On the Voronin's universality theorem for the Riemann zeta-function

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Abstract. We present a slight modification of the Voronin's proof for the universality of the Riemann zeta-function. The difference (and simplification) is that we do not use the rearrangement of terms in functional series.

1 Introduction

In [8] Voronin proved the well known universality theorem, which states: *let* $0 < r < 1/4$. *Suppose that* $f(s)$ *is an analytic function in the interior of the disc* $|s| \le r$ *and is continuous up to the boundary of this disc. If* $f(s) \neq 0$ *for* $|s| < r$*, then for every* $\varepsilon > 0$ *there exists* $T = T(\varepsilon)$ *such that*

$$\max_{|s| \le r} |g(s) - \zeta(s + 3/4 + iT)| < \varepsilon.$$

The Voronin's proof is ineffective, that means that from it an upper bound for $T = T(\varepsilon)$ can not be derived . One of the reasons is that Voronin uses the theorem of Pecherskii on the rearrangement of terms in functional series. In some partial cases the effective proof was obtained in Garunkštis [3]. The mentioned paper, works of Gonek [4] and Good [5] suggest some simplification of Voronin's proof. In the next chapter (see Proposition 1) we give a slight modification of Main Lemma from Voronin [8] (or Lemma 1 from [6], §7.1). In the proof of Proposition 1 we avoid the rearrangements of series. The remaining proof of the universality theorem for the Riemann zeta-function is the same as in §7.1 of [6]. For another way to prove universality theorems, using the method of limit theorems, see Bagchi [1], Laurinčikas [7]. In this case the proof is also ineffective.

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2 Proof

For a complex s and for a real vector $\theta = (\theta_1, \ldots, \theta_m)$ we define

$$\zeta_m(s,\theta) := \prod_{p \le m} \left(1 - \frac{e^{-2\pi i \theta_p}}{p^s}\right)^{-1}.$$

Proposition 1 *Let $0 < r < 1/4$. Suppose that $g(s)$ is analytic for $|s| < r$ and continuous for $|s| \le r$. Then for any $\varepsilon > 0$ and $y > 0$ there exist integer $m \ge y$ and a vector $\theta = (\theta_1, \ldots, \theta_n)$ with $\theta_j \in \{0, 1/4, 1/2, 3/4\}$, $j = 1, \ldots, n$ such that*

$$\max_{|s| \le r} \left| g(s) - \log \zeta_m\left(s + \frac{3}{4}, \theta\right)\right| < \varepsilon,$$

where

$$\log \zeta_m(s,\theta) = -\sum_{p \le m} \log\left(1 - \frac{e^{-2\pi i \theta_p}}{p^s}\right) = \sum_{p \le m} \left(\frac{e^{-2\pi i \theta_p}}{p^s} + \frac{1}{2}\frac{e^{-2 \cdot 2\pi i \theta_p}}{p^{2s}} + \ldots\right).$$

To proof the proposition we need few lemmas.

Lemma 2 *Let X be a linear normed space and let $D \subset X$ be a convex set, closed in view of the norm of X. Then for any vector $s \in X \setminus D$ there are $\varepsilon > 0$ and linear functional $f \in X^*$, such that*

$$\Re f(x) \le \Re f(s) - \epsilon$$

for all $x \in D$.

For a **proof** of Lemma 2 see ([2], §5.2). From Lemma 2 we obtain:

Lemma 3 *Let H be a real Hilbert space and let $D \subset X$ be a convex set, closed in view of the norm of X. If $D \ne H$ then there is a vector $e \in H$, $\|e\| = 1$, such that*

$$\sup_{x \in D} (x, e) \le +\infty.$$

Lemma 4 *Let u_n, $n \in \mathbb{N}$ be vectors of a real Hilbert space H and let the series $\sum_{n=1}^{\infty} u_n$ satisfies the condition*

