The effective universality theorem for the Riemann zeta function

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Abstract. It is known that Voronin’s universality theorem for the Riemann zeta-function is ineffective. For some partial cases we obtain the effective version of this theorem.

1 Introduction

The famous Voronin’s universality theorem states: *let* $0 < r < 1/4$. *Suppose that* $g(s)$ *is an analytic function in the interior of the disc* $|s| \leq r$ *and is continuous up to the boundary of this disc. Then for every* $\varepsilon > 0$ *there exists* $\tau = \tau(\varepsilon)$ *such that*

$$\max_{|s| \leq r} |g(s) - \log \zeta(s + 3/4 + i\tau)| < \varepsilon,$$

*where the branch of* $\log \zeta(s)$ *is taken that is real for* $s \in \mathbb{R}$ *and is extended analytically along the intervals* $[2, 2 + it], [2 + it, \sigma + it]$. *Really Voronin proved more: that for every* $\varepsilon > 0$

$$d(\varepsilon) := \liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{|s| \leq r} |g(s) - \log \zeta(s + 3/4 + i\tau)| < \varepsilon \right\} > 0.$$  

Note that the universality theorem can be formulated for $\zeta(s)$ instead of $\log \zeta(s)$ (see Karatsuba and Voronin [6], chapter 7.1, Theorem 2). For further development on universality theorems see Laurinčikas [8], Matsumoto [10]. Until now no bounds are known for $\tau(\varepsilon)$ and $d(\varepsilon)$. Voronin’s proof is ineffective. He uses the theorem of Pecherskii on the rearrangement of terms in functional series. An interesting approach to solve this problem was done by Good [4]. Extending on his ideas we prove

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Theorem 1 Let \( \varepsilon, \beta, r \) and \( R \) be such that \( 0 < \varepsilon \leq 1, \ 0 < r < R < 1/4, \ 0 < \beta + R < 1/4 \) and \( r < \delta e^{-1 - \frac{1}{4s}} \), where

\[
\delta := \frac{1}{4} - R - \beta - \frac{\log \frac{e}{2R}}{1 - 4R}.
\]

Let \( g(s) \) be analytic in \( |s| \leq R, \ \rho > 355991^{\frac{1}{145}} \) and \( \delta \log \rho \) be an integer. Let

\[
M := \max_{|s| \leq R} |g(s)| + 0.8 + \frac{3.5}{1 - 4R} \leq \frac{\rho^3}{5 \delta^3 \log^4 \rho}
\]

and let \( V, Q \) and \( T \) satisfies the following conditions:

\[
50 \leq V \leq \rho \leq Q \leq \min \left\{ \frac{0.07 \rho^\frac{1}{2} \log^\frac{3}{4} T}{V^\frac{1}{2}}, 0.99 \log T \right\}.
\]

Then

\[
\alpha := \delta \log \frac{\delta}{er} - \frac{1}{4}
\]

is positive and the measure of \( \tau \in [T/2, T] \), such that

\[
\max_{|s| \leq R} \left| \log \zeta \left( s + \frac{3}{4} + i\tau \right) - g(s) \right| < \frac{M}{\left( \frac{R}{r} - 1 \right) \rho^{\delta \log R} \tau} + \frac{3}{\rho^\alpha \log \rho}
\]

\[
+ \frac{3 \log \rho}{\rho^{(1 - 4\delta)(1 - 2\tau)}} + \frac{3}{\rho^{(1 - 2\delta)(1 - 2\tau)}} \log \rho + \epsilon + 500 \frac{\rho^{\frac{5}{4} + r}}{V \log \rho} + \frac{80}{\rho^{\frac{5}{4} - r} \log^2 \rho}
\]

is greater than

\[
\frac{T}{2} \left( 1 + V^{-\pi(\rho)} - 63 \log^2 Q \right),
\]

where \( \pi(x) \) denotes the number of primes not exceeding \( x \).

Unfortunately Theorem 1 is valid only for discs with very small radius \( r \). It is easy to calculate that \( r \) is less than 0.00036 \( (R = 0.06 \ldots, \beta = 0) \). The requirement that \( \delta \log \rho \) must be an integer is technical. It can be avoided relaxing the accuracy of the approximation. The next is an example of the theorem for concrete values of \( r \) and \( R \):

Corollary 2 Let \( 0 < \varepsilon \leq 1/2 \). Let the function \( g(s) \) be analytic in the disc \( |s| \leq 0.05 \) and assume \( \max_{|s| \leq 0.06} |g(s)| \leq 1 \). Let \( r = 0.0001 \). Then

\[
\log \tau(\varepsilon) \leq e^{\frac{10}{\varepsilon}} \quad \text{and} \quad d(\varepsilon) \geq e^{-\frac{1}{10}}.
\]
Analogously to the lower density \( d(\varepsilon) \) we can define the upper density \( D(\varepsilon) \) by replacing 'lim inf' by 'lim sup' in (1). In [13] Steuding obtained nontrivial upper bounds for \( D(\varepsilon) \) for some special class of functions.

An other way to prove universality theorems is to use the method of limit measures (see, for example, Laurinčikas [8]). In this case effectivization problems are discussed in Laurinčikas [9].

The proof of Theorem 1 is divided in to three independent parts (Sections 2, 3 and 4) so that the theorem immediately follows from Propositions 3, 8 and 16. The mentioned restriction on the radius \( r \) appears only in the first part (Section 2).

We will use the notation \( \Theta(\alpha) \) to indicate 'some number not greater in modulus than \( \alpha \')

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2 Approximation by trigonometric polynomials

The aim of this chapter is

**Proposition 3** Let \( \beta, r \) and \( R \) be such that \( 0 < r < R < 1/4 \), \( 0 < \beta + R < 1/4 \) and \( r < \delta e^{-1-\frac{1}{8}} \), where

\[
\delta := \frac{1}{4} - R - \beta.
\]

Then

\[
\alpha := \delta \log \frac{\delta}{e^r} - \frac{1}{4}
\]

is positive.

Let \( g(s) \) be analytic in \(|s| \leq R\) and let

\[
M := \max_{|s| \leq R} |g(s)| + 0.8 + \frac{3.5}{1 - 4R} \leq \frac{\rho^\beta}{5\delta^3 \log^4 \rho}.
\]

Let \( \rho > \frac{355991}{1 \times 3\times 5} \) and \( \delta \log \rho \) be an integer. Then there are real numbers \( \theta_p, p \leq \rho \), such that

\[
g(s) = -\sum_{p \leq \rho} \log \left(1 - e^{-2\pi i \theta_p} \frac{s}{p^{\frac{3}{4}}} \right) + \Theta \left( \frac{M}{R - 1} \delta \log \frac{2}{r} + \frac{3}{\rho^\alpha \log \rho} + \frac{3\log \rho}{\rho^{(1-4\delta)(\frac{3}{4}-r)}} + \frac{3}{\rho^{(1-4\delta)(\frac{7}{4}-2\delta) \log \rho}} \right)
\]

for \( |s| \leq r \). Moreover one can choose \( \theta_{p_n} = n/2 \) for \( p_n \leq \rho^{1-4\delta} \).
For the proof of the proposition the following lemmas will be useful.

