The effective universality theorem for the Riemann zeta function

Ramūnas Garunkštis*

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Abstract. It is known that Voronin's universality theorem for the Riemann zeta-function is ineffective. For some partial cases we obtain the effective version of this theorem.

1 Introduction

The famous Voronin's universality theorem states: *let* $0 < r < 1/4$. *Suppose that* $g(s)$ *is an analytic function in the interior of the disc* $|s| \leq r$ *and is continuous up to the boundary of this disc. Then for every* $\varepsilon > 0$ *there exists* $\tau = \tau(\varepsilon)$ *such that*

$$\max_{|s|\leq r} |g(s) - \log \zeta(s + 3/4 + i\tau)| < \varepsilon,$$

where the branch of $\log \zeta(s)$ *is taken that is real for* $s \in \mathbb{R}$ *and is extended analytically along the intervals* $[2, 2+it]$, $[2+it, \sigma+it]$. Really Voronin proved more: that for every $\varepsilon > 0$

$$(1) \qquad d(\varepsilon) := \liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \max_{|s|\leq r} |g(s) - \log \zeta(s + 3/4 + i\tau)| < \varepsilon\right\} > 0.$$

Note that the universality theorem can be formulated for $\zeta(s)$ instead of $\log \zeta(s)$ (see Karatsuba and Voronin [6], chapter 7.1, Theorem 2). For further development on universality theorems see Laurinčikas [8], Matsumoto [10]. Until now no bounds are known for $\tau(\varepsilon)$ and $d(\varepsilon)$. Voronin's proof is ineffective. He uses the theorem of Pecherskii on the rearrangement of terms in functional series. An interesting approach to solve this problem was done by Good [4]. Extending on his ideas we prove

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Theorem 1 *Let ε, β, r and R be such that $0 < \varepsilon \leq 1$, $0 < r < R < 1/4$, $0 < \beta + R < 1/4$ and $r < \delta e^{-1-\frac{1}{4\delta}}$, where*

$$\delta := \frac{\frac{1}{4} - R - \beta}{\log \frac{e}{2R}}.$$

Let $g(s)$ be analytic in $|s| \leq R$, $\rho > 355991^{\frac{1}{1-4\delta}}$ and $\delta \log \rho$ be an integer. Let

$$M := \max_{|s| \leq R} |g(s)| + 0.8 + \frac{3.5}{1-4R} \leq \frac{\rho^{\beta}}{5\delta^3 \log^4 \rho}$$

and let V, Q and T satisfies the following conditions:

$$50 \leq V \leq \rho \leq Q \leq \min\left\{ \frac{0.07\rho^{\frac{1}{9}} \log^{\frac{4}{9}} T}{V^{\frac{4}{9}}}, 0.99 \log T \right\}.$$

Then

$$\alpha := \delta \log \frac{\delta}{er} - \frac{1}{4}$$

is positive and the measure of $\tau \in [T/2, T]$, such that

$$\max_{|s| \leq r} \left| \log \zeta \left(s + \frac{3}{4} + i\tau \right) - g(s) \right| < \frac{M}{\left(\frac{R}{r} - 1\right) \rho^{\delta \log \frac{R}{r}}} + \frac{3}{\rho^{\alpha} \log \rho}$$

$$+ \frac{3 \log \rho}{\rho^{(1-4\delta)(\frac{3}{4}-r)}} + \frac{3}{\rho^{(1-4\delta)(\frac{1}{2}-2r)} \log \rho} + \varepsilon + 500 \frac{\rho^{\frac{1}{4}+r}}{V \log \rho} + \frac{80}{\rho^{\frac{1}{4}-r} \log^{\frac{1}{2}} \rho}$$

is greater than

$$\frac{T}{2} \left(\frac{1}{2} V^{-\pi(\rho)} - 63 \frac{\log^2 Q}{\varepsilon^2 Q^{0.249}} \right),$$

where $\pi(x)$ denotes the number of primes not exceeding x.

Unfortunately Theorem 1 is valid only for discs with very small radius r. It is easy to calculate that r is less than $0.00036\ldots$ ($R = 0.06\ldots$, $\beta = 0$). The requirement that $\delta \log \rho$ must be an integer is technical. It can be avoided relaxing the accuracy of the approximation. The next is an example of the theorem for concrete values of r and R:

Corollary 2 *Let $0 < \varepsilon \leq 1/2$. Let the function $g(s)$ be analytic in the disc $|s| \leq 0.05$ and assume $\max_{|s| \leq 0.06} |g(s)| \leq 1$. Let $r = 0.0001$. Then*

$$\log \tau(\varepsilon) \leq e^{\frac{10}{\varepsilon^{13}}} \qquad \text{and} \qquad d(\varepsilon) \geq e^{-\frac{1}{\varepsilon^{13}}}.$$

Analogously to the lower density $d(\varepsilon)$ we can define the upper density $D(\varepsilon)$ by replacing 'lim inf' by 'lim sup' in (1). In [13] Steuding obtained nontrivial upper bounds for $D(\varepsilon)$ for some special class of functions.

An other way to prove universality theorems is to use the method of limit measures (see, for example, Laurinčikas [8]). In this case effectivization problems are discussed in Laurinčikas [9].

The proof of Theorem 1 is divided in to three independent parts (Sections 2, 3 and 4) so that the theorem immediately follows from Propositions 3, 8 and 16. The mentioned restriction on the radius r appears only in the first part (Section 2).

We will use the notation $\Theta(\alpha)$ to indicate 'some number not greater in modulus than α'.

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2 Approximation by trigonometric polynomials

The aim of this chapter is

Proposition 3 *Let β, r and R be such that $0 < r < R < 1/4$, $0 < \beta + R < 1/4$ and $r < \delta e^{-1-\frac{1}{4\delta}}$, where*

$$\delta := \frac{\frac{1}{4} - R - \beta}{\log \frac{e}{2R}}.$$

Then

$$\alpha := \delta \log \frac{\delta}{er} - \frac{1}{4}$$

is positive.

Let $g(s)$ be analytic in $|s| \le R$ and let

$$M := \max_{|s| \le R} |g(s)| + 0.8 + \frac{3.5}{1-4R} \le \frac{\rho^\beta}{5\delta^3 \log^4 \rho}.$$

Let $\rho > 355991^{\frac{1}{1-4\delta}}$ and $\delta \log \rho$ be an integer. Then there are real numbers θ_p, $p \le \rho$, such that

$$\begin{aligned} g(s) = & - \sum_{p \le \rho} \log \left(1 - \frac{e^{-2\pi i \theta_p}}{p^{s+\frac{3}{4}}}\right) \\ & + \Theta\left(\frac{M}{\left(\frac{R}{r} - 1\right) \rho^{\delta \log \frac{R}{r}}} + \frac{3}{\rho^\alpha \log \rho} + \frac{3 \log \rho}{\rho^{(1-4\delta)(\frac{3}{4}-r)}} + \frac{3}{\rho^{(1-4\delta)(\frac{1}{2}-2r)} \log \rho}\right) \end{aligned}$$

for $|s| \le r$. Moreover one can choose $\theta_{p_n} = n/2$ for $p_n \le \rho^{1-4\delta}$.