$$\sum_{n=1}^{\infty} \|u_n\|^2 < \infty.$$

Let for any $e \in H$, $\|e\| = 1$ the sequence $\left\{\sum_{n=1}^{M} |(u_n, e)| : M \in \mathbb{N}\right\}$ is unbounded. Then for any number y, any $s \in H$ and any $\varepsilon > 0$ there are integer $N \ge y$ and numbers $\alpha_1, \ldots, \alpha_n$ equal to -1 or 1 such that

$$\left\| s - \sum_{n=1}^{N} \alpha_n u_n \right\| < \varepsilon.$$

Proof. Let $s \in H$, and let $\varepsilon > 0$ be arbitrary. Choose m so that $m \geq y$ and

$$\sum_{n=m}^{\infty} \|u_n\|^2 < \varepsilon^2/36.$$

Let P_m denote the set of all vectors $x \in H$ of the form

$$x = \sum_{n=m}^{N} \lambda_n u_n,$$

where $\lambda_n \in [-1, 1]$, $n = m, m+1, \ldots, N$, and $N = m, m+1, m+2, \ldots$. The set P_m is convex. Let $\overline{P_m}$ be the closure of P_m in the norm of H. Then $\overline{P_m}$ is a closed convex set. By assumption and Lemma 3 we see that $\overline{P_m} = H$. Consequently, there exist $N = N(s) \geq m$ and $|\lambda_n| \leq 1$, $n = m, m+1, \ldots, N$ such that

$$\left\| s - \sum_{n=m}^{N} \lambda_n u_n \right\| < \varepsilon/3.$$

Using induction and the property $\|x\| = (x, x)$, it is easy to construct numbers $\alpha_m, \ldots, \alpha_N$ equal to -1 or 1 such that

$$\left\| \sum_{n=m}^{N} \lambda_n u_n - \sum_{n=m}^{N} \alpha_n u_n \right\|^2 \leq 4 \sum_{n=m}^{N} \|u_n\|^2 < \varepsilon^2/9.$$

Then

$$\left\| s - \sum_{n=m}^{N} \alpha_n u_n \right\| < \varepsilon.$$

Now Lemma 4 follows, if we apply the above to

$$s - \sum_{n=1}^{m-1} u_n$$

instead of s.

Proof of Proposition 1. By the continuity of $g(s)$, there exists $\gamma > 1$ such that $\gamma^2 r < 1/4$ and

$$\max_{|s| \leq r} |g(s) - g(s/\gamma^2)| < \varepsilon.$$

The function $g(s/\gamma^2)$ belongs to the Hardy space $H_2^{(\gamma r)}$. We remind that the Hardy space $H_2^{(R)}$ is the set of functions $f(s)$ analytic in $|s| < R$ with the norm

$$\|f\| = \lim_{r \to R} \int \int_{|s|<r} |f(s)| d\sigma dt < \infty.$$

Let's define a scalar product in $H_2^{(R)}$ by the formula

$$(\varphi_1(s), \varphi_2(s)) = \Re \int\int_{|s|\le R} \varphi_1(s)\overline{\varphi_2(s)} d\sigma dt.$$

This makes $H_2^{(R)}$ into a real Hilbert space.

To prove the proposition we apply Lemma 4. Since the series

$$-\sum_{k=1}^{\infty} \log\left(1 - \frac{e^{-2\pi ik/4}}{p_k^{s+3/4}}\right)$$

differs by an absolutely convergent series from the series

$$\sum_{k=1}^{\infty} \eta_{p_k}(s) := \sum_{k=1}^{\infty} \frac{e^{-2\pi ik/4}}{p_k^{s+3/4}}$$

it suffices to verify the conditions of Lemma 4 for the last series. We have

$$\sum_{k=1}^{\infty} \|\eta_p(s)\|^2 \ll \sum_{k=1}^{\infty} \left|\frac{1}{p_k^{3/4-R}}\right|^2 < \infty,$$

since $0 < R < 1/4$. Next, let $\varphi(s) \in H_2^{(R)}$ with $\|\varphi(s)\| = 1$. Then Voronin [8] (or see [6], §7.1) proved that there exists a subseries of

$$\sum_{k=1}^{\infty} (\eta_k(s), \varphi(s))$$

diverging to ∞. Proposition 1 is proved.

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