**Lemma 4** We have that,

\[
\frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \leq \pi(x) \quad \text{for} \quad x \geq 32\,299,
\]

\[
\pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right) < 1.094 \frac{x}{\log x} \quad \text{for} \quad x \geq 355\,991,
\]

and

\[
\pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1.2762}{\log x}\right) \quad \text{for} \quad x > 1.
\]

**Proof.** The inequalities are obtained by Dusart [2, 3].

**Lemma 5** Let \( \Delta_{\lambda\rho}, \, \lambda < \rho \), denote the set of vectors \( z = (z_p)_{\lambda < p \leq \rho} \) with complex components \( z_p \) of modulus \( \leq 1 \). Let \( g_k, \, k = 0, 1, \ldots, \) be functions on \( \Delta_{\lambda\rho} \) defined by

\[
g_k(z) = \sum_{\lambda < p \leq \rho} z_p p^{-\sigma} (-\log p)^k, \quad \sigma \leq 1 - \varepsilon.
\]

Let \( K \) be a positive integer and \( w_k, \, k = 0, \ldots, K, \) complex numbers such that

\[
(2) \quad K^3 \leq 0.06 \log^2 \lambda \log \frac{\rho}{\lambda}
\]

and

\[
|w_k| \leq \frac{\lambda^{1-\sigma} \log \rho}{10K^3 \log \lambda} \left(\frac{1 - \log \lambda}{\log \rho}\right)^{K+1} \frac{1}{k!(K-k)!} \log^k \rho.
\]

Then the system of equations

\[
g_k(z) = w_k, \quad k = 0, \ldots, K,
\]

has a solution \( z \) in \( \Delta_{\lambda\rho} \) for \( \lambda \geq 355\,991 \).

**Proof.** This is the quantitative version of Lemma 9 from Good [4]. To obtain the explicit constants we use Lemma 4.
Lemma 6 Let $K$ and $L$ be positive integers and $K \leq L$. Let $a_{kl}$ and $b_k$, $1 \leq k \leq K$, $1 \leq l \leq L$, denote complex numbers. Suppose that the system of equations

$$
\sum_{l=1}^{L} a_{kl} z_l = b_k, \quad 1 \leq k \leq K,
$$

has a solution $(z_1, \ldots, z_L)$ belonging to

$$
\Delta^L = \{(z_1, \ldots, z_L) | z_L \text{ complex and } |z_l| \leq 1 \text{ for } 1 \leq l \leq L\}.
$$

Then (3) also has a solution $(z'_1, \ldots, z'_L)$ in $\Delta^L$ such that $|z'_l| = 1$ for at least $L - K$ positive integers $l \leq L$.

Proof. This is Lemma 6 from Good [4].

Lemma 7 Let $\sigma > 1/2$, $355991 \leq \lambda < \rho$ and $\theta_p \in \mathbb{R}$. Then

$$
- \sum_{\lambda < p \leq \rho} \log \left(1 - \frac{e^{-2\pi i \theta_p}}{p^s}\right) = \sum_{\lambda < p \leq \rho} \frac{e^{-2\pi i \theta_p}}{p^s} + O\left(\frac{1.11\sigma}{2\sigma - 1} \lambda^{2\sigma - 1} \log \lambda\right).
$$

Proof. We expand the logarithm by power series, then we use the partial summation and Lemma 4.

Proof of Proposition 3. Let $f(s)$ be analytic in $|s| \leq R$. Then

$$
f(s) = \sum_{i=0}^{K} a_i s^i + \Theta\left(\frac{(r/R)^K \max_{|u| \leq R} |f(u)|}{r - 1}\right), \quad |s| \leq r,
$$

where the coefficients obey the bounds

$$
|a_i| \leq \frac{\max_{|u| \leq R} |f(u)|}{R^i}, \quad i = 0, \ldots, K.
$$

1) With help of Lemmas 5 and 6, we will approximate the above polynomial by the exponential polynomial $\sum_{\lambda < p \leq \rho} e^{-2\pi i \theta_p} p^{-s-\frac{3}{4}}$ with appropriate $\lambda$, $\rho$ and $\theta_p$.

Let’s choose

$$
K := \delta \log \rho \quad \text{and} \quad \lambda := \rho \exp(-4K) = \rho^{1-4\delta}
$$

Note, that by the conditions of the proposition, $\delta$ is less than 0.1 and $r$ is less than 0.001. Hence $K$ satisfies (2).

By Lemma 4 we have that

$$
\sum_{\lambda < p \leq \rho} p^{-s-\frac{3}{4}} \leq \frac{6\rho^{\frac{1}{4}}}{\log \rho}.
$$
Also we will use the well known inequality

\[ n! > \left( \frac{n}{e} \right)^n \sqrt{2\pi n}. \]

Then, for \(|z_p| \leq 1\) and \(|s| \leq r\), we obtain that

\[
\left| \sum_{\lambda < p \leq \rho} z_p p \left(-\frac{1}{3} \right)^k \sum_{\lambda < p \leq \rho} z_p p \left(-\frac{1}{3} \right)^k \frac{(r \log p)^k}{k!} \right| 
\leq \sum_{\lambda < p \leq \rho} p^{-\frac{3}{4}} \sum_{k > K} \frac{(r \log p)^k}{k!} \leq \frac{3}{\rho^\alpha \log \rho}.
\]

If \(\sigma = 3/4\) and \(|w_k|, 0 \leq k \leq K\), are complex numbers satisfying

\[ |w_k| \leq \frac{1}{5} \rho^{\frac{1}{4}} (\log \rho)^{K-k-1} K^{-3} (K-k)! \left( \frac{e}{2} \right)^{-K}, \quad 0 \leq k \leq K, \]

then Lemma 5 shows the existence of a \(|z_p| \leq 1\) such that

\[
\sum_{k=0}^{K} s^k \sum_{\lambda < p \leq \rho} z_p p^{-\frac{3}{4}} \frac{(r \log p)^k}{k!} = \sum_{k=0}^{K} s^k w_k.
\]

If \(R > 0\) and \(\beta = 1/4 - R - \delta \log \frac{\rho}{2R} \) then

\[
\rho^{\frac{1}{4}} \left( \frac{e}{2} \right)^{-K} R^{K-k} \frac{(K-k)!}{(R \log \rho)^{K-k}} \geq \rho^{\frac{1}{4}} - R \left( \frac{2R}{e} \right)^{K} R^{-k} = \rho^{\frac{3}{4}} R^{-k}.
\]

Thus (7) is solvable for

\[ |w_k| \leq \rho^\beta \frac{5^\delta R^k}{\log^4 \rho}. \]