For the proof of the proposition the following lemmas will be useful.

Lemma 4 *We have that,*

$$\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right)\le \pi(x) \quad \textit{for} \quad x\ge 32\,299,$$

$$\pi(x)\le \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right)< 1.094\frac{x}{\log x} \quad \textit{for} \quad x\ge 355\,991,$$

and

$$\pi(x)\le \frac{x}{\log x}\left(1+\frac{1.2762}{\log x}\right) \quad \textit{for} \quad x>1.$$

Proof. The inequalities are obtained by Dusart [2, 3].

Lemma 5 *Let $\Delta_{\lambda\rho}$, $\lambda<\rho$, denote the set of vectors $z=(z_p)_{\lambda<p\le\rho}$ with complex components z_p of modulus ≤ 1. Let g_k, $k=0,1,\dots,$ be functions on $\Delta_{\lambda\rho}$ defined by*

$$g_k(z)=\sum_{\lambda<p\le\rho} z_p p^{-\sigma}(-\log p)^k, \quad \sigma\le 1-\varepsilon.$$

Let K be a positive integer and w_k, $k=0,\dots,K$, complex numbers such that

$$K^3\le 0.06\log^2\lambda\log\frac{\rho}{\lambda} \tag{2}$$

and

$$|w_k|\le \frac{\lambda^{1-\sigma}\log\rho}{10K^3\log\lambda}\left(\frac{1-\frac{\log\lambda}{\log\rho}}{2K}\right)^{K+1} k!(K-k)!\log^k\rho.$$

Then the system of equations

$$g_k(z)=w_k, \quad k=0,\dots,K,$$

has a solution z in $\Delta_{\lambda\rho}$ for $\lambda\ge 355\,991$.

Proof. This is the quantitative version of Lemma 9 from Good [4]. To obtain the explicit constants we use Lemma 4.

Lemma 6 *Let K and L be positive integers and $K \leq L$. Let a_{kl} and b_k, $1 \leq k \leq K$, $1 \leq l \leq L$, denote complex numbers. Suppose that the system of equations*

$$\sum_{l=1}^{L} a_{kl} z_l = b_k, \qquad 1 \leq k \leq K, \tag{3}$$

has a solution $(z_1, \ldots, z_L)$ belonging to

$$\Delta^L = \{(z_1, \ldots, z_L)|\ z_L \text{ complex and } |z_l| \leq 1 \text{ for } 1 \leq l \leq L\}.$$

Then (3) also has a solution $(z_1', \ldots, z_L')$ in Δ^L such that $|z_l'| = 1$ for at least $L - K$ positive integers $l \leq L$.

Proof. This is Lemma 6 from Good [4].

Lemma 7 *Let $\sigma > 1/2$, $355991 \leq \lambda < \rho$ and $\theta_p \in \mathbb{R}$. Then*

$$-\sum_{\lambda < p \leq \rho} \log\left(1 - \frac{e^{-2\pi i \theta_p}}{p^s}\right) = \sum_{\lambda < p \leq \rho} \frac{e^{-2\pi i \theta_p}}{p^s} + \Theta\left(\frac{1.11\sigma}{2\sigma - 1} \frac{1}{\lambda^{2\sigma-1} \log \lambda}\right).$$

Proof. We expand the logarithm by power series, then we use the partial summation and Lemma 4.

Proof of Proposition 3. Let $f(s)$ be analytic in $|s| \leq R$. Then

$$f(s) = \sum_{i=0}^{K} a_i s^i + \Theta\left(\frac{\left(\frac{r}{R}\right)^K \max_{|u| \leq R} |f(u)|}{\frac{R}{r} - 1}\right), \quad |s| \leq r, \tag{4}$$

where the coefficients obey the bounds

$$|a_i| \leq \frac{\max_{|u| \leq R} |f(u)|}{R^i}, \quad i = 0, \ldots, K. \tag{5}$$

1) With help of Lemmas 5 and 6, we will approximate the above polynomial by the exponential polynomial $\sum_{\lambda < p \leq \rho} e^{-2\pi i \theta_p} p^{-s-\frac{3}{4}}$ with appropriate λ, ρ and θ_p.

Let's choose

$$K := \delta \log \rho \quad \text{and} \quad \lambda := \rho \exp(-4K) = \rho^{1-4\delta}$$

Note, that by the conditions of the proposition, δ is less than 0.1 and r is less than 0.001. Hence K satisfies (2).

By Lemma 4 we have that

$$\sum_{\lambda < p \leq \rho} p^{-\frac{3}{4}} \leq \frac{6\rho^{\frac{1}{4}}}{\log \rho}.$$

Also we will use the well known inequality

(6) $$n! > \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Then, for $|z_p| \le 1$ and $|s| \le r$, we obtain that

$$\left| \sum_{\lambda<p\le\rho} z_p p^{-s-\frac{3}{4}} - \sum_{k=0}^{K} s^k \sum_{\lambda<p\le\rho} z_p p^{-\frac{3}{4}} \frac{(-\log p)^k}{k!} \right|$$
$$\le \sum_{\lambda<p\le\rho} p^{-\frac{3}{4}} \sum_{k>K} \frac{(r\log p)^k}{k!} \le \frac{3}{\rho^\alpha \log\rho}.$$

If $\sigma = 3/4$ and $|w_k|$, $0 \le k \le K$, are complex numbers satisfying

$$|w_k| \le \frac{1}{5}\rho^{\frac{1}{4}}(\log\rho)^{k-K-1}K^{-3}(K-k)!\left(\frac{e}{2}\right)^{-K}, \qquad 0 \le k \le K,$$

then Lemma 5 shows the existence of a $|z_p| \le 1$ such that

(7) $$\sum_{k=0}^{K} s^k \sum_{\lambda\le p\le\rho} z_p p^{-\frac{3}{4}} \frac{(-\log p)^k}{k!} = \sum_{k=0}^{K} s^k w_k.$$

If $R > 0$ and $\beta = 1/4 - R - \delta\log\frac{e}{2R}$ then

$$\rho^{\frac{1}{4}}\left(\frac{e}{2}\right)^{-K} R^{K-k}\frac{(K-k)!}{(R\log\rho)^{K-k}} \ge \rho^{\frac{1}{4}-R}\left(\frac{2R}{e}\right)^{K} R^{-k} = \frac{\rho^\beta}{R^k}.$$

Thus (7) is solvable for

$$|w_k| \le \frac{\rho^\beta}{5\delta^3 R^k \log^4\rho}.$$

If

$$\max_{|u|\le R} |f(u)| \le \frac{\rho^\beta}{5\delta^3\log^4\rho},$$

then, in view of (5), the coefficients a_k satisfies the same bound as w_k. Therefore there exists $|z_p| \le 1$ for $\lambda < p \le \rho$, such that

(8) $$f(s) = \sum_{\lambda<p\le\rho} \frac{z_p}{p^{s+\frac{3}{4}}} + \Theta\left(\frac{3}{\rho^\alpha\log\rho} + \frac{\max_{|u|\le R}|f(u)|}{\left(\frac{R}{r}-1\right)\rho^{\delta\log\frac{R}{r}}}\right)$$

if $|s| \le r$.

