RIEMANN'S HYPOTHESIS and the LIOUVILLE FUNCTION for THE PROOF OF RH
Another case that I SEE IT, BUT I CAN'T BELIEVE IT

FAUSTO GALETTO
Independent Researcher, Past Professor of "Industrial Quality Management" at Politecnico of Turin.
fausto.galetto@gmail.com

Abstract:
It is well known that "The Riemann hypothesis is equivalent to the statement that for every **ε** > 0 the function $L(x)/x^{1/2+\varepsilon}$ approaches zero as $x \to \infty$", where L is the "cumulative" Liouville function. Starting from the definitions of L(x) and ζ(s) we derive an integral equation about L(x): from that we derive that $|L(x)|<x^{1/2+\varepsilon}$; the proof of the Riemann hypothesis follows.

Keywords:
RH (the Riemann hypothesis), ζ(s) (the Riemann zeta function), integral equations, the Liouville function

1. Introduction

Riemann devised the *Riemann zeta function*: ζ(z), where z is a complex number z=x+iy and i is the "positive" imaginary unit such that $i^2=-1$.
The *Riemann Hypothesis* states all the ***nontrivial** zeros* are the complex numbers z=1/2 + iy, with suitable values of y [the line z=1/2 + iy in the z plane is named *Critical Line*]. All the *"known"* zeros computed up to now [up to 2018], more than 10^{12} zeros have been computed, all on the *Critical Line*.
Many people tried to prove RH and making it a theorem; the author himself, using the theory of Hilbert Spaces, tried it [2] after studying the book of Titchmarsh [1]; there one finds that *the Riemann hypothesis is equivalent to the statement that for every* **ε** *> 0 the Mertens function* $M(x)=O(x^{1/2+\varepsilon})$: using this equivalence the author provided [3] a new proof of RH.
Now the author suggest another new proof of RH, by using the fact that *the Riemann hypothesis is equivalent to the statement that for every* **ε** *> 0 the "cumulative" Liouville function* $L(x)=O(x^{1/2+\varepsilon})$.

Let n be a positive integer; we find the prime factors of n and count them: we indicate as ω(n) the number of prime factors of n, counted with their multiplicity.

The Liouville function is defined by

$$\lambda(n) = (-1)^{\omega(n)}$$

It takes only two values -1 and 1; it graph is an infinite "rectangular" wave, with steps up and down according the number ω(n).
It is a sequence like

{1,−1,−1, 1,−1, 1,−1,−1, 1, 1,−1,−1,−1, 1, 1, 1,−1,−1,−1,−1, 1, ..., 1,..., -1, ..., ..., ...}.

which is a "determination" of a random walk.
The "*cumulative*" function of Liouville is defined by the sum

$$L(n) = \sum_{1}^{n} \lambda(k)$$

where and n being integer numbers.
For x, a real number, the *cumulative* Liouville function is defined by the sum

$$L(x) = \sum_{1}^{n \leq x} \lambda(k) \tag{1}$$

The Riemann Hypothesis is ***equivalent*** to the statement that, for every fixed ε> 0,

$$\lim_{n\to\infty} \frac{L(n)}{n^{\frac{1}{2}+\varepsilon}} = 0 \tag{2}$$

2. Connections between L(x) and ζ(s)

In this section we use the standard symbols for the *Riemann zeta function* ζ(s); s is the complex variable, s=σ+it; x is a real variable.
Since

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{1}^{\infty} \frac{\lambda(n)}{n^s} \tag{3}$$

from (3) one can prove that, for σ>1

$$\frac{\zeta(2s)}{s\zeta(s)} = \int_0^\infty x^{-s-1} L(x) dx \tag{4}$$

On the other hand one can prove that

$$M(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(2s)x^s}{s\zeta(s)} ds, \quad for\ c > 1 \tag{5}$$

From (4) and (5), without assuming that RH is true, we derive the integral equation

$$L(x) = \frac{1}{2} \int_0^x \frac{L(y)}{y}\, dy \tag{6}$$

We will prove it after the introduction to the Theory of Distributions (next section).