If

\[ \max_{|u| \leq R} |f(u)| \leq \rho^\beta \frac{5^\delta \log^4 \rho}{R^k}, \]

then, in view of (5), the coefficients \(a_k\) satisfies the same bound as \(w_k\). Therefore there exists \(|z_p| \leq 1\) for \(\lambda < p \leq \rho\), such that

\[
f(s) = \sum_{\lambda < p \leq \rho} \frac{z_p}{p^{s+\frac{3}{4}}} + \Theta \left( \frac{3}{\rho^\alpha \log \rho} + \frac{\max_{|u| \leq R} |f(u)|}{\left( \frac{R}{r} - 1 \right) \rho^{\delta \log \frac{R}{r}}} \right)
\]

if \(|s| \leq r\).
Now we will show, that the $z_p$ in (8) can be replaced by arbitrary complex numbers of modulus $= 1$. We have

\[ \sum_{\eta < p \leq 2\eta} \frac{z_p}{P^{s+\frac{3}{4}}} = \sum_{0 \leq k \leq N} s^k \sum_{\eta < p \leq 2\eta} \frac{z_p (-\log p)^k}{k!} + \Theta \left( \eta^\frac{1}{4} \sum_{k > N} \frac{0.001^k \log^k 2\eta}{k!} \right) \]

for $|s| \leq r$. If $N$ is the integral part of $0.3 \log \eta$ then by (6) we have for the error term,

\[ \frac{\eta^\frac{1}{4}}{8} \sum_{k > N} \frac{0.001^k \log^k 2\eta}{k!} \leq \frac{\eta^\frac{1}{4}}{2\pi (N + 1)} \sum e^{-k (log (N + 1) - 1) - \log 2 \eta + \log 1000} \]

\[ \leq 0.3\eta^{-\frac{3}{4}}, \quad (\eta > 355991). \]

Next we apply Lemma 6 to the system

\[ b_k = \sum_{\eta < p \leq 2\eta} \frac{(-\log p)^k}{k!} z_p, \quad k = 0, \ldots, N. \]

Thus for any complex numbers $z_p$ with $|z_p| \leq 1$, $\eta < p \leq 2\eta$, there are complex numbers $z'_p$ with $|z'_p| \leq 1$, $\eta < p \leq 2\eta$, and $|z'_p| = 1$ for at least $\eta - N - 1$ primes $p$ such that

\[ \sum_{\eta < p \leq 2\eta} \frac{(-\log p)^k}{k!} z_p = \sum_{\eta < p \leq 2\eta} \frac{(-\log p)^k}{k!} z'_p, \quad k = 0, \ldots, N. \]

Hence we have by (9)

\[ \sum_{\eta < p \leq 2\eta} \frac{z_p}{P^{s+\frac{3}{4}}} = \sum_{\eta < p \leq 2\eta} \frac{z'_p}{P^{s+\frac{3}{4}}} + \Theta(0.6\eta^{-\frac{3}{4}}). \]

Therefore real numbers $\theta_p$ can be found such that

\[ \sum_{\eta < p \leq 2\eta} \frac{z_p}{P^{s+\frac{3}{4}}} = \sum_{\eta < p \leq 2\eta} e^{-2\pi i \theta_p} p^{-s-\frac{3}{4}} + \Theta \left((N + 1)\eta^{-\frac{3}{4}} + 0.6\eta^{-\frac{3}{4}}\right) \]

for $|s| \leq r$. Using the last equality with $\eta = 2^j \lambda$ for $1 \leq j \leq \log \left(\frac{\rho}{\lambda}\right)/\log 2 - 1$, in view of $\sum_{j=1}^\infty j 2^{-j} x = 2x (2^x - 1)^{-2}$, we see that for any $z_p$ with $|z_p| \leq 1$, $\lambda < p \leq \rho$, there are real numbers $\theta_p$, $\lambda < p \leq \rho$, such that

\[ \sum_{\lambda < p \leq \rho} z_p p^{-s-\frac{3}{4}} = \sum_{\lambda < p \leq \rho} e^{-2\pi i \theta_p} p^{-s-\frac{3}{4}} + \Theta \left(1.1 \lambda^{-\frac{3}{4}} \log \lambda\right) \]

for $\lambda \geq 355991$ and $|s| \leq r$.

2) To finish the proof we apply the first part of the proof to

\[ f(s) = g(s) + \sum_{p_n \leq \lambda} \log \left(1 - \frac{(-1)^n}{p_n^{s+\frac{3}{4}}}\right). \]
Expanding by power series and by partial summation we obtain that for $|s| \leq R$,

$$\left| \sum_{p_n \leq \lambda} \log \left( 1 - \frac{(-1)^n}{p_n^{s+\frac{3}{4}}} \right) \right| \leq \left| \sum_{p_n \leq \rho^{1-\delta}} \frac{(-1)^n}{p_n^{s+\frac{3}{4}}} \right| + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{n^{k(\frac{3}{4}-R)}}$$

$$\leq \frac{s + \frac{3}{4}}{\sigma + \frac{3}{4}} 2R^{-\frac{3}{4}} + \frac{0.5}{1 - 2R^{-\frac{3}{4}}} \int_1^{x^{2R^{-\frac{3}{4}}}} dx < 0.8 + \frac{3.42}{1 - 4R}.$$

Then the Proposition 3 follows from (8), (10) and Lemma 7.

3 Approximation of $\zeta(s)$ by a finite product

Let

$$\zeta_Q(s) = \prod_{p \leq Q} \left( 1 - \frac{1}{p^s} \right)^{-1},$$

where $Q \in \mathbb{N}$ and $s \in \mathbb{C}$.

Proposition 8 Let $0 < r < 1/4$ and $0 < \varepsilon \leq 1$. The measure of $\tau \in [T/2, T]$, such that

$$\max_{|s| \leq r} \left| \log \left( s + \frac{3}{4} + i\tau \right) - \log \zeta_Q \left( s + \frac{3}{4} + i\tau \right) \right| < \varepsilon$$

is greater than

$$\frac{T}{2} \left( 1 - 0.06 \frac{\log^2 Q}{(0.25 - r)^{3Q^{0.25-r^2}}} \right),$$

if

$$T \geq \pi e^{\frac{1.02(0.25+r)Q}{0.25-r}}, \quad Q > 355991.$$

The proposition we will derive from the following lemmas.

Lemma 9 Let $f(x)$ be a real-valued function on the interval $[a, b]$, and let $f'(x)$ be continuous and monotonic on $[a, b]$ and $|f'(x)| \leq \delta < 1$. Then

$$\sum_{a < n \leq b} e(f(n)) = \int_a^b e(f(x)) \, dx + \Theta \left( \frac{4\sqrt{2}\delta}{\pi(1 - \delta)} + \frac{6\sqrt{2}\delta}{\pi} + 3 \right).$$

Here $e(x) = e^{2\pi i x}$.  

