. Therefore,
$\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ and $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ always has a minimum of three different prime divisors if the odd prime bases are greater than or equal to 11. There is a minimum of four distinct prime divisors of the product of the four repunits if $q_5' \neq q_6'$.

If $q_5' = q_6'$, it is not possible for both $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ to be powers of this prime. Therefore,

$$\begin{aligned}\frac{q_k^{2\alpha_k+1}-1}{q_k-1} &= q_5'^{h_{1k}}x_k' \\ \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} &= q_5'^{h_{5\ell}}x_\ell'\end{aligned} \tag{2.62}$$

where $x_k' \neq 1$ or $x_\ell' \neq 1$. Both do not equal a power of q_1' or q_2'. Then x_k' or x_ℓ' has a prime factor q_7' that does not equal q_1', q_2' or q_5'. Consequently, a minimum of four different primes divides $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\frac{q_k^{2\alpha_k+1}-1}{q_k-1}\frac{q_\ell^{2\alpha+1}-1}{q_\ell-1}$.

The prime divisors of any remaining repunits with odd prime bases less then 11 may be listed. It has been proven that no odd perfect number can have 3, 5 and 7 as factors simultaneously [34]. Consequently, the following sets may occur amongst these three primes: $\{3\}$, $\{5\}$, $\{7\}$, $\{3,5\}$, $\{3,7\}$ and $\{5,7\}$. It is evident that the number of distinct prime factors will be greater than or equal to the number of repunits for each of these sets by Lemma 2. There could be a coincidence between these divisors and factors of repunits with larger odd prime bases. Yet the conclusions reached previously are unaffected, because these primes would divide remaining factors x_a, x_b, x_c, x_d, x_i, x_j, x_k, x_ℓ, x_m and x_r in the repunits.

Suppose that the product of $n-1$ repunits with odd prime bases and exponents $\prod_{r=1}^{n-1}\frac{q_{i_r}^{2\alpha_{i_r}+1}-1}{q_{i_r}-1}$ has a minimum of $n-1$ different prime divisors, which occurs in the sum of divisors of an odd perfect number. If there is a prime factor of $\frac{q_{i_n}^{2\alpha_{i_n}+1}-1}{q_{i_n}-1}$ that divides this product, either it is a power of $2\alpha_{i_r}+1$, $r=1,...,n-1$ or it is a primitive prime factor of the other repunits. The first condition will be considered afterwards. Secondly, it has the form $a_r(2\alpha_{i_r}+1)+1$ for any $a \in \mathbb{Z}^+$ and $r=1,...,n-1$, a set of $n+3$ repunits can be produced with a minimum of $n+3$ prime divisors by the preceding discussion. Therefore, a minimum of n distinct prime divisors exist in a product $\prod_{r=1}^{n}\frac{q_{i_r}^{2\alpha_{i_r}+1}-1}{q_{i_r}-1}$, which occurs in the sum of divisors of an odd perfect number, by induction for $n=4z$, $1+4z$, $2+4z$, $z \in \mathbb{Z}^+$ when the exponents are not prime divisors. It can be demonstrated, however, that any odd perfect number with three or four prime factors also must have a minimum of three or four, and such integers cannot satisfy the condition $\frac{\sigma(N)}{N} = 2$ [12]. It is sufficient to prove, for a given prime divisor of N, that there are three other prime factors such that the product of the four repunits with prime exponents is divisible by a minimum of four prime divisors to extend the result to all primes divisors of N. Consequently, if the prime exponents are not divisors of the repunits, there would be a minimum of n different prime divisors in the product of n repunits arising from $\sigma(\prod_{i=1}^{n} q_i^{2\alpha_i})$ for all positive integers n, if $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} q_i^{2\alpha_i}$ is an odd perfect number.

For the known solutions to the equation $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$, one of the prime bases is 2. Since $q_j - 1 = 1$, the theorem is circumvented because the bounds (2.39) are satisfied for a larger set of primes q_i and exponents t_k. Furthermore, the integers sets $\{1, 2, ..., 2\alpha_i\}$ and $\{1, 2, ..., 2\alpha_j\}$ are used entirely in the products to give the same integer, which must then be a prime. The existence of solutions to the inequalities $1 < (q_i - 1)^{2\alpha_i} < 3^{2\alpha_i}$ implies that these bounds do not exclude the equality of $\frac{2^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ for some $2\alpha_j + 1, q_i, 2\alpha_i + 1$.

If the primes, which divide only $q_i^{2\alpha_i+1} - 1$, are factors of $q_i - 1$ or $q_j - 1$, these would be partially cancelled in a comparison of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Any divisor of $q_i - 1$ which is a factor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ also must divide $2\alpha_i + 1$. When $2\alpha_i + 1$ and $2\alpha_j + 1$ are prime, this divisor would have to be $2\alpha_i + 1$ or $2\alpha_j + 1$.

An exception to the result concerning the occurrence of a distinct prime factor in one of the repunits could occur if equations of the form

$$\begin{aligned}
\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)^{\nu}\,\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\
(2\alpha_j+1)^{\nu}\,\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\
(2\alpha_j+1)^{\nu_1}\,\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)^{\nu_2}\,\frac{q_j^{2\alpha_i+1}-1}{q_j-1}
\end{aligned} \tag{2.63}$$

are satisfied, as the set of prime divisors of the repunits would be identical if $2\alpha_i+1 \Big| \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ in the first relation, $2\alpha_j + 1 \Big| \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ in the second condition and $2\alpha_i + 1 \Big| \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, $2\alpha_j + 1 \Big| \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ in the third relation. The conditions with $\nu = \nu_1 = \nu_2 = 1$ follow if different powers of the other primes do not arise. However, $q_i^p - 1 \equiv 0 \ (mod\ p^\nu)$ only if $q_i^p - 1 \equiv 0 \ (mod\ p)$, and since $q_i^p - q \equiv 0 \ (mod\ p)$, this is possible when $p|(q_i - 1)$. If $p|(q_i - 1)$, then $\frac{q_i^p - 1}{q_i - 1} \equiv p$ and $\frac{q_i^p - 1}{q_i - 1} \not\equiv 0 \ (mod\ p^\nu)$, $\nu \geq 2$. Setting $\nu = \nu_1 = \nu_2$ gives

$$\begin{aligned} \frac{q_i^{2\alpha_i+1} - 1}{q_i - 1} &= (2\alpha_i + 1)\frac{q_j^{2\alpha_j+1} - 1}{q_j - 1} \\ (2\alpha_j + 1)\frac{q_i^{2\alpha_i+1} - 1}{q_i - 1} &= \frac{q_j^{2\alpha_j+1} - 1}{q_j - 1} \\ (2\alpha_j + 1)\frac{q_i^{2\alpha_i+1} - 1}{q_i - 1} &= (2\alpha_i + 1)\frac{q_j^{2\alpha_i+1} - 1}{q_j - 1} \end{aligned} \tag{2.64}$$

with $2\alpha_i + 1 \not| \frac{q_j^{2\alpha_j+1} - 1}{q_j - 1}$ and $2\alpha_j + 1 \not| \frac{q_i^{2\alpha_i+1} - 1}{q_i - 1}$. It follows that either $2\alpha_i + 1$ or $2\alpha_j + 1$ is a prime which divides only one of the repunits.

Furthermore, $q_i - 1$ and $q_j - 1$ are integers which are not rescaled, such that distinct prime divisors arise in the products of the remaining linear factors. If the exponents $2\alpha_i + 1$ and $2\alpha_j + 1$ are unequal, these factors do not divide both repunits. When the exponents $2\alpha_i + 1$ and $2\alpha_j + 1$ are equal, it would be necessary for n_{ij} to be equal to one, which is not feasible since $q_i \neq q_j$. It follows also that $\frac{q_i^{2\alpha_i+1} - 1}{q_i - 1}$ and $\frac{q_j^{2\alpha_j+1} - 1}{q_j - 1}$ cannot be equal, and the original assumption of their inequality is valid. However, by the first two relations in Eq.(2.55), there exists a prime which is not a divisor of both repunits.

■

The non-existence of odd integers satisfying $\sigma(N) = 2N$ is related to the number of prime divisors in the square root expression $\sqrt{2(4k + 1)\sigma(N)}$. A proof of the excess of prime divisors of $\sigma(N)$ would begin with the introduction of a new factor for each repunit.

Example 2.5.

An example of a congruence with more than one solution less than P is

$$\begin{aligned} (2\alpha_i + 1) + \alpha_i(2\alpha_i + 1)x + \frac{(2\alpha_i + 1)2\alpha_i(2\alpha_i - 1)}{3!}x^2 + \frac{(2\alpha_i + 1)2\alpha_i(2\alpha_i - 1)(2\alpha_i - 2)}{4!}x^3 \\ + \frac{(2\alpha_i + 1)2\alpha_i(2\alpha_i - 1)(2\alpha_i - 2)(2\alpha_i - 3)}{5!}x^4 = 0 \ (mod\ 11) \end{aligned} \tag{2.65}$$

which is solved by $x = 2,\ 4$ when $2\alpha_i + 1 = 5$. This is consistent with the inequality (2.32), since $\epsilon = 0.074897796$ when $x_0 = 2$. Consequently, $\frac{3^5 - 1}{2} = 11 \cdot 11$ and $\frac{5^5 - 1}{4} = 11 \cdot 71$ both have the divisor 11 and the distinct prime divisor arises in the larger repunit. Indeed, $\frac{3^5-1}{2}$ would not have a different prime factor from a repunit $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ that has 11 as a divisor. A set of primes $\{3, q_k, ...\}$ and exponents $\{5, 2\alpha_k, ...\}$ does not represent an exception to the Theorem 1 if $\frac{3^5-1}{2}$ is chosen to be the first repunit in the product, introducing the prime divisor 11. Each subsequent repunit then would have a distinct prime factor from the previous repunit in the sequence.