3. Theory of Distributions Basics

Distribution theory [4] can be thought of as the completion of differential calculus, because in the theory any distribution can be differentiated any times we want (it is one of the two great revolutions in mathematical analysis in the 20th century).
It is known what a function f(x), with x real, is. The same for f(s), with s complex.
We can generalise the definition of a function f(x) by computing the integral

$$< f, \varphi >= \int f(y)\varphi(y)\ dy \tag{7}$$

The functions φ(x) are named *test functions* and are *vanishing outside a bounded subset of the space R and are such that all partial derivatives of all orders of* φ(x) *are continuous.* [for the case "monodimensional"!]
The integral is to be intended in terms of Lebesgue integration theory.
Any function gives rise to a distribution by setting $< f, \varphi >= \int f(x)\varphi(x)\ dx$, at least if the integral can be defined. This is certainly true if *f* is continuous, but actually more general functions will work.
The so called "Dirac δ-function" actually is defined only as a distribution by **<δ,φ>=φ(0).**
It cannot differentiated as a function, but it can differentiated as a distribution by **<δ′,φ>=-φ′(0).**

In the real space R we can define the Heaviside function H(x) [also named step function] which is

$$H(x) = \begin{cases} 0, & for\ x < 0 \\ 1, & for\ x \geq 0 \end{cases}$$

H(x) cannot be differentiated at x=0. On the contrary, the distribution **<H,φ>** can be differentiated and we have **<H′,φ>=<δ,φ>**: we indicate it with the symbol **δ(x).**
There is a second type of step function, actually distribution, named as well Heaviside function H(x) which is

$$H(x) = \begin{cases} 0, & for\ x < 0 \\ \frac{1}{2} & for\ x = 0 \\ 1, & for\ x > 0 \end{cases}$$

The derivative of this distribution is **<H′,φ>=<δ,φ>=1/2δ(x)**
For distributions we can define all the operations (and others) we can define for functions: e.g. Fourier transforms, Laplace transforms, integration, derivation, ...
In the next sections we will consider the Liouville function, in the sense of the Distribution Theory, in order to take advantage of the Theory.

4. The integral equation for L(x)

Inserting (4) into (5) we get [without assuming that RH is true]

$$L(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \left[\int_0^\infty y^{-s-1} L(y) dy\right] ds \tag{8}$$

Some manipulations of (8) give

$$L(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty x^s y^{-s}\frac{L(y)}{y} dy\, ds$$

$$L(x) = \int_0^\infty \frac{L(y)}{y}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{y^s} ds\, dy \tag{9}$$

and finally

$$L(x) = \frac{1}{2}\int_0^x \frac{L(y)}{y}\, dy \tag{10}$$

To prove (8) we need to consider first the integral, where x and y are real variables, and s the complex variable, s=σ+it

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{y^s}\frac{ds}{s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{x}{y}\right]^s\frac{ds}{s}$$

Integrating, when x<y, one gets 0; integrating, when x=y, one gets ½; integrating, when x>y, one gets 1.
This is the second type of step distribution, H(u-1) with u=x/y; when x=y H(u-1)=H(0).
The complex integral in (9) then is related to the distribution $\frac{1}{2}\delta(u-1)$ which is 0 for any u≠1.
That means that the integral about the "integration variable y" must be 0 above x. Therefore we get (10). L(x) is the distribution related to the Liouville function.
This is the point given in the title:

I SEE IT, BUT I CAN'T BELIEVE IT

Let's look at (10) in terms of a random walk, ruled by tossing a fair coin every 1 second; after each toss the function L(y) takes a step "upwards" if heads shows, "downwards" if tails shows; L(y) is the value of the height of a staircase, at time y: the ratio L(y)/y is the rate of height-change in the interval 0----y. Formula (10) tells us the L(x) is related to the rate of height-change L(y)/y.
If we let $\mathbf{S_n}$ be the random variable "sum of the sequence of Heads=1 and Tails=-1" we have that (Central Limit Theorem in reference [5])