8
Proof of this lemma can be found, for example, in Ivić [5]. To calculate the exact constant in the error term we use the following evaluation:

\[
\sum_{k=1}^{\infty} \frac{1}{k(k-\delta)} \leq \frac{1}{1-\delta} + \frac{1}{2(2-\delta)} + \int_{1}^{\infty} \frac{dx}{x^2} \leq \frac{1}{1-\delta} + \frac{3}{2}.
\]

Lemma 10 For \( \sigma \geq 0, x \geq |t|/\pi, \ x \geq 1, \ s = \sigma + it, \)

\[
(12) \quad \zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{s-1} + \Theta \left( \frac{7\sqrt{2}\pi^{-1} + 3}{x^\sigma} \right).
\]

Proof. For \( \sigma \geq \sigma_0 > 0 \) we have (see, for example, §3.5 of Titchmarsh [14] or §1.5 of Ivić [5]):

\[
(13) \quad \zeta(s) = \sum_{n \leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} - s \int_{N}^{\infty} \left( \frac{1}{2} - \{u\} \right) u^{-s-1} du,
\]

where the last summand in modulus is \( \leq \frac{|s|}{\sigma_0} N^{-\sigma} \). For \( u \geq x \) we set

\[
A(u) = \sum_{x < n \leq u} n^{-it}
\]

and apply Lemma 9 with \( f(x) = (2\pi)^{-1}t \log x, \ \delta = \frac{1}{2} \), provided that \( x \geq |t|/\pi \). Then

\[
A(u) = \frac{u^{1-it} - x^{1-it}}{1-it} + \Theta \left( \frac{7\sqrt{2}}{\pi} + 3 \right).
\]

For \( x \leq N \), partial summation gives

\[
\sum_{x < n \leq N} n^{-s} = \sigma \int_{x}^{N} \frac{A(u) \, du}{u^{\sigma+1}} + \frac{A(N)}{N^\sigma} - \frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + \left( \frac{7\sqrt{2}\pi^{-1} + 3}{x^\sigma} \right) + O \left( \frac{x}{N^\sigma} \right), \quad N \to \infty.
\]

From this and (13) it follows that (12) is valid for \( \sigma \geq \sigma_0 \), where \( \sigma_0 \) is as close to 0 as we like. By a continuity of \( \zeta(s) \) the formula (12) remains valid for \( \sigma \geq 0 \). Lemma 10 is proved.

Lemma 11 We have

\[
\sum_{1 \leq n \leq X} \frac{1}{n} = \log x + \gamma + \Theta \left( \frac{2}{X} \right),
\]

where \( \gamma \) is Euler’s constant.
Proof. See, for example, Chandrasekharan [1], §6.3.

Let \( \tau(n) \) denotes a number of divisors of \( n \) including 1 and \( n \).

Lemma 12 We have,

\[
D(x) := \sum_{n \leq x} \tau(n) = x \log x + \Theta(\gamma x + 2)
\]

and for \( x > 10 \),

\[
D_2(x) := \sum_{n \leq x} \tau^2(n) \leq \frac{1}{6} x \log^3 x + \frac{1}{2} (1 + 2\gamma) x \log^2 x + \left(4 + 2\gamma + \gamma^2\right) x \log x + \left(4\gamma + \gamma^2\right) x + 4 \leq 1.92 x \log^3 x.
\]

For \( x \geq 355991 \),

\[
D_2(x) \leq 0.29 x \log^3 x
\]

where \( \tau(n) \) is a number of divisors of \( n \) including 1 and \( n \).

Proof. By Lemma [11] and by

\[
0 \leq \sum_{n \leq x} \left\{ \frac{x}{n} \right\} \leq x
\]

we obtain

\[
D(x) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n \leq x} \frac{x}{n} - \sum_{n \leq x} \left\{ \frac{x}{n} \right\} = x \log x + \Theta(x\gamma + 2).
\]

From \( \tau(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}) = (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_n + 1) \) it follows, that \( \tau(mn) \leq \tau(m)\tau(n) \). Thus, in view of the first part of the proof, and using summation by parts, we have

\[
\sum_{n \leq x} \tau^2(n) = \sum_{m,n \leq x} \tau(mn) \leq \sum_{m \leq x} \tau(m) \sum_{n \leq x} \tau(n)
\]

\[
\leq \sum_{m \leq x} \tau(m) \left( \frac{x}{m} \log \frac{x}{m} + \frac{\gamma x}{m} + 2 \right)
\]

\[
= \gamma D(x) + 2D(x) - \int_{1}^{x} D(y) \left( \frac{x}{y} \log \frac{x}{y} + \frac{\gamma x}{y} \right)' dy
\]

\[
\leq (\gamma + 2)(x \log x + \gamma x + 2)
\]

\[
- \int_{1}^{x} \left( y \log y + \gamma y + 2 \right) \left( \frac{x}{y} \log \frac{x}{y} + \frac{\gamma x}{y} \right)' dy
\]

\[
= \frac{1}{6} x \log^3 x + \frac{1}{2} (1 + 2\gamma) x \log^2 x + \left(4 + 2\gamma + \gamma^2\right) x \log x + \left(4\gamma + \gamma^2\right) x + 4.
\]
The remaining inequalities of Lemma 12 easily follows from the above inequality.

**Lemma 13** Let \( a_1, \ldots, a_n \) be arbitrary complex numbers. Then

\[
\int_{\frac{T}{2}}^{T} \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt = \frac{T}{2} \sum_{n \leq N} |a_n|^2 + \Theta \left( 837 \sum_{n \leq N} n|a_n|^2 \right).
\]

**Proof.** This Lemma is a quantitative version of the well known statement. To calculate an exact constant in the error term we follow lines of the proof of Theorem 5.2 from Ivić [5]. Squaring and integrating, we obtain

\[
\int_{\frac{T}{2}}^{T} \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt = \frac{T}{2} \sum_{n \leq N} |a_n|^2 + \Theta \left( 2 \sum_{m \neq n \leq N} \frac{a_m \bar{a}_n}{\log m - \log n} \right).
\]