3. Formulation of Conditions on the Factors of Integers having the Form of an Odd Perfect Number

Let $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} q_i^{2\alpha_i}$ [18] and the coefficients $\{a_i\}$ and $\{b_i\}$ be defined by

$$a_i \frac{q_i^{2\alpha_i+1}-1}{q_i-1} = b_i \frac{(4k+1)^{4m+2}-1}{4k} \qquad gcd(a_i, b_i) = 1. \tag{3.1}$$

Bounds on the number of solutions to these equations for given a, b, q_i and q_j have been derived [45]. If $\frac{\sigma(N)}{N} \neq 2$,

$$\begin{aligned} &\sqrt{2(4k+1)} \left[\frac{q_1^{2\alpha_1+1}-1}{q_1-1} \frac{q_2^{2\alpha_2+1}-1}{q_2-1} ... \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \frac{(4k+1)^{4m+2}-1}{4k} \right]^{\frac{1}{2}} \\ &\qquad = \sqrt{2(4k+1)} \frac{(b_1...b_\ell)^{\frac{1}{2}}}{(a_1...a_\ell)^{\frac{1}{2}}} \cdot \left(\frac{(4k+1)^{4m+2}-1}{4k} \right)^{\frac{(\ell+1)}{2}} \\ &\qquad \neq 2(4k+1)^{2m+1} \prod_{i=1}^{\ell} q_i^{\alpha_i} \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} \frac{b_1...b_\ell}{a_1...a_\ell} &= \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \left(\frac{4k}{(4k+1)^{4m+2}-1} \right)^{\ell} \\ &\neq 2(4k+1)^{4m+1} \left[\frac{4k}{(4k+1)^{4m+2}-1} \right]^{\ell+1} \prod_{i=1}^{\ell-1} q_i^{2\alpha_i} \cdot \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}. \end{aligned} \tag{3.3}$$

When $\ell > 5$ is odd, there exists an odd integer ℓ_o and an even integer ℓ_e such that $\ell = 3\ell_0 + 2\ell_e$, so that

$$\begin{aligned} \frac{b_1...b_\ell}{a_1...a_\ell} = &\left(\frac{b_{13}}{a_{13}} \frac{a_2}{b_2} \right) \left(\frac{b_{46}}{a_{46}} \frac{a_5}{b_5} \right) ... \left(\frac{b_{3\ell_o-2,3\ell_o}}{a_{3\ell_o-2,3\ell_o}} \frac{a_{3\ell_o-1}}{b_{3\ell_o-1}} \right) \left(\frac{b_{3\ell_o+1} b_{3\ell_o+2}}{a_{3\ell_o+1} a_{3\ell_o+2}} \right) \\ &... \left(\frac{b_{\ell-1} b_\ell}{a_{\ell-1} a_\ell} \right) \cdot \frac{s^2}{t^2} \end{aligned} \tag{3.4}$$

with $s,\ t \in \mathbb{Z}$. It has been proven that $\frac{b_{3\bar{i}-2,3\bar{i}}}{a_{3\bar{i}-2,3\bar{i}}}\frac{a_{3\bar{i}-1}}{b_{3\bar{i}-1}} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ for any choice of $a_{3\bar{i}-2}, a_{3\bar{i}-1}, a_{3\bar{i}}$ and $b_{3\bar{i}-2}, b_{3\bar{i}-1}, b_{3\bar{i}}$ consistent with Eq.(3.1) [12]. Similarly, $\frac{b_\ell}{a_\ell} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ so that

$$\begin{aligned}\frac{b_{3\bar{i}-2,3\bar{i}}}{a_{3\bar{i}-2,3\bar{i}}}\frac{a_{3\bar{i}-1}}{b_{3\bar{i}-1}} &\equiv 2(4k+1)\frac{\bar{\rho}_{3\bar{i}-2}}{\bar{\chi}_{3\bar{i}-2}}\cdot\frac{s^2}{t^2}\\ \frac{b_{3\bar{j}+1}b_{3\bar{j}+2}}{a_{3\bar{j}+1}a_{3\bar{j}+2}} &= 2(4k+1)\frac{\hat{\rho}_{3\bar{j}+2}}{\hat{\chi}_{3\bar{j}+2}}\cdot\frac{s^2}{t^2}\end{aligned} \tag{3.5}$$

where the fractions are square-free and $gcd(\bar{\rho}_{3\bar{i}-2}, \bar{\chi}_{3\bar{i}-2}) = 1$, $gcd(\hat{\rho}_{3\bar{j}+2}, \hat{\chi}_{3\bar{j}+2}) = 1$. * Then,

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\cdot\frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{3\ell_o-2}}{\bar{\chi}_{3\ell_o-2}}\frac{\hat{\rho}_{\ell-2\ell_e+2,2}}{\hat{\chi}_{\ell-2\ell_e+2,2}}...\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\cdot\frac{s^2}{t^2}. \tag{3.6}$$

When $\ell = 2\ell_o + 3\ell_e > 4$ for odd ℓ_0 and even ℓ_e,

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\left(\frac{4k+1)^{4m+2}-1}{4k}\right)\cdot\frac{\hat{\rho}_{22}}{\hat{\chi}_{22}}...\frac{\hat{\rho}_{2\ell_o,2}}{\hat{\chi}_{2\ell_o,2}}\frac{\bar{\rho}_{\ell-3\ell_e+1}}{\bar{\chi}_{\ell-3\ell_e+1}}...\frac{\bar{\rho}_{\ell-2}}{\bar{\chi}_{\ell-2}}\cdot\frac{s^2}{t^2}. \tag{3.7}$$

If the products

$$\prod_{\bar{i}=1}^{\ell_o}\left(\frac{\bar{\rho}_{3\bar{i}-2}}{\bar{\chi}_{3\bar{i}-2}}\right)\prod_{\bar{j}=1}^{\ell_e}\left(\frac{\hat{\rho}_{3\ell_o+2\bar{j},2}}{\hat{\chi}_{3\ell_o+2\bar{j},2}}\right) \tag{3.8}$$

for odd ℓ and

$$\prod_{\bar{i}=1}^{\ell_o}\left(\frac{\hat{\rho}_{2\bar{i},2}}{\hat{\chi}_{2\bar{i},2}}\right)\prod_{j=1}^{\ell_e-1}\left(\frac{\bar{\rho}_{2\ell_o+3\bar{j}+1}}{\bar{\chi}_{2\ell_o+3\bar{j}+1}}\right) \tag{3.9}$$

for even ℓ are not the squares of rational numbers,

$$\begin{aligned}&\frac{b_1...b_\ell}{a_1...a_\ell} \neq 2(4k+1)\cdot\frac{s^2}{t^2} && \ell\ is\ odd\\ &\frac{b_1...b_\ell}{a_1...a_\ell} \neq 2(4k+1)\left(\frac{4k+1)^{4m+2}-1}{4k}\right)\cdot\frac{s^2}{t^2} && \ell\ is\ even\end{aligned} \tag{3.10}$$

which would imply the inequality (3.3) and the non-existence of an odd perfect number.

Conditions on the factors of N have been given [19]. A bound for the odd exponent of the prime $4k+1$ in the product representation of this integer will be derived.

Lemma 3.1. The exponent $4m+1$ in the prime factorization must be greater than or equal to 5 for the integer $N = (4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$ to be an odd perfect number.

---

* The notation has been changed from that of reference [12] with a different choice of index for $\bar{\rho}$, $\bar{\chi}$.

Proof. Let $I(n) = \frac{\sigma(n)}{n}$ be the abundance of an odd integer. The equality of $\sigma(N)$ with $2N$ requires the following limits [14]

$$\begin{aligned}
I\left((4k+1)^{4m+1}\right) &= \frac{1}{(4k+1)^{4m+1}} \frac{(4k+1)^{4m+2}-1}{4k} < \frac{4k+1}{4k} \\
I\left(\prod_{i=1}^{\ell} q_i^{2\alpha_i}\right) &> \frac{8k}{4k+1} \\
\frac{3(4k+1)^2 - 4(4k+1)+2}{4k(4k+1)} &< I\left((4k+1)^{4m+1}\right) + I\left(\prod_{i=1}^{\ell} q_i^{2\alpha_i}\right) \\
&\leq \frac{3(4k+1)^2+(4k+1)+1}{(4k+1)(4k+2)}.
\end{aligned} \tag{3.11}$$

and

$$I\left((4k+1)^{4m+1}\right) \leq \frac{3(4k+1)^2+(4k+1)+1}{(4k+1)(4k+2)} - \frac{8k}{4k+1} = \frac{(4k+1)^2+(4k+1)+3}{(4k+1)^2+(4k+1)}. \tag{3.12}$$

Now consider the abundance of the prime factor $(4k+1)^{4m+1}$ when $m = 0$. It follows that

$$I(4k+1) = \frac{4k+2}{4k+1} \leq \frac{(4k+1)^2+(4k+1)+3}{(4k+1)^2+(4k+1)} \tag{3.13}$$

or

$$(4k+1)^2 - 2(4k+1) \leq 0 \tag{3.14}$$

which has a solution when $0 \leq 4k+1 \leq 2$. Therefore, $4k+1$ can only occur in a square-free factor of N if $k = 0$. Then $4m+1 = 1$ may be an exponent only of 1, which yields a trivial factor. The prime factorization of N would include $(4k+1)^{4m+1}$ with $4k+1 \geq 5$ and $4m+1 \geq 5$.

