Fourier Analysis and the Riemann ζ-Function

Ben Green, January 2001

In these notes our aim is to derive the functional equation of the Riemann ζ-function in what I consider to be the most natural way. Our emphasis will be on motivation rather than completely rigourous justification of results.

The ζ-function. As the reader is probably aware we write

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} n^{-s}, \tag{1}$$

and observe that this converges to give an analytic function on the region $\Re(s) > 1$. Such a definition is extremely pertinent to the study of the primes. This can already been seen in the well-known *Euler Product* for ζ

$$\zeta(s) \;=\; \prod_p \left(1 - p^{-s}\right)^{-1}, \tag{2}$$

which is also valid for $\Re(s) > 1$. An analytic proof that there are infinitely many primes follows by supposing the contrary and deriving a contradiction from a comparison of (1) and (2) as $s \to 1^+$.

The following quite remarkable fact about the ζ-function is the object of this note.

Theorem 1 *Write $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then ξ extends to an meromorphic function on all of $\mathbb{C}$ which, furthermore, is analytic except for simple poles at $s = 0, 1$. In addition ξ satisfies the* functional equation

$$\xi(s) \;=\; \xi(1-s).$$

It would seem from the tidiness of the conclusion that ξ is a more natural function to consider than ζ. In some sense this is so; the factor $\pi^{-s/2}\Gamma(s/2)$ may be regarded as a contribution to the Euler product arising from the so-called "prime at infinity".

Fourier Analysis. It is common, in an undergraduate course, to learn about Fourier coefficients and Fourier transforms. We recall that if $f : \mathbb{T} \to \mathbb{C}$ is integrable then one defines for each $n \in \mathbb{Z}$ the Fourier coefficient

$$\hat{f}(n) \;=\; \int_{\mathbb{T}} f(\theta) e^{2\pi i n\theta}\, d\theta. \tag{3}$$

On the other hand if $f : \mathbb{R} \to \mathbb{C}$ is in $L^1(\mathbb{R})$ then for each $\lambda \in \mathbb{R}$ one sets

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{ix\lambda}\,dx \tag{4}$$

and calls the resultant function the Fourier transform. I have seen an argument which exhibits the Fourier transform on $\mathbb{R}$ as a sort of limiting case of that on $\mathbb{T}$ but there is another, deeper, way in which the two concepts are related. This we now describe.

Let G be a locally compact abelian group (LCAG). By this we mean a topological group which is Hausdorff, abelian and in which every point has a compact neighbourhood, but the exact definition is not terribly relevant to us here. Examples of LCAGs are finite groups, $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, $(\mathbb{R}_{>0}, \times)$ and $\mathbb{T}$. Any LCAG has a *Pontryagin Dual* $\hat{G}$ which consists of all continuous homomorphisms $\phi : G \to \mathbb{T}$. A natural topology can be put on $\hat{G}$ and it turns out that the resulting group is also an LCAG. It is a nice exercise to descibe the duals of some of the groups mentioned above. Whilst it is fairly clear, for example, that for each $n \in \mathbb{Z}$ there is a homomorphism $\phi_n : \mathbb{T} \to \mathbb{T}$ defined by $\phi_n(e^{2\pi i\theta}) = e^{2\pi in\theta}$, it requires a little more ingenuity to show that there are no more. Once successful, however, one will know that $\hat{\mathbb{T}} \cong \mathbb{Z}$. A deep theorem, known as Pontryagin duality, says that $G \cong \hat{\hat{G}}$.

Another rather deep result tells us that any LCAG has an essentially unique *Haar Measure* μ_G wich may be used to integrate suitable functions on G. I will not dwell on the precise definition, but the reader will gain a feeling for these things by thinking either of Lebesgue measure on $\mathbb{R}$ or the measure on $\mathbb{Z}$ defined by $\mu(\{n\}) = 1$ for all n. A crucial property of μ in both of these cases is its invariance under the action of G on itself, and this is a property that Haar measures enjoy in general.

Let now G be a LCAG and let $f : G \to \mathbb{C}$ be in $L^1(G)$. The *Fourier Transform* of f takes values in $\hat{G}$ and is defined by

$$\hat{f}(\chi) = \int_G f(x)\chi(x)\,d\mu(x).$$

In view of my earlier remark that $\hat{\mathbb{T}} = \mathbb{Z}$ the reader may now care to convince herself that this definition degenerates into (3), at least up to a constant depending on the normalization of Haar measure, in that special case. As a further exercise she might like to convince herself that $\hat{\mathbb{R}} \cong \mathbb{R}$ and thence recover (4).