Then (for details see Ivić [5], formula (5.11) and above),

\[
\left| \sum_{m \neq n \leq N} \frac{a_m \bar{a}_n}{\log m - \log n} \right| \leq \pi \sum_{n \leq N} |a_n|^2 + \sum_{k,l \geq 1} \left| \sum_{(m,n) \in I_k \times I_l} \int_{0}^{1} a_m \bar{a}_n e^{2\pi i y (\log m - \log n)} \log m - \log n \, dy \right|,
\]

where \( I_j = (N2^{-j}, N2^{1-j}] \), \( j = 1, 2, \ldots \). For \( |k - l| \geq 2 \),

\[
\left| \sum_{(m,n) \in I_k \times I_l} \int_{0}^{1} a_m \bar{a}_n e^{2\pi i y (\log m - \log n)} \log m - \log n \, dy \right| \leq \frac{1}{\pi} \sum_{(m,n) \in I_k \times I_l} |a_m a_n| \max_{(m,n) \in I_k \times I_l} (\log m - \log n)^{-2}
\]

\[
\leq \frac{9}{\pi} (k - l)^{-2} \left( \sum_{(m,n) \in I_k \times I_l} |a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{(m,n) \in I_k \times I_l} |a_n|^2 \right)^{\frac{1}{2}},
\]

\[
\leq \frac{9}{\pi} (k - l)^{-2} N^{2 - \frac{k+l}{2}} \left( \sum_{m \in I_k} |a_m|^2 \right)^{\frac{1}{2}} \left( \sum_{m \in I_l} |a_n|^2 \right)^{\frac{1}{2}} \leq \frac{9}{\pi} (k - l)^{-2} (S_k S_l)^{\frac{1}{2}},
\]

where

\[
S_j = \sum_{n \in I_j} n|a_n|^2.
\]
Here we used the Cauchy-Schwarz inequality and

\[
| \log m - \log n | \geq \log(N2^{-l}) - \log(N2^{1-l}) = (\log 2)(k-l-1) \geq \frac{1}{3}(k-l),
\]

which holds if \( k-l \geq 2 \), while the case \( l-k \geq 2 \) is analogous. A further application of the Cauchy-Schwarz inequality gives

\[
\sum_{|k-l| \geq 2} (k-l)^{-2}(S_kS_l) \leq \left( \sum_{|k-l| \geq 2} S_k(k-l)^{-2} \right)^{1/2} \left( \sum_{|k-l| \geq 2} S_l(k-l)^{-2} \right)^{1/2} \leq 1.3 \sum_{n \leq N} |a_n|^2,
\]
as \( \sum_{k>0,|k-l| \geq 2}(k-l)^{-2} \leq 2\zeta(2) - 2 \leq 1.3. \)

For the terms in (15) with \(|k-l| \leq 1\) we write

\[
\sum_{(m,n) \in I_k \times I_l} \int_0^1 a_m \bar{a}_n e^{2\pi i y (\log m - \log n)} \frac{dy}{\log m - \log n} = M \int_0^1 \sum_{(m,n) \in I_k \times I_l} \frac{a_m' \bar{a}_n'}{M \log m - M \log n} dy,
\]

where \( a_m' = a_m \exp(2\pi i y \log m) \) and \( M = N2^{4.63-(k+l)/2}. \) Here \( M \) is chosen such that if \(|k-l| \leq 1\), then for \( m > n \) and \((m,n) \in I_k \times I_l\)

\[
[M \log m] - [M \log n] \geq M(\log m - \log n) - 1 = M \log(1 + \frac{m-n}{n}) - 1 \geq \frac{2^{3.63-k+l}0.46(m-n)}{2^{1-l}} - 1 \geq 2(m-n) - 1 \geq (m-n)
\]

and

\[
M \log m - M \log n \geq 2(m-n),
\]
since \( m-n \geq 1, (m-n)/n \leq 3, \) and \( \log(1+x) \geq 0.46x \) for \( 0 \leq x \leq 3; \) the case \( n \geq m \)
is analogous. Therefore we have for \(|k-l| \leq 1, \)

\[
M \sum_{(m,n) \in I_k \times I_l} \frac{a_m' \bar{a}_n'}{M \log m - M \log n} = M \sum_{(m,n) \in I_k \times I_l} \frac{a_m' \bar{a}_n'}{[M \log m] - [M \log n]}
\]

\[
+ \Theta \left( 6.2N2^{-\frac{k+l}{2}} \sum_{(m,n) \in I_k \times I_l} \frac{|a_m a_n|}{(m-n)^2} \right).
\]

To evaluate the first term on the right of (16) we use the inequality (see Ivić [5], (5.10))

\[
\sum_{m \neq n} \frac{a_m \bar{b}_n}{q_m - q_n} \leq 3\pi \left( \sum |a_n|^2 \right)^{1/2} \left( \sum |b_n|^2 \right)^{1/2},
\]

(17)
where \( \{q_n\}_{n=1}^\infty \) is any sequence of integers such that \( q_m \neq q_n \) if \( m \neq n \). Then for the left-hand-side of (16) we have

\[
M \sum_{n = 1}^\infty \frac{a_m \overline{a}_n}{M \log m - M \log n} \leq \left( 3\pi^2 2^{3.63} + 6.2 \cdot 2 \cdot \zeta(2) \right) (S_k S_l)^{1/2}
\leq 137.09(S_k S_l)^{1/2}.
\]

Using once again the Cauchy-Schwarz inequality we obtain

\[
\sum_{|k-l| \leq 1} (S_k S_l)^{1/2} \leq 3 \sum_{n \leq N} n|a_n|^2.
\]

This completes the evaluation for the case \(|k - l| \leq 1\). Then in view of the above evaluation for the case \(|k-l| \geq 2\) and the inequalities (14) and (15) we obtain Lemma 13.

**Lemma 14** Let \( E = \{ s : |s| \leq d \} \) with \( 0 < d < 1/4 \). Let \( \omega = 1/4 - d \). Then for

\[
T \geq \pi e^{0.51(1-2\omega)Q-872}, \quad Q > 355.991,
\]

we have, that

\[
J := \int \int \int \left| \zeta \left( s + \frac{3}{4} + i \tau \right) \zeta^{-1}_Q \left( s + \frac{3}{4} + i \tau \right) - 1 \right|^2 d\tau d\sigma dt \leq 0.02 \frac{T \log^2 Q}{2 \omega^3 Q^{2\omega}}.
\]

**Proof.** By Lemma 10 for \( t \in [T/2, T] \), we have

\[
\zeta(s) = \sum_{n \leq T/\pi} \frac{1}{n^s} - \frac{(T/\pi)^{1-s}}{s-1} + \Theta \left( \frac{7\sqrt{2\pi - 1}}{T^\sigma} \right) = \sum_{n \leq T/\pi} \frac{1}{n^s} + \Theta \left( \frac{6.8\pi^\sigma}{T^\sigma} \right).
\]

Then

\[
J = \left( \int_{E + \frac{3}{4}}^{T} \int_{\frac{T}{2}}^{T} \left| \zeta (s + i \tau) \zeta^{-1}_Q (s + i \tau) - 1 \right|^2 d\tau \right) d\sigma dt
\leq \left( \int_{E + \frac{3}{4}}^{T} \int_{\frac{T}{2}}^{T} \left| \zeta^{-1}_Q (s + i \tau) \sum_{n \leq T/\pi} \frac{1}{n^s} - 1 \right|^2 d\tau \right) d\sigma dt
+ 2 \left( \int_{E + \frac{3}{4}}^{T} \int_{\frac{T}{2}}^{T} \left| \zeta^{-1}_Q (s + i \tau) \sum_{n \leq T/\pi} \frac{1}{n^s} - 1 \right|^2 d\tau d\sigma dt \right)^{1/2}
\]
\[
\left( \int_{\frac{E}{1}^2}^{E^2} \frac{6.8^2 \pi^{2\sigma}}{T^{2\sigma}} \int_{\frac{T}{2}}^{T} |\zeta^{-1}_{Q} (s + i\tau)|^2 \, d\tau \, d\sigma \right)^{\frac{1}{2}}
\]

+ \left( \int_{\frac{E}{1}^2}^{E^2} \frac{6.8^2 \pi^{2\sigma}}{T^{2\sigma}} \int_{\frac{T}{2}}^{T} |\zeta^{-1}_{Q} (s + i\tau)|^2 \, d\tau \, d\sigma \right)^{\frac{1}{2}}.