Now we will show, that the z_p in (8) can be replaced by arbitrary complex numbers of modulus = 1. We have

$$\sum_{\eta<p\le 2\eta}\frac{z_p}{p^{s+\frac{3}{4}}}=\sum_{0\le k\le N}s^k\sum_{\eta<p\le 2\eta}\frac{z_p}{p^{\frac{3}{4}}}\frac{(-\log p)^k}{k!}+\Theta\left(\eta^{\frac{1}{4}}\sum_{k>N}\frac{0.001^k\log^k 2\eta}{k!}\right)\tag{9}$$

for $|s|\le r$. If N is the integral part of $0.3\log\eta$ then by (6) we have for the error term,

$$\eta^{\frac{1}{4}}\sum_{k>N}\frac{0.001^k\log^k 2\eta}{k!}\le\frac{\eta^{\frac{1}{4}}}{\sqrt{2\pi(N+1)}}\sum_{k>N}e^{-k(\log(N+1)-1-\log\log 2\eta+\log 1000)}$$
$$\le 0.3\eta^{-\frac{3}{4}},\quad (\eta>355991).$$

Next we apply Lemma 6 to the system

$$b_k=\sum_{\eta<p\le 2\eta}\frac{(-\log p)^k}{k!}z_p,\quad k=0,\dots,N.$$

Thus for any complex numbers z_p with $|z_p|\le 1$, $\eta<p\le 2\eta$, there are complex numbers z_p' with $|z_p'|\le 1$, $\eta<p\le 2\eta$, and $|z_p'|=1$ for at least $\eta-N-1$ primes p such that

$$\sum_{\eta<p\le 2\eta}\frac{(-\log p)^k}{k!}z_p=\sum_{\eta<p\le 2\eta}\frac{(-\log p)^k}{k!}z_p',\quad k=0,\dots,N.$$

Hence we have by (9)

$$\sum_{\eta<p\le 2\eta}\frac{z_p}{p^{s+\frac{3}{4}}}=\sum_{\eta<p\le 2\eta}\frac{z_p'}{p^{s+\frac{3}{4}}}+\Theta(0.6\eta^{-\frac{3}{4}}).$$

Therefore real numbers θ_p can be found such that

$$\sum_{\eta<p\le 2\eta}\frac{z_p}{p^{s+\frac{3}{4}}}=\sum_{\eta<p\le 2\eta}e^{-2\pi i\theta_p}p^{-s-\frac{3}{4}}+\Theta\left((N+1)\eta^{r-\frac{3}{4}}+0.6\eta^{-\frac{3}{4}}\right)$$

for $|s|\le r$. Using the last equality with $\eta=2^j\lambda$ for $1\le j\le\log(\rho/\lambda)/\log 2-1$, in view of $\sum_{j=1}^{\infty}j2^{-jx}=2^x(2^x-1)^{-2}$, we see that for any z_p with $|z_p|\le 1$, $\lambda<p\le\rho$, there are real numbers θ_p, $\lambda<p\le\rho$, such that

$$\sum_{\lambda<p\le\rho}z_p p^{-s-\frac{3}{4}}=\sum_{\lambda<p\le\rho}e^{-2\pi i\theta_p}p^{-s-\frac{3}{4}}+\Theta\left(1.1\lambda^{r-\frac{3}{4}}\log\lambda\right)\tag{10}$$

for $\lambda\ge 355991$ and $|s|\le r$.

2) To finish the proof we apply the first part of the proof to

$$f(s)=g(s)+\sum_{p_n\le\lambda}\log\left(1-\frac{(-1)^n}{p_n^{s+\frac{3}{4}}}\right).$$

Expanding by power series and by partial summation we obtain that for $|s| \leq R$,

$$\left|\sum_{p_n \leq \lambda} \log\left(1 - \frac{(-1)^n}{p_n^{s+\frac{3}{4}}}\right)\right| \leq \left|\sum_{p_n \leq \rho^{1-4\delta}} \frac{(-1)^n}{p_n^{s+\frac{3}{4}}}\right| + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=2}^{\infty} \frac{1}{n^{k\left(\frac{3}{4}-R\right)}}$$

$$\leq \frac{\left|s+\frac{3}{4}\right|}{\sigma+\frac{3}{4}} 2^{R-\frac{3}{4}} + \frac{0.5}{1-2^{R-\frac{3}{4}}} \int_1^{\infty} x^{2R-\frac{3}{2}} dx < 0.8 + \frac{3.42}{1-4R}.$$

Then the Proposition 3 follows from (8), (10) and Lemma 7.

3 Approximation of $\zeta(s)$ by a finite product

Let

$$\zeta_Q(s) = \prod_{p \leq Q} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $Q \in \mathbb{N}$ and $s \in \mathbb{C}$.

Proposition 8 *Let $0 < r < 1/4$ and $0 < \varepsilon \leq 1$. The measure of $\tau \in [T/2, T]$, such that*

$$\max_{|s| \leq r} \left|\log \zeta\left(s + \frac{3}{4} + i\tau\right) - \log \zeta_Q\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon$$

is greater than

$$\frac{T}{2}\left(1 - 0.06 \frac{\log^2 Q}{(0.25-r)^5 Q^{0.25-r} \varepsilon^2}\right),$$

if

$$T \geq \pi e^{\frac{1.02(0.25+r)Q}{0.25-r}}, \quad Q > 355991.$$

The proposition we will derive from the following lemmas.

Lemma 9 *Let $f(x)$ be a real-valued function on the interval $[a, b]$, and let $f'(x)$ be continuous and monotonic on $[a, b]$ and $|f'(x)| \leq \delta < 1$. Then*

$$\text{(11)} \qquad \sum_{a<n\leq b} e(f(n)) = \int_a^b e(f(x))\, dx + \Theta\left(\frac{4\sqrt{2}\delta}{\pi(1-\delta)} + \frac{6\sqrt{2}\delta}{\pi} + 3\right).$$

Here $e(x) = e^{2\pi i x}$.

