



Robin's Inequality for Sum of Divisors Function and the Riemann Hypothesis

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Dedicated to the memory of Professor Włodzimierz Staś

Abstract. Let $\sigma(n)$ denote the sum of divisors function. In this paper we give a simple proof of the Robin inequality **(R)**: $\sigma(n) < e^\gamma n \log \log n$, for all positive integers $n \geq 5041$.

The Robin inequality **(R)** implies Riemann Hypothesis.

1. Introduction

The Riemann zeta function $\zeta(s)$ for $s = \sigma + it$ is defined by the Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which converges for $\sigma > 1$ and it has analytic continuation to the complex plane with one singularity a simple pole with residue 1.

The Riemann Hypothesis ([8]), is intimately connected with the distribution of prime numbers. The Riemann zeta-function is a special case of L -functions.

These L -functions are connected with many important and difficult problems in number theory, algebraic geometry, topology, representation theory and modern physics, see; Berry and Keating [1], Katz and Sarnak [5], and Murty [7].

In 1984 Robin [9] proved very interesting and important criterion:

Robin's criterion. *The Riemann Hypothesis is true if and only if*

$$\text{(R)} \quad \sigma(n) < e^\gamma n \log \log n,$$

for all positive integers $n \geq 5041$, where $\sigma(n) = \sum_{d|n} d$ and γ is Euler's constant.

In 2002 Lagarias [6] proved some extension of the Robin criterion. Many others criteria and important results connected with the Riemann Hypothesis have been

proved and these results have been described by Conrey in his elegant article [3]. Recently, Choie, Lichardopol, Moree and Sole' in the paper [2] proved that if $n \geq 37$ does not satisfy Robin's criterion it must be even and is neither squarefree nor squarefull, moreover that n must be divisible by fifth power > 1 .

In our paper [4]; see, Theorem 1, p. 70 has been proved of the following result:

R.1. *Let $n = 2m$, $(2, m) = 1$. Then for all odd positive integers $m > \frac{3^9}{2}$ we have*

$$(1.1) \quad \sigma(2m) < \frac{39}{40} e^\gamma 2m \log \log 2m,$$

and

$$(1.2) \quad \sigma(m) < e^\gamma m \log \log m.$$

It is easy to see that from the result of **R.1** follows that for complete proof of the Robin inequality (**R**) it suffices to prove that inequality (**R**) is true for all positive integers n such that $n = 2^\alpha m$, $(2, m) = 1$ and $\alpha \geq 2$.

In this connection in the paper [4] has been proved the following theorem (see, Theorem 2, p. 71):

R.2. *If there exists an positive integer m_0 such that for every odd positive integer $m > m_0$, we have*

$$(1.3) \quad \sigma(2m) < \frac{3}{4} e^\gamma 2m \log \log 2m,$$

then for all integers $n = 2^\alpha m$, $(2, m) = 1$, $m > m_0$ and every fixed integer $\alpha \geq 2$ we have

$$(1.4) \quad \sigma(2^\alpha m) < e^\gamma 2^\alpha m \log \log 2^\alpha m.$$

The result described in **R.1** is cited by M. Weber in his monograph [11] on page 541.

From the result of **R.2** follows that for complete proof of the Robin inequality (**R**) it suffices to prove the following theorem:

Theorem. *For all integers $n = 2m$, $(2, m) = 1$ such that $m > \frac{1}{2} e^{e^9}$ we have*

$$(*) \quad \sigma(2m) < \frac{3}{4} e^\gamma 2m \log \log 2m.$$

We prove this Theorem in part 3 of this paper by using basic lemmas given in next part of our paper.

2. Basic Lemmas

Lemma 1. Let $n = 2^\alpha m$, $(2, m) = 1$ and let $\omega(m)$ denote the number of distinct primes divisors of m . If $\omega(m) = 1$, then for every odd positive integer $m > \frac{1}{4}e^{e^2}$ and each fixed integer $\alpha \geq 2$, we have

$$(2.1) \quad \sigma(2^\alpha m) < e^\gamma 2^\alpha m \log \log 2^\alpha m.$$

Proof. By the assumption of Lemma 1 it follows that $m = p^\beta$, where $p \geq 3$ is an odd prime and $\beta \geq 1$ is positive integer.