For \( T > \pi Q \) we have

\[
\zeta^{-1}_{Q} (s + i\tau) \sum_{\frac{T}{\pi}}^{\frac{T}{\pi}} \frac{1}{n^s} - 1 = \sum_{Q < n \leq \frac{Q_{1}}{T/\pi}} \frac{b_n}{n^s},
\]

where \( Q_{1} = p_1p_2\ldots p_Q \) and \( |b_n| \leq \tau(n) \). By the inequality (see Schoenfeld [12]), \( \sum_{p \leq x} \log p < 1.000081 \) for \( x > 0 \), we see that \( Q_{1} \leq e^{1.01Q} \). Thus, in view of Lemma [13] for \( T > \pi Q \),

\[
\int_{T/2}^{T} \left| \zeta^{-1}_{Q} (s + i\tau) \sum_{n \leq \frac{T}{\pi}} \frac{1}{n^s} - 1 \right|^2 \, d\tau \leq \sum_{Q < n \leq \frac{Q_{1}}{T/\pi}} \frac{\tau^2(n)}{n^{2\sigma - 1}} + 837 \sum_{Q < n \leq \frac{Q_{1}}{T/\pi}} \frac{\tau^2(n)}{n^{2\sigma - 1}}
\]

\[
\leq 2\sigma T \int_{Q}^{\infty} \frac{D_2(x)}{x^{2\sigma + 1}} \, dx + 837(2\sigma - 1) \int_{1}^{Q_{1}/\pi} \frac{D_2(x)}{x^{2\sigma}} \, dx + \frac{837D_2(Q_{1}/\pi)}{(Q_{1}/\pi)^{2\sigma - 1}}.
\]

By Lemma [12],

\[
\int_{E^{\frac{1}{2}}}^{E^{2}} \left( \int_{Q}^{\infty} \frac{D_2(x)}{x^{2\sigma + 1}} \, dx \right) \, d\sigma \, dt \leq \frac{1}{2} \int_{\frac{1}{2} + \omega}^{1} \left( \int_{Q}^{\infty} \frac{D_2(x)}{x^{2\sigma}} \, dx \right) \, d\sigma \leq T \frac{0.003 \log^2 Q}{\omega^3 Q^{2\omega}}.
\]

If \( \frac{T}{\pi} \geq e^{0.51(1-2\omega)Q-872} \), then

\[
837 \int_{E^{\frac{1}{2}}}^{E^{2}} \left( 2\sigma - 1 \right) \int_{Q}^{Q_{1}/\pi} \frac{D_2(x)}{x^{2\sigma + 1}} \, dx + \frac{D_2(Q_{1}/\pi)}{(Q_{1}/\pi)^{2\sigma - 1}} \, d\sigma \, dt
\]

\[
\leq \frac{31}{\omega} \log^3 Q_{1}/\pi \int_{\frac{1}{2} + \omega}^{1-\omega} \left( \frac{Q_{1}/\pi}{Q} \right)^{2-2\sigma} \, d\sigma \leq 0.003T \frac{10^2 Q}{\omega^3 Q^{2\omega}}.
\]

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and, applying Lemma 13 to the Dirichlet polynomial $\zeta_Q^{-1}(s) = \sum_{n=1}^{Qn} c_n/n^s$, where $|c_n| \leq 1$, we have

$$\int \int_{E+\frac{3}{4}} \left( \frac{T}{T^2} \left| \zeta_Q^{-1}(s+i\tau) \right|^2 d\tau \right) d\sigma dt \leq 0.0001 \frac{\log^2 Q}{\omega^3 Q^{2\omega}}.$$

Then

$$J \leq 0.02 \frac{\log^2 Q}{\omega^3 Q^{2\omega}}.$$

By this the lemma is proved.

**Lemma 15** Let $z = x + iy$. If $f(z)$ is analytic in $|z - z_0| \leq R$ and

$$\int \int_{|z - z_0| \leq R} |f(z)|^2 \, dxdy = H,$

then

$$|f(z)| \leq \frac{\sqrt{H/\pi}}{R - R'}$$

for $|z - z_0| \leq R' < R$.

**Proof** of this lemma can be found in [6].

**Proof of Proposition 8.** We use notations from Lemma 14. Let $d = r + \frac{1/4 - \omega}{2}$. Then $\omega = \frac{1/4 - \omega}{2}$. Let us define a set $A_T$ by

$$A_T = \left\{ \tau : \tau \in [T/2, T], \int \int_{E} \left| \zeta \left( s + \frac{3}{4} + i\tau \right) \zeta_Q^{-1} \left( s + \frac{3}{4} + i\tau \right) - 1 \right|^2 d\sigma dt \leq (0.6\varepsilon \omega)^2 \pi \right\}.$$

By Lemma 14, for $T/\pi \geq e^{0.51(1 - 2\omega)Q^{-872}}$, $Q > 355.991$ we have

$$\text{meas } A_T > \frac{T}{2} \left( 1 - 0.02 \frac{\log^2 Q}{\omega^3 Q^{2\omega}} \frac{1}{(0.6\varepsilon \omega)^2 \pi} \right).$$

Lemma 15 gives us that, for $\tau \in A_T$,

$$\max_{|s| \leq r} \left| \zeta \left( s + \frac{3}{4} + i\tau \right) \zeta_Q^{-1} \left( s + \frac{3}{4} + i\tau \right) - 1 \right| < 0.6\varepsilon.$$

Thus, if $\varepsilon \leq 1$, then

$$\max_{|s| \leq r} \left| \log \zeta \left( s + \frac{3}{4} + i\tau \right) - \log \zeta_Q \left( s + \frac{3}{4} + i\tau \right) \right| < \varepsilon.$$

Proposition 8 is proved.
4 Application of Weyl’s criterion

Proposition 16 Let \( \rho \geq 355991 \), \( 50 \leq V \leq \rho < Q \), \( 300VQ^2(Q/\rho)^{\frac{1}{3}} \leq \log T \) and \( \theta_p \), \( p \leq \rho \) are any real numbers. Let \( |s| \leq r < 1/4 \). Then the measure of \( \tau \in [T,2T] \), such that

\[
\left| \sum_{p \leq Q} \log \left( 1 - \frac{1}{p^{\frac{4}{3}}+s+ir} \right) \right| - \sum_{p \leq \rho} \log \left( 1 - \frac{e^{2\pi i \theta_p}}{p^{\frac{4}{3}}+s} \right) \leq \frac{500\rho^{\frac{4}{3}+r}}{V \log \rho} + \frac{80}{\rho^{\frac{4}{3}-r} \log \frac{2}{\rho}}
\]

is greater than \( \frac{1}{2}TV^{-\pi(\rho)} \).