∎

The necessity of exactly one odd power of a prime in the factorization of an odd perfect number N results from $\sigma(q_i^{2\alpha_i})$ being an odd integer for all odd primes q_i and even exponents $2\alpha_i$ and the divisibility of $\sigma(N)$ by 4 when N is properly divisible by a minimum of two odd powers of primes. The occurrence of the odd power $(4k+1)^{4m+1}$ follows from the divisibility of $\sigma(N)$ by 4 if the base or the exponent are congruent to 3 *mod* 4 [18]. A recent proof of the requirement of $m = 0$ for the existence of an odd perfect number $(4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$ proceeds from the absence of any integer solutions to $\sigma(p^r m^2) = 2p^r m^2$ for any prime $p = 4k+1$, with $gcd(p,m) = 1$, and $r > 1$ [46]. The combination of Lemma 3 and this last result would be sufficient to demonstrate that there exist no odd perfect numbers with this prime decomposition. Therefore, together with the theorem of Euler, their nonexistence may be deduced.

4. Examples of Integers satisfying the Rationality Condition

A set of odd integers $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} \; q_i^{2\alpha_i}$ with $4m+1 \geq 5$, such that

$$\left[2(4k+1) \frac{q_1^{2\alpha_1+1} - 1}{q_1 - 1} ... \frac{q_\ell^{2\alpha_\ell+1} - 1}{q_\ell - 1} \frac{(4k+1)^{4m+2} - 1}{4k} \right]^{\frac{1}{2}} \tag{4.1}$$

is rational, is given in the following list:

$$\begin{aligned}
&37^5 3^2 5^2 29^2 79^2 83^2 137^2 283^2 313^2 \\
&37^5 3^2 29^2 67^2 79^2 83^2 137^2 283^2 \\
&37^5 3^2 7^2 11^2 29^2 79^2 83^2 137^2 191^2 283^2 \\
&37^5 3^2 5^2 11^2 13^2 29^2 47^2 79^2 313^2 \\
&37^5 11^2 29^2 79^2 211^2 313^2 \\
&37^5 13^2 29^2 47^2 79^2 83^2 137^2 211^2 283^2 313^2 \\
&37^5 5^2 7^2 11^2 13^2 29^2 47^2 79^2 83^2 137^2 191^2 211^2 283^2 \\
&37^5 3^2 11^2 13^2 29^2 47^2 67^2 79^2 \\
&37^5 3^2 7^2 13^2 29^2 47^2 79^2 191^2 313^2 \\
&37^5 7^2 11^2 13^2 29^2 47^2 67^2 79^2 83^2 137^2 191^2 211^2 283^2 313^2 \\
&37^5 5^2 11^2 29^2 67^2 79^2 211^2 \\
&37^5 5^2 7^2 13^2 29^2 47^2 79^2 191^2 211^2 313^2 .
\end{aligned} \tag{4.2}$$

None of these integers satisfy the condition $\frac{\sigma(N)}{N} = 2$. For example, the sum of divisors of the first integer is

$$\sigma(37^5 3^2 5^2 29^2 79^2 83^2 137^2 283^2 313^2) = 2 \cdot 37 \cdot 3^4 \cdot 7^4 \cdot 13^2 \cdot 19^2 \cdot 31^2 \cdot 43^2 \cdot 67^2 \cdot 73^2 \cdot 181^2 \cdot 367^2 \tag{4.3}$$

such that new prime factors $7, 13, 19, 31, 43, 67, 73, 181, 367$ are introduced. The inclusion of these prime factors in the integer N leads to yet additional primes, and then the lack of closure of the set of prime factors renders it impossible for them to be paired to give even powers in $\sigma(N)$ with the exception of $2(4k+1)$.

It may be noted that the decompositions of repunits with prime bases of comparable magnitude and exponent 6 include factors that are too large and cannot be matched easily with factors of other repunits. It also can be established that repunits with prime bases and other exponents do not have square-free factors that can be easily matched to provide a closed sequence of such pairings. This can be ascertained from the partial list

$$\begin{aligned}
U_5(6,5) &= \frac{5^5-1}{4} = 781 \\
U_6(6,5) &= 3906 = 2\cdot 7\cdot 31\cdot 3^2 \\
U_5(8,7) &= 2801 \\
U_5(12,11) &= 16105 \\
U_5(14,13) &= 30941 \\
U_5(18,17) &= 88741 \\
U_7(4,3) &= 1093 \\
U_7(6,5) &= 19531 \\
U_9(4,3) &= 9841 \\
U_3(32,31) &= 993 = 3\cdot 331.
\end{aligned} \tag{4.4}$$

The entire set of integers satisfying the rationality condition is therefore restricted to a ring of prime bases and powers and exponents $4m+1=5$, $2\alpha_i = 1$ with a given set of prime factors occurring in the sum-of-divisors function.

The first prime of the form $4k+1$ which arises as a coefficient D in the equality

$$\frac{q^3-1}{q-1} = Dy^2 \qquad\qquad q\ prime \tag{4.5}$$

is 3541. This prime is too large to be the basis for the factor $\frac{(4k+1)^{4m+2}-1}{4k}$, since it would give rise to many other unmatched prime factors in the product of repunits and the rationality condition would not be satisfied. Amongst the coefficients D which are composite, the least integer with a prime divisor of the form $4k+1$ is 183, obtained when $q = 13$. However, the repunit with base 61 still gives rise to factors which cannot be matched since

$$\frac{61^6-1}{60} = 858672906 = 2\cdot 3\cdot 7\cdot 13\cdot 31\cdot 97\cdot 523. \tag{4.6}$$

Therefore, the rationality condition provides confirmation of the nonexistence of odd perfect numbers, which, however, can be proven with certainty only through the methods described in this work.

5. On the Non-Existence of Coefficients of Repunits satisfying the Perfect Number Condition

Based on Theorem 2.2, it is proven that additional prime divisors arise in the sum-of-divisors function and a relation equivalent to the perfect number condition cannot be satisfied by the primes $4k+1$ and $q_i,\ i = 1, ..., \ell$.

Theorem 5.1. There does not exist any set of odd primes $\{4k+1; q_1, ..., q_\ell\}$ such that there are coefficients $\{a_i\}$ and $\{b_i\}$ with

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\left[\frac{4k}{(4k+1)^{4m+2}-1}\right]^{\ell+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$$

if each pair of repunits in the sum-of-divisors function, $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, satisfies the inequalities

$$\begin{aligned}
\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &\neq (2\alpha_i+1)\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\\
(2\alpha_j+1)\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &\neq \frac{q_j^{2\alpha_j+1}-1}{q_i-1}\\
(2\alpha_j+1)\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &\neq (2\alpha_i+1)\frac{q_j^{2\alpha_i+1}-1}{q_j-1}.
\end{aligned}$$

Proof. It has been observed that $\frac{b_1}{a_1} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ by the non-existence of multiply perfect numbers with less than four prime factors [9][10],

$$\frac{b_1 b_2}{a_1 a_2} \neq 2(4k+1)\left[\frac{4k}{(4k+1)^{4m+2}-1}\right]^3 q_1^{2\alpha_1} q_2^{2\alpha_2} \tag{5.1}$$

by the non-existence of perfect numbers with three prime divisors [10], and proven that $\frac{b_1 b_2 b_3}{a_1 a_2 a_3} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ so that the inequality is valid for $\ell = 1, 2, 3$ [12].

Suppose that there are no odd integers N of the form $(4k+1)^{4m+1}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i}$ with $\frac{\sigma(N)}{N} = 2$ so that

$$\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} \neq 2(4k+1)\left[\frac{4k}{(4k+1)^{4m+2}-1}\right]^{\ell}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i}. \tag{5.2}$$

If there exists a perfect number with prime factors $\{4k+1, q_1, ..., q_\ell\}$, then $\frac{b_1...b_\ell}{a_1...a_\ell}$ must have $1+\ell+\tau\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) - \tau(U_{4m+2}(4k+2,4k+1), \prod_{i=1} q_i)$ distinct prime factors, where $U_{4m+2}(4k+2,4k+1)$ is the Lucas number $\frac{(4k+1)^{4m+2}-1}{4k}$ and $\tau(U_{4m+2}(4k+2,4k+1), \prod_{i=1} q_i)$ denotes the number of common divisors of the two integers. However, equality $\sigma(N)$ and 2N also implies that $\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ has at least $\ell+2-\tau\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)$ different prime divisors and a maximum of $\ell+1$ prime factors. If there is no cancellation between $\frac{4k}{(4k+1)^{4m+2}-1}$ and $\prod_{i=1}^{\ell} q_i^{2\alpha_i}$, multiplication of $\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1,q_i) = \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ by $U_{4m+2}(4k+2,4k+1) = \left(\frac{4k}{(4k+1)^{4m+2}-1}\right)^{\ell}$ introduces $\tau\left(\frac{4k}{(4k+1)^{4m+2}-1}\right)$ new prime divisors and $\frac{b_1...b_\ell}{a_1...a_\ell}$ would have $\ell+2$ distinct prime factors. However, if $gcd(U_{4m+2}(4k+2,4k+1), U_{2\alpha_i+1}(q_i+1,q_i)) = 1$ for all i, the repunit $U_{4m+2}(4k+2,4k+1)$ must introduce additional prime divisors. It follows that

$$\frac{(4k+1)^{4m+2}-1}{8k} = \prod_{\bar{i}\in I} q_i^{2\alpha_i} \tag{5.3}$$

would be required for some integer set I. There are no positive integer solutions to

$$\frac{x^n-1}{x-1} = 2y^2 \qquad x \equiv 1\ (mod\ 4),\ n \equiv 2\ (mod\ 4),\ n \geq 6. \tag{5.4}$$