Moreover, by the assumption that $m > \frac{1}{4}e^{e^2}$ it follows that

$$(2.2) \quad m = p^\beta > \frac{1}{4}e^{e^2}.$$

On the other hand we have

$$(2.3) \quad \sigma(2^\alpha m) = \sigma(2^\alpha p^\beta) = \sigma(2^\alpha)\sigma(p^\beta) = (2^{\alpha+1} - 1) \frac{p^{\beta+1} - 1}{p - 1}.$$

Since $p - 1 \geq \frac{2}{3}p$ and $2^{\alpha+1} - 1 < 2^{\alpha+1}$, then from (2.3) and (2.2) it follows that

$$(2.4) \quad \sigma(2^\alpha m) < 2^{\alpha+1} \cdot \frac{p^{\beta+1}}{\frac{2}{3}p} = 3 \cdot 2^\alpha m.$$

From the assumption follows that $2^\alpha m \geq 2^2 m > e^{e^2}$, hence

$$(2.5) \quad e^\gamma \log \log 2^\alpha m > 1.6 \log \log e^{e^2} > 1.6 \times 2 > 3.$$

By (2.5) and (2.4) it follows that the inequality (2.1) holds and the proof of Lemma 1 is finished. \square

Lemma 2. Let $n = 2m_1$, $(2, m_1) = 1$ and $\omega(m_1) = 2$. Then for every odd positive integer m_1 such that $m_1 > \frac{1}{2}e^{e^3}$, we have

$$(2.6) \quad \frac{\sigma(2m_1)}{2m_1} < \frac{3}{4}e^\gamma \log \log 2m_1.$$

Proof. By the assumption of Lemma 2 it follows that $m_1 = p_1^{\alpha_1} p_2^{\alpha_2}$. Therefore by the properties of the function σ we get,

$$(2.7) \quad \frac{\sigma(2m_1)}{2m_1} = \frac{\sigma(2)\sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2})}{2p_1^{\alpha_1} p_2^{\alpha_2}} = \frac{3(p_1^{\alpha_1+1} - 1)(p_2^{\alpha_2+1} - 1)}{2p_1^{\alpha_1} p_2^{\alpha_2} (p_1 - 1)(p_2 - 1)}.$$

Since $p_1 \geq 3$ and $p_2 \geq 5$ then we have,

$$(2.8) \quad p_1 - 1 \geq \frac{2}{3}p_1, \quad p_2 - 1 \geq \frac{4}{5}p_2.$$

From (2.7) and (2.8) we obtain

$$(2.9) \quad \frac{\sigma(2m_1)}{2m_1} < \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{3}{4} \cdot \frac{15}{4}.$$

Since $e^\gamma > 1.6$ and $2m_1 > e^{e^3}$, then we get

$$(2.10) \quad e^\gamma \log \log 2m_1 > 1.6 \log \log e^{e^3} = 1.6 \times 3 = 4.8 > \frac{15}{4}.$$

From (2.10) and (2.9) we obtain the inequality (2.6) and the proof of Lemma 2 is complete. \square

Lemma 3. Let $n = \prod_{j=1}^k p_j^{\alpha_j}$ and let $I(n) = \prod_{j=1}^k \left(1 - \frac{1}{p_j^{1+\alpha_j}}\right)$, where p_j are primes and $\alpha_j \geq 1$ are integers for $j = 1, 2, \dots, k$ and let σ , φ be the sum of divisors function and Euler's totient function, respectively. Then we have

$$(2.11) \quad \frac{\sigma(n)}{n} = I(n) \frac{n}{\varphi(n)}.$$

The proof of Lemma 3 is given in our paper [4], see; Lemma 2 on page 69.

Lemma 4 (Rosser-Schoenfeld's inequality [1], Theorem 15; Cf. [4], Lemma 1, p. 69). Let φ be the Euler's totient function. Then for all positive integers $n \geq 3$, the following inequality is true

$$(R-S) \quad \frac{n}{\varphi(n)} \leq e^\gamma \left(\log \log n + \frac{2.5}{e^\gamma \log \log n} \right),$$

except, when $n = 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 = 223\,092\,870$

and this case the constant $c = 2.5$ must be replaced by the constant $c_1 = 2.50637 < 2.51$.

Now, we prove the following Lemma:

Lemma 5. Let $n = 2m_1$, $(2, m_1) = 1$ and $\omega(m_1) = 2$. Then for all integers $2m_1 > e^{e^9}$ we have

$$(2.12) \quad I(m_1) < \frac{50}{51}.$$

Proof. From Lemma 3 and the definition of the function $I(n)$ for case $n = 2m_1 = 2 \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2}$, we have

$$(2.13) \quad I(n) = I(2m_1) = \left(1 - \frac{1}{2^{1+\alpha_1}}\right) \left(1 - \frac{1}{p_1^{1+\alpha_1}}\right) \left(1 - \frac{1}{p_2^{1+\alpha_2}}\right) = I(2) \cdot I(m_1).$$

By (2.13) it follows that

$$(2.14) \quad 0 < I(m_1) < 1 \iff I(m_1) \in (0, 1) = \left(0, \frac{50}{51}\right) \cup \left[\frac{50}{51}, 1\right).$$

From (2.14) and the assumption of Lemma 5 and Lemma 3 follows that the inequality (2.6) is true, so denote that we have

$$(2.15) \quad \text{If } \sigma(2m_1) < \frac{3}{4}e^\gamma 2m_1 \log \log 2m_1, \text{ then } I(m_1) \in \left(0, \frac{50}{51}\right) \text{ or } I(m_1) \in \left[\frac{50}{51}, 1\right).$$