Proof of this lemma can be found, for example, in Ivić [5]. To calculate the exact constant in the error term we use the following evaluation:

$$\sum_{k=1}^{\infty} \frac{1}{k(k-\delta)} \leq \frac{1}{1-\delta} + \frac{1}{2(2-\delta)} + \int_1^{\infty} \frac{dx}{x^2} \leq \frac{1}{1-\delta} + \frac{3}{2}.$$

Lemma 10 *For $\sigma \geq 0$, $x \geq |t|/\pi$, $x \geq 1$, $s = \sigma + it$,*

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{s-1} + \Theta\left(\frac{7\sqrt{2}\pi^{-1} + 3}{x^{\sigma}}\right). \tag{12}$$

Proof. For $\sigma \geq \sigma_0 > 0$ we have (see, for example, §3.5 of Titchmarsh [14] or §1.5 of Ivić [5]):

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} - s\int_N^{\infty} \left(\frac{1}{2} - \{u\}\right) u^{-s-1} du, \tag{13}$$

where the last summand in modulus is $\leq \frac{|s|}{\sigma_0} N^{-\sigma}$. For $u \geq x$ we set

$$A(u) = \sum_{x < n \leq u} n^{-it}$$

and apply Lemma 9 with $f(x) = (2\pi)^{-1} t \log x$, $\delta = \frac{1}{2}$, provided that $x \geq |t|/\pi$. Then

$$A(u) = \frac{u^{1-it} - x^{1-it}}{1-it} + \Theta\left(\frac{7\sqrt{2}}{\pi} + 3\right).$$

For $x \leq N$, partial summation gives

$$\begin{aligned} \sum_{x < n \leq N} n^{-s} &= \sigma \int_x^N \frac{A(u) du}{u^{\sigma+1}} + \frac{A(N)}{N^{\sigma}} \\ &= \frac{N^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + \frac{(7\sqrt{2}\pi^{-1} + 3)}{x^{\sigma}} + O\left(\frac{x}{N^{\sigma}}\right), \quad N \to \infty. \end{aligned}$$

From this and (13) it follows that (12) is valid for $\sigma \geq \sigma_0$, where σ_0 is as close to 0 as we like. By a continuity of $\zeta(s)$ the formula (12) remains valid for $\sigma \geq 0$. Lemma 10 is proved.

Lemma 11 *We have*

$$\sum_{1 \leq n \leq X} \frac{1}{n} = \log x + \gamma + \Theta\left(\frac{2}{X}\right),$$

where γ is Euler's constant.

Proof. See, for example, Chandrasekharan [1], §6.3.

Let $\tau(n)$ denotes a number of divisors of n including 1 and n.

Lemma 12 *We have,*

$$D(x) := \sum_{n\leq x} \tau(n) = x\log x + \Theta(\gamma x + 2)$$

and for $x > 10$*,*

$$\begin{aligned} D_2(x) := \sum_{n\leq x} \tau^2(n) &\leq \frac{1}{6} x\log^3 x + \frac{1}{2}(1+2\gamma)\, x\log^2 x + \left(4 + 2\gamma + \gamma^2\right) x\log x \\ &\quad + \left(4\gamma + \gamma^2\right) x + 4 \leq 1.92\, x\log^3 x. \end{aligned}$$

For $x \geq 355991$*,*

$$D_2(x) \leq 0.29 x\log^3 x$$

where $\tau(n)$ *is a number of divisors of* n *including* 1 *and* n*.*

Proof. By Lemma 11 and by

$$0 \leq \sum_{n\leq x} \left\{\frac{x}{n}\right\} \leq x$$

we obtain

$$D(x) = \sum_{n\leq x} \left[\frac{[x]}{n}\right] = \sum_{n\leq x} \left[\frac{x}{n}\right] = \sum_{n\leq x} \frac{x}{n} - \sum_{n\leq x} \left\{\frac{x}{n}\right\} = x\log x + \Theta(x\gamma + 2).$$

From $\tau(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}) = (\alpha_1+1)(\alpha_2+1)\dots(\alpha_n+1)$ it follows, that $\tau(mn) \leq \tau(m)\tau(n)$. Thus, in view of the first part of the proof, and using summation by parts, we have

$$\begin{aligned} \sum_{n\leq x} \tau^2(n) &= \sum_{\substack{m,n\leq x \\ mn\leq x}} \tau(mn) \leq \sum_{m\leq x} \tau(m) \sum_{n\leq \frac{x}{m}} \tau(n) \\ &\leq \sum_{m\leq x} \tau(m) \left(\frac{x}{m}\log\frac{x}{m} + \frac{\gamma x}{m} + 2\right) \\ &= \gamma D(x) + 2D(x) - \int_1^x D(y) \left(\frac{x}{y}\log\frac{x}{y} + \frac{\gamma x}{y}\right)'_y dy \\ &\leq (\gamma+2)(x\log x + \gamma x + 2) \\ &\quad - \int_1^x (y\log y + \gamma y + 2) \left(\frac{x}{y}\log\frac{x}{y} + \frac{\gamma x}{y}\right)'_y dy \\ &= \frac{1}{6} x\log^3 x + \frac{1}{2}(1+2\gamma)\, x\log^2 x + \left(4 + 2\gamma + \gamma^2\right) x\log x \\ &\quad + \left(4\gamma + \gamma^2\right) x + 4. \end{aligned}$$

The remaining inequalities of Lemma 12 easily follows from the above inequality.

Lemma 13 *Let $a_1, \dots, a_n$ be arbitrary complex numbers. Then*

$$\int\limits_{\frac{T}{2}}^{T} \left| \sum_{n \le N} a_n n^{it} \right|^2 dt = \frac{T}{2} \sum_{n \le N} |a_n|^2 + \Theta \left(837 \sum_{n \le N} n|a_n|^2 \right).$$

Proof. This Lemma is a quantitative version of the well known statement. To calculate an exact constant in the error term we follow lines of the proof of Theorem 5.2 from Ivić [5]. Squaring and integrating, we obtain

$$\int\limits_{\frac{T}{2}}^{T} \left| \sum_{n \le N} a_n n^{it} \right|^2 dt = \frac{T}{2} \sum_{n \le N} |a_n|^2 + \Theta \left(2 \sum_{m \ne n \le N} \frac{a_m \bar{a}_n}{\log m - \log n} \right). \tag{14}$$

Then (for details see Ivić [5], formula (5.11) and above),

$$\begin{aligned} &\left| \sum_{m \ne n \le N} \frac{a_m \bar{a}_n}{\log m - \log n} \right| \\ &\qquad \le \pi \sum_{n \le N} |a_n|^2 + \sum_{k,l \ge 1} \left| \sum_{(m,n) \in I_k \times I_l} \int\limits_0^1 a_m \bar{a}_n \frac{e^{2\pi i y (\log m - \log n)}}{\log m - \log n} dy \right|, \end{aligned} \tag{15}$$