Then $\frac{x^{2m+1}-1}{x-1} = y_1^2 z$ and $\frac{x^{2m+1}+1}{2} = y_2^2 z'$, where $zz' = y_3^2$ and $y = y_1y_2y_3$. Since $y_3^2 = z^2\frac{z'}{z}$, $\frac{z'}{z} = \hat{z}^2$ for an integer $\hat{z}$ and $\frac{x^{2m+1}-1}{x-1} \cdot \frac{2}{x^{2m+1}+1} = \frac{y_1^2 z}{y_2^2 z'} = \frac{y_1^2}{y_2^2}\frac{1}{\hat{z}^2}$, Then $\frac{y_1}{y_2\hat{z}} = \left(\frac{x^{2m+1}-1}{x^{2m+1}+1}\right)^{\frac{1}{2}}\left(\frac{2}{x-1}\right)^{\frac{1}{2}}$. The factors of $x-1$ do not cancel with all of the prime divisors of $x^{2m+1}-1$, while $\frac{x^{2m+1}-1}{2}$ and $\frac{x^{2m+1}+1}{2}$ have no common divisors and both cannot be squares of integers for $x \geq 2$ and $m \geq 1$. Finally, $\left(\frac{x^{2m+1}-1}{x-1}\right)^{\frac{1}{2}}$ is rational only for $x = 3$ and $2m+1 = 5$, while $\left(\frac{3^5+1}{2}\right)^{\frac{1}{2}}$ is not rational, and relation for y_1, y_2 and $\hat{z}$ has no integer solution for the given values of x and m. There is no odd integer $4k+1 \geq 5$ satisfying the condition in Eq.(5.3) for $m \geq 1$. A variant of this proof has been obtained by demonstrating the irrationality of $\left[\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \left(\frac{8k(4k+1)}{(4k+1)^{4m+2}-1}\right)\right]^{\frac{1}{2}}$ when $gcd(U_{2\alpha+1}(q_i+1,q_i), U_{4m+2}(4k+2,4k+1)) = 1$ [12].

When the number of prime factors of $\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ is less than $\ell+1$, there must be at least two divisors of $U_{4m+2}(4k+2,4k+1)$ which do not arise in the decomposition of this product. While one of the factors is 2, $4k+1$ is not a divisor, implying that any other divisor must be $q_{\bar{j}}$ for some $\bar{j}$. It has been assumed that there are no prime sets

$\{4k+1; q_1, ..., q_{\ell-1}\}$ satisfying the perfect number condition. Either $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$ does not have $\ell+1$ factors, or if it does have $\ell+1$ factors, then $\prod_{i=1}^{\ell-1} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ has $\ell-1$ factors but

$$\prod_{i=1}^{\ell-1} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq (4k+1)^{4m+1} \prod_{i\neq\bar{i},\bar{j}} q_i^{2\alpha_i - t_i} \tag{5.5}$$

for some prime $q_{\bar{i}}$, where $q_i^{t_i} \parallel U_{4m+2}(4k+2, 4k+1)$. Multiplication by $U_{2\alpha_\ell+1}(q_\ell+1, q_\ell)$ must contain the factor $q_{\bar{i}}^{2\alpha_{\bar{i}}}$, because $\frac{(4k+1)^{4m+2}-1}{4k}$ only would introduce the two primes 2, $q_{\bar{j}}$ and not $q_{\bar{i}}$, since $\prod_{i=1}^{\ell-1} U_{2\alpha_i+1}(q_i+1, q_i)$ contains $\ell-1$ primes, including $4k+1$ and excluding $q_{\bar{i}}$. Interchanging the roles of the primes in the set $\{q_i, i=1, ..., \ell\}$, it follows that the prime equations

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (4k+1)^{h_i} q_{j_i}^{2\alpha_{j_i}} \\ \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} &= (4k+1)^{h_\ell} q_{\bar{i}}^{2\alpha_{\bar{i}}} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2q_{\bar{j}}^{2\alpha_{\bar{j}}} \end{aligned} \tag{5.6}$$

must hold, where $h_\ell \neq 0$. Since the second equation has no positive integer solution, $k \geq 1$ and $m \geq 1$ [12], it follows that $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$ must not have $\ell+1$ prime factors.

There cannot be less than $\ell+1$ different factors of $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$, as each new repunit $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ introduces at least one distinct prime divisor by Theorem 1 and the factorization of $\frac{(4k+1)^{4m+2}-1}{4k}$ contains at least two new primes. Consequently, this would imply $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$ has at least $\ell+2$ prime factors and $\frac{b_1...b_\ell}{a_1...a_\ell}$ has a minimum of $\ell+3$ prime divisors, which is larger than the number necessary for equality between $U_{4m+2}(4k+2, 4k+1) \prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)$ and $2(4k+1)\prod_{i=1}^{\ell} q_i^{2\alpha_i}$.

If the maximum number of prime factors, $\ell+1$, is attained for $\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)$, then the only additional prime divisor arising from multiplication with $U_{4m+2}(4k+2, 4k+1)$ is 2. However, since both 2 and $2k+1$ divide the repunit with base $4k+1$, this property does not hold unless $2k+1$ and prime factors of $\frac{1}{2k+1}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)$ can be included in the set $\{q_i, i=1..., \ell\}$. There are then a total of $\ell+2$ prime factors in $\frac{b_1...b_\ell}{a_1...a_\ell}$ only if $\left[\frac{8k}{(4k+1)^{4m+2}-1}\right]^{\ell+1}$ can be absorbed into $\prod_{i=1}^{\ell} q_i^{2\alpha_i}$.

The repunit then can be expressed as

$$\frac{(4k+1)^{4m+2}-1}{4k} = 2\prod_{j\in\{K\}} q_j^{t_j} \tag{5.7}$$

where $\{K\} \subseteq \{1, ..., q_\ell\}$. By the non-existence of odd perfect numbers with prime factors $4k+1, q_i, i = 1..., \ell-1$, either $U_{4m+2}(4k+2, 4k+1) \prod_{i=1}^{\ell-1} U_{2\alpha_i+1}(q_i+1, q_i)$ has less than $\ell+1$ factors or

$$\prod_{i=1}^{\ell-1} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq (4k+1)^{4m+1} \prod_{i \in \overline{\{K\}-\{q_{\bar{i}}\}}} q_i^{2\alpha_i} \prod_{\{K\}-\{q_{\bar{i}}\}} q_j^{2\alpha_j - t_j} \tag{5.8}$$

for some $\bar{i} \in \{1, ..., \ell\}$. If the product of repunits for the prime basis $\{4k+1, q_1, ..., q_{\ell-1}\}$ has less than $\ell+1$ factors, $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ introduces at least two new prime factors. Even if one of these divisors is $4k+1$, the other factor must be $q_{\bar{i}}$ for some $\bar{i} \neq \ell$. Interchange of the primes $q_i, i = 1, ..., \ell$ again yields the relations in equation (5.6). If the product (5.8) includes ℓ primes, and not $q_{\bar{i}}$, then $q_{\bar{i}}^{2\alpha_{\bar{i}}}$ must be a factor of $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$. Interchanging the primes, it follows that the product (5.8) includes $\prod_{i\neq\bar{i}} q_i^{2\alpha_i}$.

The prime divisors in $U_{2\alpha_\ell+1}(q_\ell+1, q_\ell)$ either can be labelled q_{j_ℓ} for some $j_\ell \in \{1, ..., \ell-1\}$ or equals $4k+1$. While $4k+1$ does not divide $U_{4m+2}(4k+2, 4k+1)$, it can occur in the other repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ for $i = 1, ..., \ell-1$. The prime $q_{j_\ell} = q_{\bar{i}}$ also may not be a divisor of $\frac{(4k+1)^{4m+2}-1}{4k}$.

One choice for q_ℓ is a prime divisor of $2k+1$. If $2k+1$ is prime, the factors of $\frac{(2k+1)^{2\alpha_\ell+1}-1}{2k}$ and $\frac{(4k+1)^{4m+2}-1}{4k}$ can be compared. Since the latter quotient is equal to $\frac{(4k+1)^{2m+1}-1}{4k} \cdot [(4k+1)^{2m+1}+1]$, consider setting m equal to α_ℓ. By Theorem 1, a divisor P divides both repunits with different powers if $4k = nP + 2k$ or $2k = nP$ implying $n = 2$, $P = k$. However, if $k \Big| \frac{(2k+1)^{2\alpha_\ell+1}-1}{2k}$, $1 + (2k+1) + ... + (2k+1)^{2\alpha_\ell} \equiv 2\alpha_\ell + 1 \equiv 0 \ (mod\ k)$ and $k = 2\alpha_\ell + 1$ for prime exponents $2\alpha_\ell + 1$, which would imply that P is an imprimitive divisor. Suppose $\frac{(2k+1)^k-1}{2k} = k \cdot \prod_i P_i^{m_i}$ and $\frac{(4k+1)^k-1}{4k} = k \cdot \prod_j \tilde{P}_j^{n_j}$. If a primitive divisor $rk+1$, $r \in \mathbf{Z}$, divides both repunits, $(2k+1)^k - 1 = (rk+1)(x-1)$, $x \in \mathbf{Z}$, $(4k+1)^k - 1 = (rk+1)(x'-1)$, $x' \in \mathbf{Z}$, so that

$$\frac{(2k+1)^k-1}{x-1} = \frac{(4k+1)^k-1}{x'-1}. \tag{5.9}$$

This relation would imply

$$x'(2k+1)^k - (2k+1)^k - x' = x(4k+1)^k - (4k+1)^k - x \tag{5.10}$$

and therefore $x - 1 = b(4k+1)^k$, $x' - 1 = a(4k+1)^k$. However, the equation is then

$$a(4k+1)^k(2k+1)^k - a(4k+1)^k \ = \ b(4k+1)^k(2k+1)^k - b(2k+1)^k \tag{5.11}$$

which cannot be satisfied by any integers a, b. Not all primitive divisors of the two repunits are identical. For the exception, $k = 1$, $2\alpha_\ell + 1 = 5$, $4m + 2 = 10n$, $\frac{(2k+1)^{2\alpha_\ell+1}-1}{2k} = 11^2$, $\frac{(4k+1)^{4m+2}-1}{4k}$ has other prime factors in addition to 11.