To prove the proposition we will need the quantitative form of Weyl’s criterion.

Lemma 17 Let \( L \) be a positive integer and \( a_1, \ldots, a_L, b_1, \ldots, b_L \) be real numbers such that \( b_l \leq a_l + 1 \) for \( l = 1, \ldots, L \). Let \( U \) denote the set of points \( x = (x_1, \ldots, x_L) \) in \( \mathbb{R}^L \), the \( L \)-dimensional real vector-space, satisfying \( a_l \leq x_l < b_l \) for \( l = 1, \ldots, L \). Let \( \alpha = (\alpha_1, \ldots, \alpha_L) \) be an \( L \)-tuple of real numbers and let \( n_\alpha(U) \) denote the number of points \( n\alpha = (n\alpha_1, \ldots, n\alpha_L), 1 \leq n \leq N \), which, modulo 1, lie in \( U \). Let \( H_l, l = 1, \ldots, L, \) be greater than 1 and

\[
\gamma_{h,t} = \begin{cases} 
  b_l - a_l + \frac{75}{H_l}, & \text{for } h = 0, \\
  \min \left( \gamma_{0,t}, \frac{30}{|h|} \right), & \text{for } h \neq 0.
\end{cases}
\]

Then we have

\[
\left| \frac{n_\alpha(U)}{N} - \prod_{i=1}^{L} (b_i - a_i) \right| \leq \prod_{l=1}^{L} (b_l - a_l) \left( \prod_{l=1}^{L} \left( 1 + \frac{75}{H_l(b_l - a_l)} \right) - 1 \right) + \\
+ \frac{1}{N} \sum_{(h)} \left| \sum_{n=1}^{N} \exp \left( 2\pi i n \sum_{l=1}^{L} \alpha_l h_l \right) \right| \prod_{l=1}^{L} \gamma_{h_l,l}
\]

where the sum \( \sum_{(h)} \) extends over all \( h = (h_1, \ldots, h_L) \neq (0, \ldots, 0) \) such that \( h_l \) is integral and \( |h_l| \leq H_l(1 + \min(\log H_l, \log L)) \), \( l = 1, \ldots, L \).

Proof. The Lemma was proved by J. F. Koksma in [7].

We need the continuous version of Lemma [17].

Lemma 18 Let the conditions of Lemma [17] be satisfied. Let \( T_\alpha(U) \) denote the Jordan measure of the set \( t \in (0,T) \), such that \( t\alpha = (t\alpha_1, \ldots, t\alpha_L) \in U \mod 1 \). Then we have

\[
\left| \frac{T_\alpha(U)}{T} - \prod_{i=1}^{L} (b_i - a_i) \right| \leq \prod_{l=1}^{L} (b_l - a_l) \left( \prod_{l=1}^{L} \left( 1 + \frac{75}{H_l(b_l - a_l)} \right) - 1 \right) + \\
+ \frac{1}{T} \sum_{(h)} \min \left( \pi \sum_{l=1}^{L} \alpha_l h_l \right)^{-1} \prod_{l=1}^{L} \gamma_{h_l,l}
\]

where \( \sum_{(h)} \) means the same as in Lemma [17].
Note. Since the statement of the lemma does not change if we take \( a_l - \alpha_l T, b_l - \alpha_l T \) instead of \( a_l, b_l \), thus the lemma remains true if we define \( T_\alpha(U) \) as the Jordan measure of the set \{ \( t \in (T, 2T) : t\alpha = (t\alpha_1, \ldots, t\alpha_L) \in U \mod 1 \) \}.

Proof. Clearly there is the sequence of integers \( \{A_k\} \) such that

\[
\left| T - \frac{A_k}{k} \right| \leq \frac{1}{k}.
\]

In Lemma 17 we take \( \frac{\alpha}{k} \) instead of \( \alpha \) and \( A_k \) instead of \( N \). Then in view of limits

\[
\lim_{k \to \infty} \frac{N_k(U)}{A_k} = \frac{T}_\alpha(U)
\]

and

\[
\left| \lim_{k \to \infty} \frac{1}{A_k} \sum_{n=1}^{A_k} \exp \left( 2\pi i n \sum_{l=1}^L \frac{\alpha_l}{k} h_l \right) \right| = \left| \frac{1}{T} \int_0^T \exp \left( 2\pi i x \sum_{l=1}^L \alpha_l h_l \right) dx \right|
\]

\[
\leq \frac{1}{T} \min \left( \pi \left| \sum_{l=1}^L \alpha_l h_l \right|^{-1}, T \right)
\]

Lemma 18 follows.

Lemma 19 Let \( |\Delta x| \leq 0.01 \) and \( M \geq 355\,991 \). Then for \( |s| \leq r < 1/4 \) we have

\[
\left| \sum_{p \leq M} \log \left( 1 - \frac{e^{-2\pi i (x+\Delta x)}}{p^{s+\frac{3}{4}}} \right) - \sum_{p \leq M} \log \left( 1 - \frac{e^{-2\pi i x}}{p^{s+\frac{3}{4}}} \right) \right| \leq 993 \Delta x M^{\frac{1}{4}+r} \log M.
\]

Proof. For \( |\Delta x| \leq 0.01 \) we obtain

\[
\left| \log \left( 1 - \frac{e^{-2\pi i (x+\Delta x)}}{p^{s+\frac{3}{4}}} \right) - \log \left( 1 - \frac{e^{-2\pi i x}}{p^{s+\frac{3}{4}}} \right) \right| \\
\leq \sqrt{\log^2 \left( 1 + \frac{|e^{-2\pi i (x+\Delta x)} - e^{-2\pi i x}|}{p^{s+\frac{3}{4}} - 1} \right) + \arctan^2 \left( \frac{|e^{-2\pi i (x+\Delta x)} - e^{-2\pi i x}|}{p^{s+\frac{3}{4}} - 1} \right)} \\
\leq \frac{14 \Delta x}{p^{s+\frac{3}{4}} - 1}.
\]

Thus,

\[
\left| \sum_{p \leq M} \log \left( 1 - \frac{e^{-2\pi i (x+\Delta x)}}{p^{s+\frac{3}{4}}} \right) - \sum_{p \leq M} \log \left( 1 - \frac{e^{-2\pi i x}}{p^{s+\frac{3}{4}}} \right) \right| \\

14 \Delta x \sum_{p \leq 355\,990} \frac{1}{\sqrt{p} - 1} + 14 \Delta x \sum_{355\,991 \leq p \leq M} \frac{1}{p^{\frac{3}{2} - r} - 1}.
\]
Calculating the first sum by computer and the second sum by partial summation, in view of inequalities from Lemma 4 and

\[
\int_{355991}^{M} \frac{x^{r-\frac{3}{4}}}{\log x} dx \leq 5.83 \frac{M^{\frac{1}{4}+r}}{\log M},
\]

we obtain the lemma.