where $I_j = (N2^{-j}, N2^{1-j}]$, $j = 1, 2, \dots$. For $|k - l| \ge 2$,

$$\begin{aligned} &\left| \sum_{(m,n) \in I_k \times I_l} \int\limits_0^1 a_m \bar{a}_n \frac{e^{2\pi i y (\log m - \log n)}}{\log m - \log n} dy \right| \\ &\qquad \le \frac{1}{\pi} \sum_{(m,n) \in I_k \times I_l} |a_m a_n| \max_{(m,n) \in I_k \times I_l} (\log m - \log n)^{-2} \\ &\qquad \le \frac{9}{\pi} (k - l)^{-2} \left(\sum_{(m,n) \in I_k \times I_l} |a_m|^2 \right)^{\frac{1}{2}} \left(\sum_{(m,n) \in I_k \times I_l} |a_n|^2 \right)^{\frac{1}{2}} \\ &\qquad \le \frac{9}{\pi} (k - l)^{-2} N 2^{-\frac{k+l}{2}} \left(\sum_{m \in I_k} |a_m|^2 \right)^{\frac{1}{2}} \left(\sum_{m \in I_l} |a_n|^2 \right)^{\frac{1}{2}} \le \frac{9}{\pi} (k - l)^{-2} (S_k S_l)^{\frac{1}{2}}, \end{aligned}$$

where

$$S_j = \sum_{n \in I_j} n|a_n|^2.$$

Here we used the Cauchy-Schwarz inequality and

$$|\log m - \log n| \geq \log(N2^{-l}) - \log(N2^{1-k}) = (\log 2)(k-l-1) \geq \frac{1}{3}(k-l),$$

which holds if $k-l \geq 2$, while the case $l-k \geq 2$ is analogous. A further application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} \sum_{|k-l|\geq 2} (k-l)^{-2}(S_k S_l)^{\frac{1}{2}} &\leq \left(\sum_{|k-l|\geq 2} S_k (k-l)^{-2}\right)^{\frac{1}{2}} \left(\sum_{|k-l|\geq 2} S_l (k-l)^{-2}\right)^{\frac{1}{2}} \\ &\leq 1.3 \sum_{n\leq N} n|a_n|^2, \end{aligned}$$

as $\sum_{k>0,|k-l|\geq 2}(k-l)^{-2} \leq 2\zeta(2) - 2 \leq 1.3$.

For the terms in (15) with $|k-l| \leq 1$ we write

$$\sum_{\substack{(m,n)\in I_k\times I_l \\ m\neq n}} \int_0^1 a_m \bar{a}_n \frac{e^{2\pi i y(\log m - \log n)}}{\log m - \log n} dy = M \int_0^1 \sum_{\substack{(m,n)\in I_k\times I_l \\ m\neq n}} \frac{a'_m \bar{a'}_n}{M\log m - M\log n} dy,$$

where $a'_m = a_m \exp(2\pi i y \log m)$ and $M = N2^{3.63-(k+l)/2}$. Here M is chosen such that if $|k-l| \leq 1$, then for $m > n$ and $(m,n) \in I_k \times I_l$

$$\begin{aligned} &[M\log m] - [M\log n] \geq M(\log m - \log n) - 1 = M\log(1 + \frac{m-n}{n}) - 1 \\ &\geq \frac{2^{3.63-\frac{k+l}{2}} 0.46(m-n)}{2^{1-l}} - 1 \geq 2(m-n) - 1 \geq (m-n) \end{aligned}$$

and

$$M\log m - M\log n \geq 2(m-n),$$

since $m-n \geq 1$, $(m-n)/n \leq 3$, and $\log(1+x) \geq 0.46x$ for $0 \leq x \leq 3$; the case $n \geq m$ is analogous. Therefore we have for $|k-l| \leq 1$,

$$\begin{aligned} (16) \qquad M \sum_{\substack{(m,n)\in I_k\times I_l \\ m\neq n}} \frac{a'_m \bar{a'}_n}{M\log m - M\log n} &= M \sum_{\substack{(m,n)\in I_k\times I_l \\ m\neq n}} \frac{a'_m \bar{a'}_n}{[M\log m] - [M\log n]} \\ &+\Theta\left(6.2N2^{-\frac{k+l}{2}} \sum_{\substack{(m,n)\in I_k\times I_l \\ m\neq n}} \frac{|a_m a_n|}{(m-n)^2}\right). \end{aligned}$$

To evaluate the first term on the right of (16) we use the inequality (see Ivić [5], (5.10))

$$(17) \qquad \left|\sum_{m\neq n} \frac{a_m \bar{b}_n}{q_m - q_n}\right| \leq 3\pi \left(\sum |a_n|^2\right)^{\frac{1}{2}} \left(\sum |b_n|^2\right)^{\frac{1}{2}},$$

where $\{q_n\}_{n=1}^{\infty}$ is any sequence of integers such that $q_m \neq q_n$ if $m \neq n$. Then for the left hand-side of (16) we have

$$\left| M \sum_{\substack{(m,n)\in I_k\times I_l \\ m\neq n}} \frac{a'_m \bar{a}'_n}{M\log m - M\log n} \right| \leq \left(3\pi 2^{3.63} + 6.2\cdot 2\cdot\zeta(2)\right)(S_k S_l)^{\frac{1}{2}} \qquad (18)$$

$$\leq 137.09 (S_k S_l)^{\frac{1}{2}}.$$

Using once again the Cauchy-Schwarz inequality we obtain

$$\sum_{|k-l|\leq 1} (S_k S_l)^{\frac{1}{2}} \leq 3 \sum_{n\leq N} n|a_n|^2. \qquad (19)$$

This completes the evaluation for the case $|k-l| \leq 1$. Then in view of the above evaluation for the case $|k-l| \geq 2$ and the inequalities (14) and (15) we obtain Lemma 13.

Lemma 14 *Let $E = \{s : |s| \leq d\}$ with $0 < d < 1/4$. Let $\omega = 1/4 - d$. Then for*

$$T \geq \pi e^{\frac{0.51(1-2\omega)Q-872}{\omega}}, \quad Q > 355\,991,$$

we have, that

$$J := \int_{\frac{T}{2}}^{T} \iint_{E} \left|\zeta\left(s + \frac{3}{4} + i\tau\right)\zeta_Q^{-1}\left(s + \frac{3}{4} + i\tau\right) - 1\right|^2 d\tau d\sigma dt \leq 0.02 \frac{T}{2}\frac{\log^2 Q}{\omega^3 Q^{2\omega}}.$$

Proof. By Lemma 10 for $t \in [T/2, T]$, we have

$$\zeta(s) = \sum_{n\leq \frac{T}{\pi}} \frac{1}{n^s} - \frac{(T/\pi)^{1-s}}{s-1} + \Theta\left(\frac{(7\sqrt{2}\pi^{-1}+3)\pi^{\sigma}}{T^{\sigma}}\right) = \sum_{n\leq\frac{T}{\pi}} \frac{1}{n^s} + \Theta\left(\frac{6.8\pi^{\sigma}}{T^{\sigma}}\right).$$