If $m \neq \alpha_\ell$, it can be demonstrated that there is a primitive divisor of $\frac{(2k+1)^{2\alpha_\ell+1}-1}{2k}$ which is not a factor of $\frac{(4k+1)^{4m+2}-1}{4k}$ by using the comparison of the linear factors in the decomposition of each numerator in Theorem 1, since $4k \neq (2k)^n,\ n \geq 2$ and $4k + 2 \neq (2k + 2)^n,\ n \geq 2$ for any $k > 1$.

In general, either there exists a prime factor of $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ which does not divide $\frac{(4k+1)^{4m+2}-1}{4k}$ or there are more than $\ell + 2$ primes in the decomposition of the product of repunits. The existence of a distinct prime divisor requires a separate demonstration for composite exponents. Let $p_1(4m + 2)$, $p_2(4m + 2)$ be two prime divisors of $4m + 2$. Then $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ has a prime factor different from the divisors of $\frac{(4k+1)^{p_1(4m+2)}-1}{4k}$ and $\frac{(4k+1)^{p_2(4m+2)}-1}{4k}$ by Theorem 1. While the union of the sets of prime divisors of the two repunits is contained in the factorization of $\frac{(4k+1)^{p_1(4m+2)p_2(4m+2)}-1}{4k}$, the repunit with the composite exponent will have a primitive divisor, which does not belong to the union of the two sets and equals $a_1p_1p_2+1$. If this prime divides $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$, $a_1p_1p_2+1 = a_2(2\alpha_\ell+1)+1$. Either $2\alpha_\ell + 1|4m + 2$ or the primitive divisor equals $a_{12}(2\alpha_\ell + 1)p_1p_2 + 1$. Suppose

$$(a_{12}(2\alpha_\ell + 1) + 1)(x + 1) = \frac{(4k + 1)^{p_1p_2} - 1}{4k} \equiv p_1p_2\ (mod\ 4). \tag{5.12}$$

Since $p_1,\ p_2$ are odd primes, either $a_{12} \equiv 0\ (mod\ 4)$ or $a_{12} \equiv 2\ (mod\ 4)$, so that the primitive divisor equals $4c(2\alpha_\ell + 1)p_1p_2 + 1$ when $p_1p_2 \equiv 1\ (mod\ 4)$ or $(4c + 2)(2\alpha_\ell + 1)p_1p_2 + 1$ if $p_1p_2 \equiv 3\ (mod\ 4)$. Let the primitive divisor P have the form $4c(2\alpha_\ell + 1)p_1p_2 + 1$ so that

$$(4c(2\alpha_\ell + 1)p_1p_2 + 1)(y + 1) = \frac{q_\ell^{2\alpha_\ell+1} - 1}{q_\ell - 1} \equiv 2\alpha_\ell + 1\ (mod\ q_\ell - 1) \tag{5.13}$$

which implies that $y + 1 = \kappa_2[2\alpha_\ell + \chi_2(q_\ell - 1)]$ is either the multiple of an imprimitive divisor, $\kappa_2'(2\alpha_\ell + 1)$ or it is the multiple of a primitive divisor. Consider the equation

$$(4c(2\alpha_\ell + 1)p_1p_2 + 1)\kappa_2(2\alpha_\ell + 1) = \frac{q_\ell^{2\alpha_\ell+1} - 1}{q_\ell - 1} \tag{5.14}$$

with $2\alpha_\ell|q_\ell - 1$. If

$$(4c(2\alpha_\ell + 1)p_1p_2 + 1)\kappa_2(2\alpha_\ell + 1) \equiv 2\alpha_\ell + 1\ (mod\ q_\ell - 1), \tag{5.15}$$

$q_\ell - 1|(4c(2\alpha_\ell+1)p_1p_2+1)\kappa_2 - 1$ or $(4c(2\alpha_\ell+1)p_1p_2+1)\kappa_2 - 1 \equiv \frac{q_\ell-1}{2\alpha_\ell-1}$ $(mod\ q_\ell - 1)$. Since $2\alpha_\ell+1|\kappa_2-1$, $\kappa_2 = \kappa_3(2\alpha_\ell+1)+1$. Let

$$(z(q_\ell-1)+1)(2\alpha_\ell+1) = \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}. \tag{5.16}$$

Then

$$z = 2\alpha_\ell + \frac{q_\ell-1}{2\alpha_\ell+1}(2\alpha_\ell - 1 + (2\alpha_\ell-2)(q_\ell+1) + ... + q_\ell^{2\alpha_\ell-2} + ... + 1) \tag{5.17}$$

is integer and $z(q_\ell-1) = 4c(2\alpha_\ell+1)^2\kappa_3p_1p_2 + \kappa_3(2\alpha_\ell+1) + 4c(2\alpha_\ell+1)p_1p_2$ and $2\alpha_\ell+1|\kappa_2-1$ so that $\kappa_2 = \kappa_3(2\alpha_\ell+1)+1$. Let

$$(z(q_\ell-1)+1)(2\alpha_\ell+1) = \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}. \tag{5.18}$$

Then

$$z = 2\alpha_\ell + \frac{q_\ell-1}{2\alpha_\ell+1}(2\alpha_\ell - 1 + (2\alpha_\ell-2)(q_\ell+1) + ... + q_\ell^{2\alpha_\ell-2} + ... + 1) \tag{5.19}$$

is integer and $z(q_\ell-1) = 4c(2\alpha_\ell+1)^2\kappa_3p_1p_2 + \kappa_3(2\alpha_\ell+1) + 4c(2\alpha_\ell+1)p_1p_2$

In the latter case, the product of the primitive divisors takes the form $b(2\alpha_\ell+1)+1$ so that $\kappa_2[(2\alpha_\ell+1)+\chi_2(q_\ell-1)] = b(2\alpha_\ell+1)+1$ since $2\alpha_\ell+1$ is prime. The two congruence relations

$$\begin{aligned} [(b-1)+(b(2\alpha_\ell+1)4cp_1p_2](2\alpha_\ell+1)+1 &\equiv 0\ (mod\ q_\ell-1) \\ (b-\kappa_2)(2\alpha_\ell+1)+1 &\equiv 0\ (mod\ q_\ell-1) \end{aligned} \tag{5.20}$$

imply

$$[\kappa_2 - 1 + (b(2\alpha_\ell+1)+1)4cp_1p_2](2\alpha_\ell+1) \equiv 0\ (mod\ q_\ell-1). \tag{5.21}$$

When $2\alpha_i+1 \not| \ q_\ell-1$, it follows that $q_\ell-1|\kappa_2-1+(b(2\alpha_\ell+1)+1)4cp_1p_2$. Since every primitive divisor is congruent to 1 modulo $2\alpha_\ell+1$, $\kappa_2 = \kappa_3(2\alpha_\ell+1)+1$ and $q_\ell-1|\kappa_3(2\alpha_\ell+1)+(b(2\alpha_\ell+1)+1)4cp_1p_2$.

The factorizations

$$\begin{aligned} (4k+1)^{4m+2}-1 &= \prod_{k=0}^{4m+1}((4k+1)-\omega_{4m+2}^k) \\ q_\ell^{2\alpha_\ell+1}-1 &= \prod_{k'=0}^{2\alpha_i}(q_\ell-\omega_{2\alpha_i+1}^{k'}) \end{aligned} \tag{5.22}$$

yield real factors which can be identified only if $((4k+1)-1)^{t_k} = (4k)^{t_k} = (q_\ell - 1)^{t'_k}$ or $(4k+2)^{t_k} = (q_\ell+1)^{t'_k}$. Then, $4k = (q_\ell-1)^{n_\ell}$, $t_k = n_\ell t'_k$ or $4k+2 = (q_\ell+1)^{n_\ell}$, $t_k = n_\ell t'_k$. Since $\sum_k \ t_k = 4m+2$, $\sum_k \ t'_k = 2\alpha_i + 1$, $4m+2 = n_\ell(2\alpha_\ell+1)$. If $4m+2$ is the product of two primes $p_1 p_2$, $2\alpha_\ell + 1$ must equal one of the primes, p_2.

