

How Euler discovered the zeta function

Keith Devlin

Euler's zeta function is defined for any real number s greater than 1 by the infinite sum:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

(Provided s is a real number bigger than 1, this infinite sum has a finite answer.) After Euler defined this function, he showed that it has a deep and profound connection with the pattern of the primes. Namely, he proved that:

$$\zeta(s) = \frac{1}{1 - (1/2^s)} \times \frac{1}{1 - (1/3^s)} \times \frac{1}{1 - (1/5^s)} \times \frac{1}{1 - (1/7^s)}$$

where the product on the right is taken over all terms $\frac{1}{1 - (1/p^s)}$ where p is a prime.

How did Euler discover this amazing connection? The story begins with the familiar theorem that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

has an infinite sum. Knowing this fact, Euler wondered about the “prime harmonic series”

$$PH = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots$$

you get by adding the reciprocals of all the primes. Does this sum have a finite or an infinite answer? A natural way to try to answer this question — if you are Euler, that is — is to split up the harmonic series into two parts, collecting all the prime terms together and all the composite terms together, i.e.,

$$\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots \right] + \left[\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \right]$$

and then try to show that the second part has a finite answer. This would mean that the first part is what causes the harmonic series to have an infinite answer. But the first part is just PH , so this would show that PH must have an infinite answer. It's a good idea but there is a snag. Because the harmonic series has an infinite answer, you cannot split it into two separate sums like this. (As every child learns in elementary school, you can split up a *finite* sum any way you like, and the answer will always remain the same. But the same is only true for infinite sums if all their answers are finite.) So Euler adopted a roundabout way instead.

Suppose, said Euler, that we take some positive real number s slightly greater than 1, and instead of looking at the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

we look at the related sum

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

you get by raising each term in the harmonic series to the power s . Provided s is bigger than 1, this sum does have a finite answer, and so you *can* split it up into two infinite parts, the first part being all the prime terms, the second all the nonprime terms, like this:

$$\zeta(s) = \left[1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \dots \right] + \left[\frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \dots \right]$$

The idea then is to show that, if you were to take s closer and closer to 1, the first sum

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \dots$$

here increases without bound, and hence that, taking $s = 1$,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots$$

is infinite.

A key step in this argument was to establish the celebrated equation

$$\zeta(s) = \frac{1}{1 - (1/2^s)} \times \frac{1}{1 - (1/3^s)} \times \frac{1}{1 - (1/5^s)} \times \frac{1}{1 - (1/7^s)}$$

where the product on the right is taken over all terms $\frac{1}{1 - (1/p^s)}$ where p is a prime.

Euler's idea was to start with the familiar formula for the sum of a geometric series:

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots \quad (0 < x < 1)$$

For any prime number p and any $s > 1$, we can set $x = 1/p^s$ to give:

$$\frac{1}{1 - (1/p^s)} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots$$

This expression on the left is a typical term in Euler's infinite product, of course, so the above equation provides an infinite sum expression for each term in the infinite product. What Euler did next was multiply together all of these infinite sums to give an alternative expression for his infinite product. Using the ordinary algebraic rules for multiplying (a finite number of finite) sums, but applying them this time to an infinite number of infinite sums, you see that when you write out the right-hand side as a single infinite sum, its terms are all the expressions of the form

$$\frac{1}{p_1^{k_1 s} \dots p_n^{k_n s}}$$

where p_1, \dots, p_n are different primes and k_1, \dots, k_n are positive integers, and each such combination occurs exactly once. But by the fundamental theorem of algebra, every positive integer can be expressed in the form $p_1^{k_1 s} \dots p_n^{k_n s}$. Hence the right-hand side is just a rearrangement of the sum

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

i.e., $\zeta(s)$. (You have to be a bit careful how you do this, to avoid getting into difficulties with infinities. The details are not particularly difficult.)

Now, from the point of view of the subsequent development of mathematics it was not so much the fact that the prime harmonic series has an infinite sum that is important, even though it did provide a completely new proof of Euclid's result that there are infinitely many primes. Rather, Euler's infinite product formula for $\zeta(s)$ marked the beginning of analytic number theory.

In 1837, the French mathematician Lejeune Dirichlet generalized Euler's method to prove that in any arithmetic progression $a, a + k, a + 2k, a + 3k, \dots$, where a and k have no common factor, there are infinitely many primes. (Euclid's theorem can be regarded as the special case of this for the arithmetic progression $1, 3, 5, 7, \dots$ of all odd numbers.) The principal modification to Euler's method that Dirichlet made was to modify the zeta function so that the primes were separated into separate categories, depending on the remainder they left when divided by k . His modified zeta function had the form

$$L(s, \chi) = \frac{\chi(1)}{1^s} + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(4)}{4^s} + \dots$$

where $\chi(n)$ is a special kind of function — which Dirichlet called a “character” — that splits the primes up in the required way. In particular, it must be the case that $\chi(mn) = \chi(m)\chi(n)$ for any m, n . (The other conditions are that $\chi(n)$ depends only on the remainder you get when you divide n by k , and that $\chi(n) = 0$ if n and k have a common factor.)

Any function of the form $L(s, \chi)$ where s is a real number greater than 1 and χ is a character is known as a Dirichlet L-series. The Riemann zeta function is the special case that arises when you take $\chi(n) = 1$ for all n .

Mathematicians subsequent to Dirichlet generalized the notion by allowing the variable s and the numbers $\chi(n)$ to be complex numbers, and used the generalized versions to prove a great many results about prime numbers, thereby demonstrating that the L-series provide an extremely powerful tool for the study of the primes.

A key result about L-functions is that, as with the zeta function, they can be expressed as an infinite product over the prime numbers (sometimes known as an Euler product), namely:

$$L(x, \chi) = \frac{1}{1 - (\chi(2)/2^s)} \times \frac{1}{1 - (\chi(3)/3^s)} \times \frac{1}{1 - (\chi(5)/5^s)} \times \frac{1}{1 - (\chi(7)/7^s)} \times \dots$$

(provided the real part of s is not negative), where the product is taken over all expressions of the form

$$\frac{1}{1 - (\chi(p)/p^s)}$$

where p is a prime number.