Then

$$\begin{aligned} J &= \iint_{E+\frac{3}{4}} \left(\int_{\frac{T}{2}}^{T} \left|\zeta(s+i\tau)\zeta_Q^{-1}(s+i\tau) - 1\right|^2 d\tau\right) d\sigma dt \\ &\leq \iint_{E+\frac{3}{4}} \left(\int_{\frac{T}{2}}^{T} \left|\zeta_Q^{-1}(s+i\tau)\sum_{n\leq\frac{T}{\pi}}\frac{1}{n^s} - 1\right|^2 d\tau\right) d\sigma dt \\ &+ 2\left(\iint_{E+\frac{3}{4}}\int_{\frac{T}{2}}^{T} \left|\zeta_Q^{-1}(s+i\tau)\sum_{n\leq\frac{T}{\pi}}\frac{1}{n^s} - 1\right|^2 d\tau d\sigma dt\right)^{\frac{1}{2}} \end{aligned}$$

$$\times\left(\iint_{E+\frac{3}{4}}\frac{6.8^2\pi^{2\sigma}}{T^{2\sigma}}\int_{\frac{T}{2}}^{T}\left|\zeta_Q^{-1}(s+i\tau)\right|^2 d\tau d\sigma dt\right)^{\frac{1}{2}}$$

$$+\quad\iint_{E+\frac{3}{4}}\frac{6.8^2\pi^{2\sigma}}{T^{2\sigma}}\int_{\frac{T}{2}}^{T}\left|\zeta_Q^{-1}(s+i\tau)\right|^2 d\tau d\sigma dt.$$

For $T>\pi Q$ we have

$$\zeta_Q^{-1}(s+i\tau)\sum_{n\le\frac{T}{\pi}}\frac{1}{n^s}-1=\sum_{Q<n\le Q_1T/\pi}\frac{b_n}{n^s},$$

where $Q_1=p_1p_2\dots p_Q$ and $|b_n|\le\tau(n)$. By the inequality (see Schoenfeld [12]), $\sum_{p\le x}\log p<1.000081x$ for $x>0$, we see that $Q_1\le e^{1.01Q}$. Thus, in view of Lemma 13, for $T>\pi Q$,

$$\int_{\frac{T}{2}}^{T}\left|\zeta_Q^{-1}(s+i\tau)\sum_{n\le\frac{T}{\pi}}\frac{1}{n^s}-1\right|^2 d\tau=\int_{\frac{T}{2}}^{T}\left|\sum_{Q<n\le Q_1T/\pi}\frac{b_n}{n^s}\right|^2 d\tau$$

$$\le\frac{T}{2}\sum_{Q<n\le Q_1T/\pi}\frac{\tau^2(n)}{n^{2\sigma}}+837\sum_{Q<n\le Q_1T/\pi}\frac{\tau^2(n)}{n^{2\sigma-1}}$$

$$\le 2\sigma\frac{T}{2}\int_Q^{\infty}\frac{D_2(x)}{x^{2\sigma+1}}dx+837(2\sigma-1)\int_1^{Q_1T/\pi}\frac{D_2(x)}{x^{2\sigma}}dx+\frac{837D_2\left(\frac{Q_1T}{\pi}\right)}{\left(\frac{Q_1T}{\pi}\right)^{2\sigma-1}}.$$

By Lemma 12,

$$\iint_{E+\frac{3}{4}}\left(\sigma T\int_Q^{\infty}\frac{D_2(x)}{x^{2\sigma+1}}dx\right)d\sigma dt\le\frac{1}{2}\int_{\frac{1}{2}+\omega}^{1}\left(\sigma T\int_Q^{\infty}\frac{D_2(x)}{x^{2\sigma+1}}dx\right)d\sigma\le T\frac{0.003\log^2 Q}{\omega^3Q^{2\omega}}.$$

If $\frac{T}{\pi}\ge e^{\frac{0.51(1-2\omega)Q-872}{\omega}}$, then

$$837\iint_{E+\frac{3}{4}}\left((2\sigma-1)\int_Q^{Q_1T/\pi}\frac{D_2(x)}{x^{2\sigma}}dx+\frac{D_2\left(\frac{Q_1T}{\pi}\right)}{\left(\frac{Q_1T}{\pi}\right)^{2\sigma-1}}\right)d\sigma dt$$

$$\le\frac{31}{\omega}\log^3\frac{Q_1T}{\pi}\int_{\frac{1}{2}+\omega}^{1-\omega}\left(\frac{Q_1T}{\pi}\right)^{2-2\sigma}d\sigma\le 0.003T\frac{\log^2 Q}{\omega^3Q^{2\omega}}$$

and, applying Lemma 13 to the Dirichlet polynomial $\zeta_Q^{-1}(s) = \sum_{n=1}^{Q_1} c_n/n^s$, where $|c_n| \leq 1$, we have

$$\iint\limits_{E+\frac{3}{4}} \frac{6.8^2\pi^{2\sigma}}{T^{2\sigma}} \left(\int\limits_{\frac{T}{2}}^{T} \left|\zeta_Q^{-1}(s+i\tau)\right|^2 d\tau \right) d\sigma dt \leq 0.0001 \frac{\log^2 Q}{\omega^3 Q^{2\omega}}.$$

Then

$$J \leq 0.02 \frac{T}{2} \frac{\log^2 Q}{\omega^3 Q^{2\omega}}.$$

By this the lemma is proved.

Lemma 15 *Let $z = x + iy$. If $f(z)$ is analytic in $|z - z_0| \leq R$ and*

$$\iint\limits_{|z-z_0|\leq R} |f(z)|^2 dxdy = H,$$

then

$$|f(z)| \leq \frac{\sqrt{H/\pi}}{R - R'}$$

for $|z - z_0| \leq R' < R$.

Proof of this lemma can be found in [6].

Proof of Proposition 8. We use notations from Lemma 14. Let $d = r + \frac{1/4-r}{2}$. Then $\omega = \frac{1/4-r}{2}$. Let us define a set A_T by

$$A_T = \left\{ \tau : \tau \in [T/2, T],\ \iint\limits_{E} \left|\zeta\left(s + \frac{3}{4} + i\tau\right)\zeta_Q^{-1}\left(s + \frac{3}{4} + i\tau\right) - 1\right|^2 d\sigma dt \leq (0.6\varepsilon\omega)^2 \pi \right\}.$$

By Lemma 14, for $\frac{T}{\pi} \geq e^{\frac{0.51(1-2\omega)Q-872}{\omega}}$, $Q > 355\,991$ we have

$$\text{meas}\, A_T > \frac{T}{2}\left(1 - 0.02 \frac{\log^2 Q}{\omega^3 Q^{2\omega}} \frac{1}{(0.6\varepsilon\omega)^2 \pi}\right).$$

Lemma 15 gives us that, for $\tau \in A_T$,

$$\max_{|s|\leq r} \left|\zeta\left(s + \frac{3}{4} + i\tau\right)\zeta_Q^{-1}\left(s + \frac{3}{4} + i\tau\right) - 1\right| < 0.6\varepsilon.$$

Thus, if $\varepsilon \leq 1$, then

$$\max_{|s|\leq r} \left|\log\zeta\left(s + \frac{3}{4} + i\tau\right) - \log\zeta_Q\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon.$$

Proposition 8 is proved.