The congruence (5.15) becomes

$$[(4c(2\alpha_\ell+1)^2 p_1 + 1)\kappa_2 - 1](2\alpha_\ell+1) \equiv 0 \ (mod \ q_\ell - 1). \tag{5.23}$$

If $4k = (q_\ell-1)^{p_1}$, then $\frac{(4k+1)^{p_1p_2}-1}{4k} = \frac{((q_\ell-1)^{p_1}+1)^{p_1p_2}-1}{(q_\ell-1)^{p_1}} = p_1p_2 + \binom{p_1p_2}{2}(q_\ell-1) + ... + p_1p_2(q_\ell-1)^{p_1(p_1p_2-2)} + (q_\ell-1)^{p_1(p_1p_2-1)}$ When $2\alpha_i+1|q_\ell-1$, $2\alpha_i+1|p_1p_2+\binom{p_1p_2}{2}(q_\ell-1) + ... + p_1p_2(q_\ell-1)^{p_1(p_1p_2-2)} + (q_\ell-1)^{p_1(p_1p_2-1)}$. However, $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \equiv 2\alpha_\ell+1 \ (mod \ q_\ell-1)$, whereas $\frac{(4k+1)^{p_1p_2}-1}{4k} \equiv p_1(2\alpha_\ell+1) \ (mod \ q_\ell-1)$, $p_1 > 1$. The repunits have the same prime divisors only if $p_1 \equiv 1 \ (mod \ q_\ell-1)$, which implies that $p_1 = \rho(q_\ell-1)+1$. Then $\frac{((q_\ell-1)^{\rho(q_\ell-1)}+1)^{\rho(q_\ell-1)(2\alpha_\ell+1)-1}}{(q_\ell-1)^{\rho(q_\ell-1)}}$ includes at least seven new factors. If $4k+2 = (q_\ell+1)^{p_1}$, then $\frac{(4k+1)^{p_1p_2}-1}{4k} \equiv \frac{(2^{p_1}-1)^{p_1p_2}-1}{2^{p_1}-2} \ (mod \ q_\ell-1)$. If $2^{p_1}-1 \equiv 1 \ (mod \ q_\ell-1)$, $1 + (2^{p_1}-1) + ... + (2^{p_1}-1)^{p_1p_2-1} \equiv p_1p_2 \equiv 2\alpha_\ell+1 \ (mod \ q_\ell-1)$ only if $p_1 \equiv 1 \ (mod \ q_\ell-1)$. When $2^{p_1}-1 \equiv 2n+1 \ (mod \ q_\ell-1)$, $\frac{(2^{p_1}-1)^{p_1p_2}-1}{2^{p_1}-2} \equiv \frac{(2n+1)^{p_1p_2}-1}{2n}$ which is not divisible by $2\alpha_i+1$ if $p_2 \not| \, 2n$. When $2\alpha_i+1|n$, $\frac{(2n)^{p_1p_2}-1}{2n} \equiv p_1p_2 \ (mod \ 2n)$ which would be congruent to $2\alpha_i+1$ only if $p_1 \equiv 1 \ (mod \ 2n)$ so that $p_1 \geq 4\alpha_i+3$. The exponent is a product of a minimum of three primes and more than six different divisors are introduced in the product of repunits. A similar conclusion is obtained if $4m+2 = p_1...p_s, \ s \geq 3$.

If every prime divisor of $\frac{(4k+1)^{4m+2}-1}{4k}$ is distinct from the factors of $\prod_{i=1}^{\ell} \ \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, the following equations are obtained

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (4k+1)^{h_i} q_{j_i}^{2\alpha_{j_i}} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2 \\ \sum_{i=1}^{\ell} \ h_i &= 4m+1 \end{aligned} \tag{5.24}$$

which only has the solution $k=0$.

The number of equal prime divisors in $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ and $\frac{(4k+1)^{4m+2}-1}{4k}$ can be chosen to be greater than 1, but equality of $\sigma(N)$ and $2N$ implies that each distinct prime divisor q_{j_i} of the repunit $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ appears in the factorization with the exponent $2\alpha_{j_i}$. Consequently, it would be inconsistent with the perfect number condition for divisors of $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ other than $q_{j_{i_0}}$ to exist.

If the prime divisor $q_{j_{i_0}}$ of $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ is a factor of $\frac{(4k+1)^{4m+2}-1}{4k}$, then one formulation of the perfect number condition for the integer $N=(4k+1)^{4m+1}q_i^{2\alpha_i}$ is

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (4k+1)^{h_i} q_{j_i}^{2\alpha_{j_i}} \qquad\qquad i \neq i_0 \\ \frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1} &= q_{j_{i_0}}^{h'_{j_{i_0}}} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2q_{i_0}^{2\alpha_{j_{i_0}}-h'_{j_{i_0}}}. \end{aligned} \tag{5.25}$$

The last relation in equation (5.25) has no solutions with $k \geq 1$ since $\frac{(4k+1)^{4m+2}-1}{4k}$ has a minimum o. f four different prime divisors, which include 2, $2k+1$, the primitive divisors of $\frac{(4k+1)^{2m+1}-1}{4k}$ and the other prime factors of $\frac{(4k+1)^{2m+1}+1}{2}$. Furthermore, if the number of prime divisors of $\frac{(4k+1)^{4m+2}-1}{4k}$ is less than seven, $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ also should have a different prime factor which is contrary to the equation (5.25).

By Theorem 2.2, there exists a primitive divisor which is not a common divisor of two repunits with different bases q_i and q_j. The new prime divisor may be denoted q_{j_ℓ} if it does not equal $4k+1$, and interchanging q_ℓ with $q_{\bar{i}}$ and $4k+1$, it can be deduced that $U_{4m+2}(4k+2, 4k+1)\prod_{\substack{i=1\\ i\neq \bar{i}}}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)$ will not be divisible by $j_{\bar{i}} \in 1, ..., \ell$, $j_{\bar{i}} \neq \bar{i}$, with the exception of one value i_0. The prime q_{j_0} will not be a factor of $\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)$. Then

$$\begin{aligned} \frac{q_{\bar{i}}^{2\alpha_{\bar{i}}+1}-1}{q_{\bar{i}}-1} &= q_{j_{\bar{i}}}^{h_{j_{bari}}} \qquad\qquad \bar{i} \neq i_0 \\ \frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1} &= (4k+1)^{4m+1} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2q_{j_{i_0}}^{h_{j_{i_0}}}. \end{aligned} \tag{5.26}$$

The last equation is equivalent to

$$\begin{aligned} \frac{(4k+1)^{2m+1}-1}{4k} &= y_1^2 \\ \frac{(4k+1)^{2m+1}+1}{2} &= y_2^2 \\ y_1 y_2 &= q_{j_{i_0}}^{\frac{h_{j_{i_0}}}{2}} \end{aligned} \tag{5.27}$$

which has no integer solutions for $k \geq 1$ and $m \geq 1$. There are no prime sets $\{4k+1; q_1, ..., q_\ell\}$ which satisfy these conditions with $h_{j_{\bar{i}}}$ and $h_{j_{i_0}}$ even.

Since $k \geq 1$ in the decomposition $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} q_i^{2\alpha_i}$, the quotient $\frac{(4k+1)^{2m+1}-1}{4k}$, $k \geq 1$ has a distinct prime divisor from the factors of $\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ because of its existence in the factorization of $\frac{(4k+1)^{p_1(2m+1)}-1}{4k}$ by Theorem 2.2. There would be then at least $\ell+3$ prime factors of $\sigma(N)$ implying that N is not an odd perfect number.

■

6. An Infinite Sequence of Increasing Prime Factors

Theorem 6.1. There are no odd perfect numbers if there exists a pair of repunits in the sum-of-divisor function satisfying one of the equations

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1) \cdot \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\ (2\alpha_j+1) \cdot \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= \frac{q_j^{2\alpha_j+1}-1}{q_i-1} \\ (2\alpha_j+1) \cdot \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1) \cdot \frac{q_j^{2\alpha_j+1}-1}{q_j-1}. \end{aligned}$$

Proof. The conditions on the quotients $\left\{\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right\}$ represent the complementary case to the constraints in Theorem 5.1. The exponents $2\alpha_i+1$, $i = 1, ..., \ell$ can be selected to be prime, since repunits with composite exponents have a minimum of three prime divisors [12].

The third constraint cannot be satisfied if $\alpha_i = \alpha_j$, with $q_i \neq q_j$, and a new prime divisor occurs in one of the repunits derived from the other two constraints equations or $\alpha_i \neq \alpha_j$ in the third condition.

Imprimitive prime divisors can be introduced into equations generically relating the two unequal repunits and minimizing the number of prime factors in the product $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Given an odd integer $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} q_i^{2\alpha_i}$, by Theorem 2.2, the least number of unmatched prime divisors in $\sigma(N)$ will be attained if there are pairs (q_i, q_j) satisfying one of the three relations

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)\cdot\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\ (2\alpha_j+1)\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\ (2\alpha_j+1)\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)\cdot\frac{q_j^{2\alpha_j+1}-1}{q_j-1}. \end{aligned} \tag{6.1}$$

As $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ would not introduce any additional prime divisors if the first relation in equation (6.1) is satisfied, the product of two pairs of repunits of this kind, with prime bases (q_i, q_j), $(q_k, q_{k'})$, yields a minimum of three distinct prime factors, if two of the repunits are prime powers. In the second relation, $(q_i, 2\alpha_i+1)$ and $(q_j, 2\alpha_j+1)$ are interchanged, whereas in the third relation, an additional prime divisor is introduced when $2\alpha_i+1 \neq 2\alpha_j+1$. Equality of $\sigma(N)$ and 2N is possible only when the prime divisors of the product of repunits also arise in the decomposition of N. If the two repunits $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}$ are prime, they will be bases for new repunits

$$\begin{aligned} \frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} &= \frac{q_j-1}{q_j(q_j^{2\alpha_j}-1)}\cdot\left[\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1\right] \\ \frac{\left[\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}\right]^{2\alpha_{n_2}+1}-1}{\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}-1} &= \frac{q_{k'}-1}{q_{k'}(q_{k'}^{2\alpha_{k'}}-1)}\cdot\left[\left[\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}\right]^{2\alpha_{n_2}+1}-1\right]. \end{aligned} \tag{6.2}$$