Suppose that

$$(2.16) \quad \text{If } \sigma(2m_1) < \frac{3}{4}e^\gamma 2m_1 \log \log 2m_1, \text{ then } I(m_1) \in \left[\frac{50}{51}, 1\right),$$

so denote that (2.16) is equivalent to

$$(2.17) \quad \text{If } \sigma(2m_1) < \frac{3}{4}e^\gamma 2m_1 \log \log 2m_1, \text{ then } \frac{50}{51} \leq I(m_1) < 1.$$

Applying to (2.17) well-known the law of contraposition we get,

$$\text{if } I(m_1) < \frac{50}{51}, \text{ then } \sigma(2m_1) \geq \frac{3}{4}e^\gamma 2m_1 \log \log 2m_1.$$

From the identity (2.11) of Lemma 3 and the fact that the function $I(n)$ is multiplicative function, we obtain

$$(2.18) \quad \frac{\sigma(2m_1)}{2m_1} = \left(1 - \frac{1}{2^2}\right) I(m_1) \frac{2m_1}{\varphi(2m_1)} = \frac{3}{4} I(m_1) \frac{2m_1}{\varphi(2m_1)},$$

because, in this case $n = 2m_1$.

Applying to the $\frac{2m_1}{\varphi(2m_1)}$ of (2.18) the Rosser-Schoenfeld's inequality (R-S) from Lemma 4, with constant $c_1 < 2.51$ we get,

$$(2.19) \quad \frac{\sigma(2m_1)}{2m_1} < \frac{3}{4} I(m_1) e^\gamma \log \log 2m_1 \left(1 + \frac{2.51}{e^\gamma (\log \log 2m_1)^2}\right).$$

Since $2m_1 > e^{e^9}$ then we have

$$(2.20) \quad e^\gamma (\log \log 2m_1)^2 > 1.6 \times 81 = 129.6 > 125.5 = 50 \times 2.51.$$

From (2.20) we obtain

$$(2.21) \quad 1 + \frac{2.51}{e^\gamma (\log \log 2m_1)^2} < 1 + \frac{2.51}{50 \times 2.51} = 1 + \frac{1}{50} = \frac{51}{50}.$$

By (2.19) and (2.21) it follows that

$$(2.22) \quad \frac{\sigma(2m_1)}{2m_1} < \frac{3}{4} I(m_1) \frac{51}{50} e^\gamma \log \log 2m_1.$$

From the assumption that $I(m_1) < \frac{50}{51}$ and (2.22) we get that the (2.16) is impossible.

The proof of Lemma 5 is complete. \square

3. Proof of the Theorem

The proof of the Theorem we give by the induction with respect to $k = \omega(m)$.

From Lemma 2 it follows that the Theorem is true when $k = 2$. Suppose that the Theorem is true for all m such that $\omega(m) < k$.

We prove our Theorem for m , when $\omega(m) = k$.

Let $n = 2m = 2m_1 \cdot M$, $(2, m) = 1$, $m = m_1 \cdot M$; $(m_1, M) = 1$, where $\omega(m_1) = 2$, $\omega(m) = \omega(m_1 \cdot M) = k > 2$.

Then from the identity (2.11) of Lemma 3 and the Rosser-Schoenfeld's inequality (R-S) of Lemma 4 we obtain

$$(3.1) \quad \frac{\sigma(n)}{n} = \frac{\sigma(2m_1 \cdot M)}{2m_1 \cdot M} < \frac{3}{4} \cdot I(m_1 \cdot M) e^\gamma \log \log 2m_1 \cdot M \left(1 + \frac{2,51}{e^\gamma (\log \log 2m_1 \cdot M)^2} \right).$$

By Lemma 5 and the formula (2.11) of Lemma 3 it follows that

$$(3.2) \quad I(m) = I(m_1 \cdot M) = I(m_1) \cdot I(M) < \frac{50}{51},$$

because the function $I(m)$ is multiplicative function. Hence, from (3.1) and (3.2) follows that

$$(3.3) \quad \frac{\sigma(n)}{n} = \frac{\sigma(2m)}{2m} < \frac{3}{4} e^\gamma \cdot \frac{50}{51} \left(1 + \frac{2,51}{e^\gamma (\log \log 2m)^2} \right) \log \log 2m.$$

From (3.3) and (2.21) it follows that for all integers m , such that $2m > e^{e^9}$, we have

$$(3.4) \quad \frac{\sigma(2m)}{2m} < \frac{3}{4} e^\gamma \log \log 2m,$$

and the proof of the Theorem is complete.

Remark. By Proposition 1 and 4 of [9] it is enough, to prove RH, to derive Robin inequality (R) for n large enough and therefore no need of computer check.

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