4 Application of Weyl's criterion

Proposition 16 *Let $\rho \geq 355\,991$, $50 \leq V \leq \rho < Q$, $300VQ^2 (Q/\rho)^{\frac{1}{4}} \leq \log T$ and θ_p, $p \leq \rho$ are any real numbers. Let $|s| \leq r < 1/4$. Then the measure of $\tau \in [T, 2T]$, such that*

$$\left|\sum_{p\leq Q} \log\left(1-\frac{1}{p^{\frac{3}{4}+s+i\tau}}\right) - \sum_{p\leq\rho} \log\left(1-\frac{e^{2\pi i\theta_p}}{p^{\frac{3}{4}+s}}\right)\right| \leq \frac{500\rho^{\frac{1}{4}+r}}{V\log\rho} + \frac{80}{\rho^{\frac{1}{4}-r}\log^{\frac{1}{2}}\rho}$$

is greater than $\frac{1}{2}TV^{-\pi(\rho)}$.

To prove the proposition we will need the quantitative form of Weyl's criterion.

Lemma 17 *Let L be a positive integer and $a_1, \ldots, a_L$, $b_1, \ldots, b_L$ be real numbers such that $b_l \leq a_l + 1$ for $l = 1, \ldots, L$. Let U denote the set of points $x = (x_1, \ldots, x_L)$ in $\mathbb{R}^L$, the L-dimensional real vector-space, satisfying $a_l \leq x_l < b_l$ for $l = 1, \ldots, L$. Let $\alpha = (\alpha_1, \ldots, \alpha_L)$ be an L-tuple of real numbers and let $N_\alpha(U)$ denote the number of points $n\alpha = (n\alpha_1, \ldots, n\alpha_L)$, $1 \leq n \leq N$, which, modulo 1, lie in U. Let H_l, $l = 1, \ldots, L$, be greater than 1 and*

$$\gamma_{h,l} = \begin{cases} b_l - a_l + \frac{75}{H_l}, & for\ h = 0, \\ \min\left(\gamma_{0,l}, \frac{30}{|h|}\right), & for\ h \neq 0. \end{cases}$$

Then we have

$$\left|\frac{N_\alpha(U)}{N} - \prod_{i=1}^{L}(b_l - a_l)\right| \leq \prod_{l=1}^{L}(b_l - a_l)\left(\prod_{l=1}^{L}\left(1+\frac{75}{H_l(b_l-a_l)}\right) - 1\right) + $$
$$+\frac{1}{N}\sum_{(h)}^{*}\left|\sum_{n=1}^{N}\exp\left(2\pi i n\sum_{l=1}^{L}\alpha_l h_l\right)\right|\prod_{l=1}^{L}\gamma_{h_l,l}\,,$$

where the sum $\sum_{(h)}^{*}$ extends over all $h = (h_1, \ldots, h_L) \neq (0, \ldots, 0)$ such that h_l is integral and $|h_l| \leq H_l(1 + \min(\log H_l, \log L))$, $l = 1, \ldots, L$.

Proof. The Lemma was proved by J. F. Koksma in [7].

We need the continuous version of Lemma 17.

Lemma 18 *Let the conditions of Lemma 17 be satisfied. Let $T_\alpha(U)$ denote the Jordan measure of the set $t \in (0, T)$, such that $t\alpha = (t\alpha_1, \ldots, t\alpha_L) \in U \mod 1$. Then we have*

$$\left|\frac{T_\alpha(U)}{T} - \prod_{i=1}^{L}(b_l - a_l)\right| \leq \prod_{l=1}^{L}(b_l - a_l)\left(\prod_{l=1}^{L}\left(1+\frac{75}{H_l(b_l-a_l)}\right) - 1\right) + $$
$$+\frac{1}{T}\sum_{(h)}^{*}\min\left(\left|\pi\sum_{l=1}^{L}\alpha_l h_l\right|^{-1}, T\right)\prod_{l=1}^{L}\gamma_{h_l,l}\,,$$

where $\sum_{(h)}^{*}$ means the same as in Lemma 17.

Note. Since the statement of the lemma does not change if we take $a_l - \alpha_l T, b_l - \alpha_l T$ instead of a_l, b_l, thus the lemma remains true if we define $T_\alpha(U)$ as the Jordan measure of the set $\{t \in (T, 2T) : t\alpha = (t\alpha_1, \dots, t\alpha_L) \in U \mod 1\}$.

Proof. Clearly there is the sequence of integers $\{A_k\}$ such that

$$\left|T - \frac{A_k}{k}\right| \le \frac{1}{k}.$$

In Lemma 17 we take $\frac{\alpha}{k}$ instead of α and A_k instead of N. Then in view of limits

$$\lim_{k\to\infty} \frac{N_{\frac{\alpha}{k}}(U)}{A_k} = \frac{T_\alpha(U)}{T}$$

and

$$\left|\lim_{k\to\infty} \frac{1}{A_k} \sum_{n=1}^{A_k} \exp\left(2\pi i n \sum_{l=1}^{L} \frac{\alpha_l}{k} h_l\right)\right| = \left|\frac{1}{T} \int\limits_0^T \exp\left(2\pi i x \sum_{l=1}^{L} \alpha_l h_l\right) dx\right|$$
$$\le \frac{1}{T} \min\left(\left|\pi \sum_{l=1}^{L} \alpha_l h_l\right|^{-1}, T\right)$$

Lemma 18 follows.

Lemma 19 *Let* $|\Delta x| \le 0.01$ *and* $M \ge 355\,991$*. Then for* $|s| \le r < 1/4$ *we have*

$$\left|\sum_{p\le M} \log\left(1 - \frac{e^{-2\pi i(x+\Delta x)}}{p^{s+\frac{3}{4}}}\right) - \sum_{p\le M} \log\left(1 - \frac{e^{-2\pi i x}}{p^{s+\frac{3}{4}}}\right)\right| \le 993 \frac{\Delta x M^{\frac{1}{4}+r}}{\log M}.$$

Proof. For $|\Delta x| \le 0.01$ we obtain

$$\left|\log\left(1 - \frac{e^{-2\pi i(x+\Delta x)}}{p^{s+\frac{3}{4}}}\right) - \log\left(1 - \frac{e^{-2\pi i x}}{p^{s+\frac{3}{4}}}\right)\right|$$
$$\le \sqrt{\log^2\left(1 + \frac{|e^{-2\pi i(x+\Delta x)} - e^{-2\pi i x}|}{p^{\sigma+\frac{3}{4}} - 1}\right) + \arctan^2\left(\frac{|e^{-2\pi i(x+\Delta x)} - e^{-2\pi i x}|}{p^{\sigma+\frac{3}{4}} - 1}\right)}$$
$$\le \frac{14\Delta x}{p^{\sigma+\frac{3}{4}} - 1}.$$

Thus,

$$\left|\sum_{p\le M} \log\left(1 - \frac{e^{-2\pi i(x+\Delta x)}}{p^{s+\frac{3}{4}}}\right) - \sum_{p\le M} \log\left(1 - \frac{e^{-2\pi i x}}{p^{s+\frac{3}{4}}}\right)\right|$$
$$14\Delta x \sum_{p\le 355990} \frac{1}{\sqrt{p} - 1} + 14\Delta x \sum_{355991\le p\le M} \frac{1}{p^{\frac{3}{4}-r} - 1}.$$

Calculating the first sum by computer and the second sum by partial summation, in view of inequalities from Lemma 4 and

$$\int_{355991}^{M} \frac{x^{r-\frac{3}{4}}}{\log x} dx \leq 5.83 \frac{M^{\frac{1}{4}+r}}{\log M},$$

we obtain the lemma.