These new quotients are either additional prime powers or satisfy relations of the form

$$\frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} = (2\alpha_{n_1}+1)\frac{q_{t_1}^{2\alpha_{t_1}+1}-1}{q_{t_1}-1} \tag{6.3}$$

to minimize the introduction of new primes. Furthermore, $2\alpha_{n_1}+1$ is the factor of lesser magnitude,

$$\frac{q_t^{2\alpha_t+1}-1}{q_t-1} > \left[\frac{q_j-1}{q_j(q_j^{2\alpha_j}-1)}\cdot\left[\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1\right]\right]^{\frac{1}{2}}. \tag{6.4}$$

The number of prime divisors of N is minimized if the repunit $\frac{q_t^{2\alpha_t+1}-1}{q_t-1}$ is a prime power. However, if it is prime, it is then the basis of another repunit

$$\frac{\left[\frac{q_t^{2\alpha_t+1}-1}{q_t-1}\right]^{2\alpha_{n_3}+1}-1}{\left[\frac{q_t^{2\alpha_t+1}-1}{q_t-1}\right]-1} > \frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} \tag{6.5}$$

since $\alpha_{n_3} \geq 1$. An infinite sequence of repunits of increasing magnitude is generated, implying the non-existence of odd integers N with prime factors satisfying one of the relations in equation (6.1).

If the third relation in equation (6.1) holds,

$$(2\alpha_t+1)\frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} = (2\alpha_{n_1}+1)\frac{q_{t_1}^{2\alpha_{t_1}+1}-1}{q_{t_1}-1}. \tag{6.6}$$

The inequality (6.4) is still satisfied, and again, repunits of increasing magnitude are introduced in the sum of divisors.

Suppose that one of the repunits in (6.1) is a prime power, with an exponent greater than or equal to 2, rather than a prime. When the first of the relations is satisfied,

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)q'^{h_j} \\ \frac{q_j^{2\alpha_j+1}-1}{q_j-1} &= q'^{h_j} \qquad\qquad h_j \geq 2. \end{aligned} \tag{6.7}$$

Then

$$\begin{aligned} q' &= \left(\frac{1+...+q_i^{2\alpha_i}}{2\alpha_i+1}\right)^{\frac{1}{h_j}} \\ &= \left(1+...+q_j^{2\alpha_j}\right)^{\frac{1}{h_j}} \end{aligned} \tag{6.8}$$

which yield the approximations

$$\begin{aligned} q' &\approx \frac{q_i^{\frac{2\alpha_i}{h_j}}}{(2\alpha_i+1)^{\frac{1}{h_j}}}\left[1+\frac{(q_i^{2\alpha_i}-1)}{q_i^{2\alpha_i}\cdot h_j(q_i-1)}+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n\cdot h_j}\frac{1}{(q_i-1)^n}\right] \\ q' &\approx q_j^{\frac{2\alpha_j}{h_j}}\left[1+\frac{(q_j^{2\alpha_j}-1)}{q_j^{2\alpha_j}h_j(q_j-1)}+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n\cdot h_j}\frac{1}{(q_j-1)^n}\right]. \end{aligned} \tag{6.9}$$

Rationality of the two series in Eq.(6.9) cannot be attained because h_j does not cancel both primes q_i and q_j. Therefore, Eq. (6.7) will not have odd prime solutions for $q_i,\ q_j,\ q',\ 2\alpha_i+1$ and $2\alpha_j+1$. A similar conclusion is found for the other two relations in Eq. (6.1).

If the repunits $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_{\ell'}^{2\alpha_{\ell'}+1}-1}{q_{\ell'}-1}$ are not prime powers, then the product of the four repunits with bases $q_i, q_j, q_k, q_{k'}$ would have a minimum of five different prime divisors. The product of $\frac{(4k+1)^{4m+2}-1}{4k}$, with at least two distinct prime divisors, and $\prod_{i=1}^{\ell}\ \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ possesses a minimum of $\ell+3$ different prime factors and the perfect number condition cannot be satisfied.

■

7. Further Ramifications of the Odd Perfect Number Conjecture

Dividing the constraint $\sigma(N)=2N$ by N yields the relation

$$1+\frac{1}{4k+1}+\frac{1}{q_1}+...+\frac{1}{q_\ell}+\frac{1}{(4k+1)^2}+\frac{1}{q_1^2}+...+\frac{1}{q_\ell^2}+...+\frac{1}{N}=2 \tag{7.1}$$

or equivalently

$$\begin{aligned}&\frac{1}{4k+1}+\frac{1}{q_1}+...+\frac{1}{(4k+1)^2}+\frac{1}{q_1}+...+\frac{1}{q_\ell}+\frac{1}{(4k+1)^2}+\frac{1}{q_1^2}+...+\frac{1}{q_\ell^2}\\ &\quad+...+\frac{1}{(4k+1)^{4m+1}q_1^{2\alpha_1+1}q_2^{2\alpha_2+1}...q_\ell^{2\alpha_\ell+1}}\\ &\quad=1-\frac{1}{N}.\end{aligned} \tag{7.2}$$

Consider a set of distinct integers $n_1,\ ...\ n_\ell$ such that

$$\frac{1}{n_1}+...+\frac{1}{n_\ell}+\frac{1}{n_1n_2}+...+\frac{1}{n_{\ell-1}n_\ell}+...+\frac{1}{n_1...n_{\ell-1}}+...+\frac{1}{n_2...n_\ell}=1-\frac{1}{n_1...n_\ell}. \tag{7.3}$$

For distinct primes, $n_\ell \sim \frac{\ell}{ln\ \ell}$ and

$$\begin{aligned}\sum_{m=(2\ell)ln(2\ell)}^{(2\ell)ln(2\ell)+\ell}\frac{1}{n_m}&\approx\frac{(ln(2\ell ln(2\ell)+\ell)^2}{2}-\frac{(ln((2\ell)ln(2\ell))^2}{2}\\ &\approx\frac{1}{2}+\frac{ln\ ln(2\ell)}{2\ ln(2\ell)}+\frac{1}{8(ln(2\ell))^2}.\end{aligned} \tag{7.4}$$

When $\ell\geq 66$,

$$exp\left[\frac{1}{2}+\frac{ln\ ln(2\ell)}{2\ ln(2\ell)}+\frac{1}{8(ln(2\ell))^2}\right]-1<1. \tag{7.5}$$

Consequently, Eq.(7.2) cannot be satisfied and there would exist no odd perfect number with ℓ square-free factors and $\ell \geq 66$, such that the least prime divisor is larger than 2ℓ.

General results based on the method of the sum of fractions tend to have a more limited range of validity than the conclusions derived from analytic techniques. Large lower bounds for N, nevertheless, have been attained through an identification of the sets of primes which could yield a value of $\frac{\sigma(N)}{N}$ closest to 2 [11].

Partitions of unity are necessary in the global definition of functions. On a smooth manifold, there is no evident constraint on the size of the neighbourhoods representing the support of a function. Instead, it is necessary to replace a smooth manifold with a metric by a choice of conformal field theory, there are conformal field theories which do not correspond to manifolds, and the classification of rational conformal field theories can be related to modular categories. The dimensions of simple objects $\{X_i,\ i = 1, ..., n\}$ of integral categories $\mathcal{C}$ [15][16] satisfy

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{x_i} &= 1 \qquad n = rank\ of\ \mathcal{C} \\ x_i &= \frac{dim\ \mathcal{C}}{(d_i)^2} = \frac{\sum_i (d_i)^2}{(d_i)^2}. \end{aligned} \tag{7.6}$$

When the dimensions of the simple objects are products of prime powers, the conditions on the sum of fractions can be identified with the condition for the existence of an odd perfect number, if the set of prime powers is spanned by a set of primes. When two simple objects of an integral category have coprime dimension, each is a projective centralizer of the other. Consequently, the fusion category corresponding to the set of prime power dimensions would be weakly group-theoretical and given by a sequence of categories containing extensions of preceding categories by groups of prime power order [17]. The non-existence of odd integers satisfying the rationality condition would be equivalent to the absence of integral categories with simple objects of odd prime power dimension.

8. Conclusion

The equivalence of the relation $\sigma(N) = 2N$ with a rationality condition containing a product of repunits is sufficient to establish the non-existence of odd perfect numbers from the prime divisors of the quotients. In the proof of the first theorem, it is shown that each repunit $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, with q_i and $2\alpha_i + 1$ being primes, introduces a new prime factor provided that three inequalities are satisfied by any pair of repunits. It is necessary to establish that there are no solutions to the exponential Diophantine equation $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ with odd prime bases and exponents is required. This result is proven in the first lemma by establishing a correspondence with the solutions to $I(q_i^{2\alpha_i} q_j^{2\alpha_j}) = I(q_i^{2\alpha_i'} q_j^{2\alpha_j'})$, where $I(n) = \frac{\sigma(n)}{n}$ and then proving that there are no primes $q_i,\ q_j \geq 3$ satisfying the latter equation. The nonexistence of two quotients with unequal odd prime bases and exponents represented by different powers of one prime is proven in the second lemma, and therefore, the product of two repunits must consist of a minimum of two prime divisors. A linear factorization of $q_i^{2\alpha_i+1} - 1$ and $q_j^{2\alpha_j+1} - 1$ for different primes q_i and q_j requires the introduction of a new primitive divisor in the cancellation resulting from selected combinations of the terms. The quotients by $q_i - 1$ and $q_j - 1$ respectively allow for potential equalities between the products of the two repunits and imprimitive divisors $2\alpha_i + 1$ and $2\alpha_j + 1$. The proof, therefore, is given for those primes and exponents which do not satisfy any of these three conditions.