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} > x\right\} = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-t^2/2}\,dt$$

proving that all random walks of the type considered are governed by the same probabilities for large n: the same is for the numbers λ(n), which are one of the possible paths of a random walk, ruled by tossing a fair coin. By choosing $x=n^{\varepsilon/2}$ we get

$$\lim_{n\to\infty} P\left\{\frac{S_n}{n^{\frac{1}{2}+\varepsilon}} < 1\right\} = \lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{n^{\varepsilon/2}} e^{-t^2/2}\,dt = 1$$

that is it is

$$\boldsymbol{L(x) < x^{\frac{1}{2}+\varepsilon}}$$

with probability 1.

5. The integral inequality for the absolute value |L(x)|

In order to prove the Riemann Hypothesis RH, we must find the distribution **|L(x)|** related to the integral equation (10).
For the absolute value we have, being ε>0,

$$|L(x)| \le \frac{1}{2}\left|\int_0^x \frac{L(y)}{y}\,dy\right| \le \frac{1}{2}\int_0^x \frac{|L(y)|}{y}\,dy$$
$$< (\frac{1}{2}+\varepsilon)\int_0^x \frac{|L(y)|}{y}\,dy \qquad (11)$$

Let's put g(t)=|L(t)|, considered as a distribution, and find its Laplace transform (p is the complex variable)

$$G(p) = L[g(t)] = \int_0^{\infty} e^{-pt} g(t)dt$$

Then

$$L\left[\int_0^t \frac{g(t)}{t}dt\right] = \frac{1}{p}\int_p^{\infty} G(\omega)d\omega$$

Therefore

$$G(p) < (\frac{1}{2}+\varepsilon)\frac{1}{p}\int_p^{\infty} G(\omega)d\omega$$

If we seek a solution

$$\Gamma(\alpha)/p^{\alpha} = L[t^{\alpha-1}] \text{ with } \alpha > 0$$

we get

$$\frac{1}{p^{\alpha}} < (\frac{1}{2}+\varepsilon)\frac{1}{-p^{\alpha}}\frac{1}{(-\alpha+1)}$$

that is

$$(\alpha - 1) < (\frac{1}{2} + \varepsilon) \qquad (12)$$

By choosing α=3/2 we have α-1<1/2+ε, that is $t^{1/2+\varepsilon}$ as solution of the integral inequality (9), and eventually

$$|L(x)| < x^{\frac{1}{2}+\varepsilon} \qquad (13)$$

6. Conclusion

Since we know that "The Riemann hypothesis is equivalent to the statement that for every **ε** > 0 the function ***L(x)=O(x^{1/2+ε})***", where L is the "cumulative" Liouville function and we found that $|L(x)|<x^{1/2+\varepsilon}$ is the solution of the integral inequality about |L(x)| [derived from integral equation about L(x), found from L(x) and ζ(2s)/ζ(s)], the proof of the Riemann hypothesis is proved. (see also [2, 3]). It is catching the probabilistic interpretation of L(x), even though the "cumulative" Liouville function is not random...

References

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Biography

Fausto Galetto (born Italy 1942) got Electronics Engineering and Mathematics degrees (Bologna University, 1967, 1973).
1992-2012 Professor of "Industrial Quality Management" at Politecnico of Turin. 1998-2001 Chairman of the Committee "AICQ-Università" (Quality in Courses about Quality in Universities).
Eight books, 200 papers on Reliability, Quality (Management, DOE, Statistics, Testing, Process Control).
Reliability Engineer (General Electric, 2 years), 1975-1982 Reliability Manager (Fiat Auto, now FCA); Quality Director (Philco 3 years). 1985-1990 Quality/Reliability Director (Iveco).
Since 1990 Quality Management consultant. Lecturer with AICQ/COREP (1980-2012). He is one of the very few who take care of "*Quality of Quality Methods used for making Quality*".