Pleased though we might feel to have unified what we previously called Fourier coefficients and Fourier transforms, there is more to be gained by looking at further examples. When $G = \mathbb{Z}/p\mathbb{Z}$, for example, one recovers the discrete Fourier transform

$$\hat{f}(r) = \sum_x f(x)\omega^{rx}.$$

Here $\omega = e^{2\pi i/p}$ and r takes values in $\mathbb{Z}/p\mathbb{Z}$, since in this case G is isomorphic to its dual. When $G = \mathbb{R}^{\times}_{>0}$ we obtain the Mellin transform

$$\hat{f}(s) = \int_0^\infty f(x) x^{is} d^*x.$$

In this equation $d^*x = dx/x$ is the Haar measure on G and s ranges over $\mathbb{R}$. Indeed G is isomorphic to the additive group $\mathbb{R}$ via the logarithm map, and so what we have here is just the ordinary Fourier transform on $\mathbb{R}$ in disguise. The reader may also care to consider the case $G = (\mathbb{Z}/2\mathbb{Z})^n$, an example of interest to combinatorialists and (I have heard) computer scientists.

Over and above the obvious aesthetic pleasure to be derived from unifying so many familiar concepts the Fourier transform is extremely useful. There are inversion formulæ, Parseval identities and spectral analysis on all manner of LCAGs. We, however, will do little more than look admiringly at the Fourier transform and take this opportunity to refer the reader to Rudin's *Fourier Analysis on Groups* or Ramakrishnan and Valenza's *Fourier Analysis on Number Fields.*

Our objective here is to show that everything related to ζ is very natural. To this end we remark that the Gamma function, which appears in the functional equation, is itself a kind of Mellin transform. Recall indeed that

$$\Gamma(s) = \int_0^\infty e^{-x} x^s d^*x$$

for $\Re(s) > 0$, and that Γ satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$.

Let us conclude this section by deriving a result that will be needed later.

Proposition 2 (Poisson Summation) *Let $f : \mathbb{R} \to \mathbb{C}$ be a function satisfying "suitable" smoothness and decay conditions. Let the hat symbol denote Fourier transform on $\mathbb{R}$. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(2\pi n).$$

Sketch Proof We consider two functions F and G defined on $\mathbb{T}$. Set

$$F(e^{2\pi i\theta}) = \sum_{n \in \mathbb{Z}} \hat{f}(2\pi n) e^{-2\pi i n\theta}$$

and

$$G(e^{2\pi i\theta}) = \sum_{k \in \mathbb{Z}} f(\theta + k).$$

It is clear that these functions are well-defined provided there are no convergence problems. We now compute their Fourier transforms; since these are functions on $\mathbb{T}$ these transforms will be defined on $\hat{\mathbb{T}} = \mathbb{Z}$. Fully aware of the possibility of horrible confusion, we will nonetheless use the hat symbol to denote this type of Fourier transform as well as the one on $\mathbb{R}$. Now we have

$$\begin{aligned}
\hat{F}(m) &= \int_0^1 \left(\sum_{n\in\mathbb{Z}} \hat{f}(2\pi n) e^{-2\pi i n\theta} \right) e^{2\pi i m\theta}\, d\theta \\
&= \sum_{n\in\mathbb{Z}} \hat{f}(2\pi n) \int_0^1 e^{2\pi i(m-n)\theta}\, d\theta \\
&= \hat{f}(2\pi m)
\end{aligned}$$

and

$$\begin{aligned}
\hat{G}(m) &= \int_0^1 \left(\sum_{k\in\mathbb{Z}} f(\theta+k) \right) e^{-2\pi i m\theta}\, d\theta \\
&= \int_0^1 \left(\sum_{k\in\mathbb{Z}} f(\theta+k) \right) e^{-2\pi i m(\theta+k)}\, d\theta \\
&= \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x) e^{-2\pi i m x}\, dx \\
&= \hat{f}(2\pi m).
\end{aligned}$$

We have shown then that $\hat{F}(m) = \hat{G}(m)$. If F and G are suitably nice functions on $\mathbb{T}$ then this equality of their Fourier coefficients implies that $F = G$ identically (certainly this will be the case if F and G can be developed as Fourier series, but it will also be true under wider assumptions). The fact that $F(0) = G(0)$ is exactly the Poisson Summation Formula. □

The statement of this result is rather at odds with our desire to place the Fourier transform in a properly general context. However it turns out that the Poisson Summation Formula can be massively generalised to the point where it must be accepted as an extremely natural fact. If G is a LCAG and H is a suitable subgroup of G then one has, purely formally, a result of the form

$$\int_{\widehat{G/H}} \hat{f}(\overline{\phi})\, d\mu_{\widehat{G/H}}(\phi) = \int_H f(h)\, d\mu_H(h).$$

Here $\overline{\phi}$ denotes the lift of the character ϕ (which is defined on G/H) to a character on G (i.e. an element of $\hat{G}$). I shall say no more about this, save for a remark that it should not, in the form stated above, be taken too literally.