The existence of new prime factors in different repunits with odd prime bases and exponents is used in the second theorem to demonstrate that an excess of prime divisors will occur in $\sigma(N)$ under the conditions of the first theorem. The nonexistence of a positive integer solutions to the equation $\frac{x^n-1}{x-1} = 2y^2,\ x \equiv 1\ (mod\ 4),\ n \equiv 2\ (mod\ 4),\ n \geq 6$ is derived. The lower bound for the exponent n is equivalent to the condition $m \geq 1$ in the prime factorization $N = (4k+1)^{4m+1} q_1^{2\alpha_1} ... q_\ell^{2\alpha_\ell}$. The latter inequality is derived from a set of inequalities for $I\left((4k+1)^{4m+1}\right)$ and $I(\prod_{i=1}^{\ell} q_i^{2\alpha_i})$ in the third lemma. When the three inequalities are replaced by equalities, of which one must be valid for a pair of repunits, it is proven in the third theorem that the special constraints either require an infinite sequence of increasingly large repunits, prime power relations that do not have odd prime solutions or additional prime factors in the sum-of-divisors function.

References

[1] A. S. Bang, *Taltheoretiske Undersogelser*, Tidsskrift for Mathematik **5** (1886), 265-284.

[2] Yu. Bilu, G. Hanrot and P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers*, J. für die reine und angewandte Mathematik **539** (2001), 75-122.

[3] F. de Bessy (1657), in: Oeuvres de Huygens, II, Correspondence, No. 389; Ouvres de Fermat, 3, Gauthier-Villars, Paris, 1896, p.567.

[4] G. D. Birkhoff and H. S. Vandiver, *On the integral of* $a^n - b^n$, Ann. Math. **5** no. 2 (1904) 173-180.

[5] R. P. Brent, G. L. Cohen and H. J. J. te Riele, *Improved Techniques for Lower Bounds for Odd Perfect Numbers*, Math. Comp. **57** (1991) 857-868.

[6] Y. Bugeaud, M. Mignotte and Y. Roy, *On the Diophantine Equation* $\frac{x^n-1}{x-1} = y^q$, Pac. J. Math. **193**(2) (2000) 257-268.

[7] Y. Bugeaud and T. N. Shorey, *On the Diophantine Equation* $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$, Pac. J. Math. **207**(1) (2002) 61-75.

[8] E. Catalan, Mathesis **8** (1888) 112-113; Mem. Soc. Sc. Liège (2) **15** (1888) 205-207.

[9] R. D. Carmichael, *Multiply Perfect Odd Numbers with Three Prime Factors*, Amer. Math. Monthly **13** (1906) 35-36.

[10] R. D. Carmichael, *Multiply Perfect Numbers of Four Different Primes*, Ann. Math. **8** (1907) 149-158.

[11] G. L. Cohen, *On the Largest Component of an Odd Perfect Number*, J. Austral. Math. Soc. **42** (1987) 280-286.

[12] S. Davis, *A Rationality Condition for the Existence of Odd Perfect Numbers*, International Journal of Mathematics and Mathematical Sciences (2003) 1261-1293.

[13] R. Descartes to M. Mersenne (1638), Oeuvres de Descartes, II, eds. C. Adam and P. Tannery, Cerf, Paris, 1898, p. 429.

[14] J. A. B. Dris, *The Abundancy Index of Divisors of Odd Perfect Numbers*, J. Integer Sequences **15** (2012) 12.4.4.

[15] P. Etingof and S. Gelaki, *Some Properties of Finite-Dimensional Semisimple Hopf Algebras*, Math. Res. Lett. **5** (1998) 191-197.

[16] P. Etinghof, D. Nikshych and V. Ostrik, *On Fusion Categories*, Ann. Math. **162** (2005) 581-642.

[17] P. Etinghof, D. Nikshych and V. Ostrik, *Weakly Group-Theoretical and Solvable Fusion Categories*, Adv. Math. **226** (2011) 176-205.

[18] L. Euler, *Tractatus de Numerorum Doctrina*, §109 in: Opera Omnia I 5 Auctoritate et Impensis Societatis Scientarum Naturalium Helveticae, Genevae, MCMXLIV.

[19] J. A. Ewell, *On the Multiplicative Structure of Odd Perfect Numbers*, J. Number Theory **12** (1980) 339-342.

[20] R. Goormaghtigh, *Nombres tel que* $A = 1 + x + x^2 + ... + x^m = 1 + y + y^2 + ... + y^n$, L'Intermediare des Mathematiciens **3** (1892), 265-284.

[21] T. Goto and Y. Ohno, *Odd Perfect Numbers have a Prime Factor Exceeding* 10^8, Math. Comp. **77** (2008) 1859-1868.

[22] O. Grun, *Über ungerade Volkommene Zahlen*, Math. Zeit. **55** (1952) 353-354.

[23] P. Hagis Jr., *An Outline of a Proof that Every Odd Perfect Number has at Least Eight Prime Factors*, Math. Comp. **34** (1983) 1027-1032.

[24] P. Hagis Jr., *Sketch of a Proof that an Odd Perfect Number Relatively Prime to 3 has at Least Eleven Prime Factors*, Math. Comp. **40** (1983) 399-404.

[25] K. G. Hare, *More on the Total Number of Prime Factors of an Odd Perfect Number*, Math. Comp. **74** (2005) 1003-1008.

[26] K. G. Hare, *New Techniques for Bounds on the Total Number of Prime Factors of an Odd Perfect Number*, Math. Comp. **74** (2005) 1003-1008.

[27] B. He and A. Togbé, *On the Number of Solutions of Goormaghtigh Equation for given x and y*, Indag. Math. **19** (2008) 65-72.

[28] B. He, *A Remark on the Diophantine Equation* $\frac{x^3-1}{x-1} = \frac{y^n-1}{y-1}$, Glas. Mat. **44** (2009) 1-6.

[29] D. E. Iannucci, *The Second Largest Prime Divisors of an Odd Perfect Number exceeds Ten Thousand*, Math. Comp. **68** (1999) 1749-1760.

[30] D. E. Iannnucci, *The Third Largest Prime Divisor of an Odd Perfect Number exceeds One Hundred*, Math. Comp. **69** (2000) 867-879.

[31] M. Jenkins, *Odd Perfect Numbers have a Prime Factor Exceeding* 10^7, Math. Comp. **72** (2003) 1549-1554.

[32] M. Kishore, *Odd Integers N with Five Distinct Prime Factors for which* $2 - 10^{-12} < \frac{\sigma(N)}{N} < 2 + 10^{-12}$, Math. Comp. **32** (1978) 303-309.

[33] M. Kishore, *Odd Perfect Numbers not Divisible by* 3, Math. Comp. **40** (1983) 405-411.

[34] U. Kühnel, *Vershärfung der notwendigen Bedingungen für die Existenz von ungeraden vollkommenen Zahlen*, Math. Zeit. **52** (1950) 202-211.

[35] N. Koblitz, *p-adic Numbers, p-adic Analysis and Zeta-Functions* Springer Science, New York, 1977.

[36] W. Ljunggren, Some Theorems on Indeterminate Equations of the Form $\frac{x^n-1}{x-1} = y^q$, Norsk Mat. Tidsskr. **25** (1943) 17-20.

[37] P. P. Nielsen, *An Upper Bound for Odd Perfect Numbers*, Integers **3** (2003) A14.

[38] P. P. Nielsen, *Odd Perfect Numbers have at Least Nine Different Prime Factors*, Math. Comp. **76** (2007) 2109-2126.

[39] K. K. Norton, *Remarks on the Number of Factors of an Odd Perfect Number*, Acta Arith. **6** (1960) 365-374.

[40] T. Nagell, Sur l'équation indéterminée $\frac{(x^n-1)}{(x-1)} = y^2$, Norsk Mat. Forenings Skr. **1** (1921) 1-17.

[41] Y. Nesterenko and T. N. Shorey, *On the Diophantine Equation* $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$, Acta Arith. **83** (1998) 381-399.

[42] P. Ochem and M. Rao, *Odd Perfect Numbers are Greater than* 10^{1500}, Math. Comp. **81** (2012) 1869-1877.

[43] O. Ore, *On the Averages of the Divisors of a Number*, Amer. Mat. Monthly, **55** (1948) 615-619.

[44] R. F. Ryan, *An Abundancy Result for the Two Prime Power Case and Results for an Equations of Goormaghtigh*, Int. Math. Forum **8** (2013) 427-432.

[45] T. N. Shorey. *On the Equation* $ax^m - by^n = k$, Indag. Math. **48** (1986) 353-358.

[46] A. N. de Sousa, Where do Odd Perfect Numbers Live?, arXiv.1801.06182.

[47] C. L. Stewart, *On Divisors of Fermat, Fibonacci, Lucas and Lehmer Numbers*, Proc. London Math. Soc. Series 3 **35** (1977) 425-477.

[48] J. Sylvester, *Sur les Nombres Parfaits*, C. R. CVI (1888) 403-405.

[49] T. Yamada, *Odd Perfect Numbers of a Special Form*, Colloq. Math. **103** (2005) 303-307.

[50] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. **3** (1892) 265-284.