Proof of Proposition 16. Let J be equal to the integral part of $V\left(Q/\rho\right)^{1/4+r}$. In $\mathbb{R}^{\pi(Q)}$ we define the set

$$U(\theta) := \Big\{ x = (x_p)_{p\leq Q} \mid \theta_p - \frac{1}{2V} \leq x_p < \theta_p + \frac{1}{2V} \text{ for } p \leq \rho \text{ and } \theta_p - \frac{1}{2J} \leq x_p < \theta_p + \frac{1}{2J} \text{ for } \rho < p \leq Q \Big\}, \tag{20}$$

where θ_p for $\rho < p \leq Q$ will be specified later. We now apply Lemma 18 (also see Note below this lemma) with $L = \pi(Q)$, $U = U(\theta)$ and $\alpha = \frac{1}{2\pi}(\log p)_{p\leq Q}$. Thus we have

$$T_\alpha\left(U(\theta)\right) \geq 0.52 T V^{-\pi(\rho)} J^{\pi(\rho)-\pi(Q)}, \tag{21}$$

if the inequalities

$$\left| \prod_{p\leq\rho} \left(1 + \frac{75V}{H_p}\right) \prod_{\rho<p\leq Q} \left(1 + \frac{75J}{H_p}\right) - 1 \right| \leq 0.47 \tag{22}$$

and

$$\sum_{(h)}^{*} \left| \sum_{p\leq Q} \frac{h_p \log p}{2} \right|^{-1} \prod_{p\leq Q} \gamma_{h_p,p} \leq 0.01 T V^{-\pi(\rho)} J^{\pi(\rho)-\pi(Q)} \tag{23}$$

hold with some positive numbers $H_p > 1$, $p \leq Q$. We set $H_p = H/\log Q$ for $p \leq Q$ and assume that $H \geq Q$. By the inequality $(1+x)^y - 1 \leq xye^{xy}$, for $x, y \geq 0$, we have that the left hand-side of (22) is smaller than

$$75J\frac{\log Q}{H}\pi(Q) \exp\left(75J\frac{\log Q}{H}\pi(Q)\right).$$

Then, in view of Lemma 4, we deduce that (22) is true if

$$\frac{JQ}{H} \leq 0.004 \tag{24}$$

On the other hand

$$\left| \frac{1}{2} \sum_{p\leq Q} h_p \log p \right|^{-1} = \left| \frac{1}{2} \log \prod_{p\leq Q} p^{h_p} \right|^{-1} \leq 2 \prod_{p\leq Q} p^{|h_p|}. \tag{25}$$

By the choice of H_p and by the inequality for the prime counting function (Lemma 4) we obtain

$$H_p\left(1+\min\left(\log H_p, \log \pi(Q)\right)\right) \le \frac{H}{\log Q}\left(1+\min\left(\log\frac{H}{\log Q}, \log\frac{1.1Q}{\log Q}\right)\right) \le 1.1H.$$

Thus by (25) we have that the left-hand side of (23) is less than

$$2\sum_{(h)}^{*}\prod_{p\le Q} p^{|h_p|}\gamma_{h_p,p} \le 2\prod_{p\le Q}\left(1+60\sum_{1\le h_p\le 1.1H}\frac{p^{|h_p|}}{h_p}\right)$$

$$\le 200\prod_{p\le Q}\frac{p^{2H}}{1.1H\log p} = 200\frac{\exp\left\{1.1H\sum_{p\le Q}\log p\right\}}{1.1H\exp\left\{\sum_{p\le Q}\log\log p\right\}} \le e^{1.12HQ-355991},$$

here we used inequalities $\sum_{p\le x}\log p < 1.01624x$, $x>0$ and $\pi(x)\ge x/\log x$, $x\ge 17$, obtained by Rosser and Schoenfeld ([11], Theorem 9 and Corollary 1). Therefore we have that (23) is true if

$$e^{1.12HQ} \le TV^{-\pi(\rho)}J^{\pi(\rho)-\pi(Q)}.$$

The last inequality is valid provided $1.2HQ \le \log T$. By this and (24) we have that (21) is true if

$$300VQ^2\left(\frac{Q}{\rho}\right)^{\frac{1}{4}} \le \log T. \tag{26}$$

Now let $\sum_{(j)}$ denotes the summation over all $j=(j_p)_{\rho<p\le Q}$, where the j_p are positive integers not exceeding J. Then, using Lemma 4, we obtain that

$$\sum_{(j)}\left|\sum_{\rho<p\le Q}\log\left(1-\frac{e^{-2\pi i j_p/J}}{p^{\frac{3}{4}+s}}\right)\right|^2 = J^{\pi(Q)-\pi(\rho)}\sum_{\rho<p\le Q}\sum_{n=1}^{\infty}\frac{1}{n^2p^{2n(\sigma+3/4)}}$$

$$\le J^{\pi(Q)-\pi(\rho)}\zeta(2)\sum_{p>\rho}\frac{1}{p^{2(\sigma+3/4)}} \le J^{\pi(Q)-\pi(\rho)}\frac{2.4}{\rho^{\frac{1}{2}-2r}\log\rho}.$$

Further we assume that the θ_p, $\rho<p\le Q$, are of the form j_p/J, where the positive integers j_p do not exceed J. Hence (20) and the last inequality imply that we can find more than $0.98J^{\pi(Q)-\pi(\rho)}$ disjoint sets $U(\theta)$ which agree in all the θ_p for $p\le\rho$ and satisfy

$$\left|\sum_{\rho<p\le Q}\log\left(1-\frac{e^{-2\pi i j_p/J}}{p^{\frac{3}{4}+s}}\right)\right| \le \frac{0.02\sqrt{2.4}}{\rho^{\frac{1}{4}-r}\sqrt{\log\rho}}; \tag{27}$$

the sum of such $U(\theta)$ we will denote by A. Thus in view of (21) we have that the measure of the set $\tau\in(0,T)$ such that $\frac{\tau}{2\pi}(\log p)_{p\le Q}$ belongs to A mod 1 is not less than $0.98J^{\pi(Q)-\pi(\rho)}\times 0.52TV^{\pi(\rho)}J^{\pi(\rho)-\pi(Q)}$. Then Proposition 16 follows by Lemma 19 and by the inequalities (26) and (27).

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Ramūnas Garunkštis
Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
2600 Vilnius
Lithuania
e-mail: ramunas.garunkstis@maf.vu.lt