The θ-function. Set

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 t}.$$

This defines an analytic function on the upper half-plane $\mathbb{H}$. Doubtless any physicists who may have chanced upon this will already have recognised the pertinence of such a definition to the heat equation on $\mathbb{T}$, but we shall show that θ is an extremely natural object from a number theorist's viewpoint too. In fact, what we will prove about θ in the next proposition is in some sense equivalent to the functional equation of ζ.

Proposition 3 *θ satisfies the relation*

$$\theta(is) = \frac{1}{\sqrt{s}}\theta\left(\frac{i}{s}\right)$$

for all $s \in \mathbb{R}_{>0}$.

Why should such a conclusion set a number theorist's mind working? If one extends the above relation analytically to the whole upper half plane one has something like

$$\theta\left(-\frac{1}{t}\right) = \sqrt{t}\theta(t),$$

where we have been deliberately vague about the branch of square root to be taken. It is fairly trivial to see that $\theta(t) = \theta(t+2)$ directly from the definition. Rather roughly speaking this means that the function ψ defined by $\psi(t) = \theta(2t)$ is a kind of *modular form* for the group $\Gamma_0(4)$ of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ and $4|c$. Modular forms are of immense importance throughout mathematics and nowhere more so than in number theory.

The proof of Proposition 3 is a swift exercise in the use of the Poisson Summation Formula. We apply it in fact to yet another very natural object, a function related to the Gaussian distribution. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{-\pi x^2 s}$. Recalling the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$$

it is an easy exercise to show that the Fourier transform of f is given by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{s}} e^{-\lambda^2/4\pi s}.$$

The Poisson Summation Formula is thus precisely Proposition 3 in this case. □

ξ as the Mellin transform of θ. We come now to the final part of our story. We will show that the function $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is basically the Mellin transform of θ. Finally we will demonstrate that Proposition 3 translates very cleanly into the functional equation of ξ.

Proposition 4 *ξ is essentially the Mellin transform of θ; in fact if $\Re(s) > 1$ then*

$$\xi(s) = \int_0^\infty \left(\frac{\theta(ix)-1}{2}\right) x^{s/2} d^*x.$$

Proof A simple substitution shows that

$$\begin{aligned}\int_0^\infty e^{-\pi n^2 x} x^{s/2}\, d^*x &= \pi^{-s/2} n^{-s} \int_0^\infty e^{-u} u^{s/2}\, d^*u \\ &= \pi^{-s/2}\Gamma(s/2) n^{-s}.\end{aligned}$$

Summing this over $n = 1, 2, \ldots$ gives the Proposition. $\square$

The next, rather simple, calculation is the key to seeing how the functional equation of ξ arises from that for θ. Using Proposition 3 we have

$$\begin{aligned}\int_0^1 \left(\frac{\theta(ix)-1}{2}\right) x^{s/2} d^*x &= \frac{1}{2}\int_0^1 \theta(ix) x^{s/2} d^*x - \frac{1}{s} \\ &= \frac{1}{2}\int_0^1 \theta\left(\frac{i}{x}\right) x^{(s-1)/2} d^*x - \frac{1}{s} \\ &= \frac{1}{2}\int_1^\infty \theta(iu) u^{(1-s)/2} d^*u - \frac{1}{s} \\ &= \int_1^\infty \left(\frac{\theta(ix)-1}{2}\right) u^{(1-s)/2} d^*u - \frac{1}{s} - \frac{1}{1-s}.\end{aligned}$$

It follows from Proposition 4 that

$$\xi(s) = \int_1^\infty \left(\frac{\theta(ix)-1}{2}\right) \left(x^{s/2} + x^{(1-s)/2}\right) d^*x - \frac{1}{s} - \frac{1}{1-s}. \tag{5}$$

Something slightly miraculous has now occured. The function $\theta(ix)$ decays extremely rapidly as $x \to \infty$ and so the integral here defines a function which is analytic on all of $\mathbb{C}$. Thus ξ may be extended to a meromorphic function on $\mathbb{C}$ whose only singularities are poles at $s = 0, 1$. Furthermore the whole right hand side of (5) is manifestly invariant under the substitution $s \mapsto 1-s$. This proves Theorem 1. $\square$