**Proof of Proposition 16.** Let \( J \) be equal to the integral part of \( V(Q/\rho)^{1/4+r} \). In \( \mathbb{R}^{\pi(Q)} \) we define the set

\[
U(\theta) := \left\{ x = (x_p)_{p \leq Q} \mid \theta_p - \frac{1}{2V} \leq x_p < \theta_p + \frac{1}{2V} \text{ for } p \leq \rho \text{ and } \theta_p - \frac{1}{2J} \leq x_p < \theta_p + \frac{1}{2J} \text{ for } \rho < p \leq Q \right\},
\]

where \( \theta_p \) for \( \rho < p \leq Q \) will be specified later. We now apply Lemma 18 (also see Note below this lemma) with \( L = \pi(Q) \), \( U = U(\theta) \) and \( \alpha = \frac{1}{2\pi}(\log p)\rho \leq Q \). Thus we have

\[
T_{\alpha}(U(\theta)) \geq 0.52TV^{-\pi(\rho)}J^{\pi(\rho)-\pi(Q)},
\]

if the inequalities

\[
\left| \prod_{p \leq \rho} \left( 1 + \frac{75V}{H_p} \right) \prod_{\rho < p \leq Q} \left( 1 + \frac{75J}{H_p} \right) - 1 \right| \leq 0.47
\]

and

\[
\sum_{(h)} \left| \sum_{p \leq Q} \frac{h_p \log p}{2} \right|^{-1} \prod_{p \leq Q} \gamma_{h_p, p} \leq 0.01TV^{-\pi(\rho)}J^{\pi(\rho)-\pi(Q)}
\]

hold with some positive numbers \( H_p > 1, p \leq Q \). We set \( H_p = H/\log Q \) for \( p \leq Q \) and assume that \( H \geq Q \). By the inequality \((1 + x)^y - 1 \leq xy e^{xy}\), for \( x, y \geq 0 \), we have that the left hand-side of (22) is smaller than

\[
75J \frac{\log Q}{H} \pi(Q) \exp \left( 75J \frac{\log Q}{H} \pi(Q) \right).
\]

Then, in view of Lemma 4, we deduce that (22) is true if

\[
\frac{JQ}{H} \leq 0.004
\]

On the other hand

\[
\left| \frac{1}{2} \sum_{p \leq Q} h_p \log p \right|^{-1} = \left| \frac{1}{2} \log \prod_{p \leq Q} p^{h_p} \right|^{-1} \leq 2 \prod_{p \leq Q} p^{|h_p|}.
\]
By the choice of \( H_p \) and by the inequality for the prime counting function (Lemma 4) we obtain
\[
H_p \left(1 + \min \left(\log H_p, \log \pi(Q)\right)\right) \leq \frac{H}{\log Q} \left(1 + \min \left(\log \frac{H}{\log Q}, \log \frac{1.1Q}{\log Q}\right)\right) \leq 1.1H.
\]
Thus by (25) we have that the left-hand side of (23) is less than
\[
2 \sum_{(h)} \prod_{p \leq Q} p^{\left| h_p \right|} \gamma_{h_p, p} \leq 2 \prod_{p \leq Q} \left(1 + 60 \sum_{1 \leq h_p \leq 1.1H} \frac{p^{\left| h_p \right|}}{h_p}\right)
\leq 200 \prod_{p \leq Q} \frac{p^{2H}}{1.1H \log p} = 200 \frac{\exp \left\{1.1H \sum_{p \leq Q} \log p\right\}}{1.1H \exp \left\{\sum_{p \leq Q} \log \log p\right\}} \leq e^{1.12HQ-355991},
\]
here we used inequalities \( \sum_{p \leq x} \log p < 1.01624x, \ x > 0 \) and \( \pi(x) \geq x/\log x, \ x \geq 17, \) obtained by Rosser and Schoenfeld ([11], Theorem 9 and Corollary 1). Therefore we have that (23) is true if
\[
e^{1.12HQ} \leq TV^{-\pi(\rho)} J^{\pi(\rho)-\pi(Q)}.
\]
The last inequality is valid provided \( 1.2HQ \leq \log T \). By this and (24) we have that (21) is true if
\[
(26) \quad 300VQ^2 \left(\frac{Q}{\rho}\right)^{\frac{4}{3}} \leq \log T.
\]
Now let \( \sum_{(j)} \) denotes the summation over all \( j = (j_p)_{\rho < p \leq Q} \), where the \( j_p \) are positive integers not exceeding \( J \). Then, using Lemma 4 we obtain that
\[
\sum_{(j)} \left\| \sum_{\rho < p \leq Q} \log \left(1 - \frac{e^{-2\pi i j_p/J}}{p^{2+s}}\right)\right\|^2 = J^{\pi(Q)-\pi(\rho)} \sum_{\rho < p \leq Q} \sum_{n=1}^{\infty} \frac{1}{n^2 p^{2n(\sigma+3/4)}}
\leq J^{\pi(Q)-\pi(\rho)} \zeta(2) \sum_{\rho < p \leq Q} \frac{1}{p^{2(\sigma+3/4)}} \leq J^{\pi(Q)-\pi(\rho)} \frac{24}{\rho^{\frac{1}{2}-2\sigma} \log \rho}.
\]
Further we assume that the \( \theta_p, \ \rho < p \leq Q, \) are of the form \( j_p/J \), where the positive integers \( j_p \) do not exceed \( J \). Hence (20) and the last inequality imply that we can find more than \( 0.98J^{\pi(Q)-\pi(\rho)} \) disjoint sets \( U(\theta) \) which agree in all the \( \theta_p \) for \( p \leq \rho \) and satisfy
\[
(27) \quad \left\| \sum_{\rho < p \leq Q} \log \left(1 - \frac{e^{-2\pi i j_p/J}}{p^{2+s}}\right)\right\| \leq 0.02 \sqrt{2.4} \sqrt{\log \rho};
\]
the sum of such \( U(\theta) \) we will denote by \( A \). Thus in view of (21) we have that the measure of the set \( \tau \in (0, T) \) such that \( \frac{\tau}{\pi} \left(\log p\right)_{p \leq Q} \) belongs to \( A \) mod 1 is not less than \( 0.98J^{\pi(Q)-\pi(\rho)} \times 0.52TV^{\pi(\rho)} J^{\pi(\rho)-\pi(Q)} \). Then Proposition 16 follows by Lemma 19 and by the inequalities (26) and (27).
References


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