Bost-Connes-Marcolli systems for Shimura varieties. I. Definitions and formal analytic properties.

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Abstract

We construct a Quantum Statistical Mechanical system (A, σ_t) analogous to the Bost-Connes-Marcolli system of [CM04] in the case of Shimura varieties. Along the way, we define a new Bost-Connes system for number fields which has the "correct" symmetries and the "correct" partition function. We give a formalism that applies to general Shimura data (G, X). The object of this series of papers is to show that these systems have phase transitions and spontaneous symmetry breaking, and to classify their KMS states, at least for low temperature.

Contents

1 Introduction

A few years ago, Bost and Connes [BC95] discovered a surprising relationship between the class field theory of $\mathbb{Q}$ and quantum statistical mechanics.

Mathematically, a quantum statistical mechanical system consists of a pair $(\mathcal{A}, \sigma_t)$, where $\mathcal{A}$ is a C*-algebra and σ_t is a one-parameter group of automorphisms of $\mathcal{A}$; physically, $\mathcal{A}$ is the algebra of observables and σ_t is the time evolution of the physical system. The physical states of the system are given by certain linear functionals on $\mathcal{A}$.

The analogy between classical and quantum statistical mechanics can be described by the following array:

	CLASSICAL	QUANTUM
Observables	$a \in C^\infty(X)$ (X, ω) $2n$-dim symplectic manifold (phase space)	$a \in \mathcal{A}$ $\mathcal{A}$: C*-algebra, a=a*
Bracket	Poisson bracket $\{a_1, a_2\} = \omega(\xi_{a_1}, \xi_{a_2})$ with $da_i + \omega(\xi_{a_i}, \_) = 0$	Commutator $[a_1, a_2]$
Hamiltonian	$H: X \to \mathbb{R}$	H unbounded selfadjoint on $\mathcal{H}$ Representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
Time evolution	Solution of $\{H, a\}(x) = (\frac{d}{dt})_{t=0} a(\sigma_t(x))$	$\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ $e^{itH}\pi(a)e^{-itH} = \pi(\sigma_t(a))$
States	Probability measure μ on X $\Phi(a) = \int_X a d\mu$	Linear functional of norm 1 $\Phi : \mathcal{A} \to \mathbb{C}$
Partition function	$\zeta(\beta) = \int_X e^{-\beta H} d\Omega$ with $\Omega = \omega^{\wedge n}$ the volume form	$\zeta(\beta) = \mathrm{Tr}(e^{-\beta H})$
Equilibrium States	Canonical ensemble $d\mu = \frac{e^{-\beta H} d\Omega}{\zeta(\beta)}$	KMS condition: $\Phi(ab) = \Phi(\sigma_{it}(b)a)$ example: $\Phi(a) = \frac{\mathrm{Tr}(ae^{-\beta H})}{\zeta(\beta)}$

The statistical content means that one singles out the equilibrium states at a given temperature $T = 1/\beta$ on $\mathcal{A}$, and these are characterized by the so called KMS_β condition. The set of these equilibrium states may have symmetries. Changing the temperature of a system can produce a phase transition phenomenon with spontaneous symmetry breaking, meaning that the symmetry changes radically with an arbitrary small change of temperature. For example, the formation at zero temperature of a snowflake from water is a phase transition, for which we can observe a symmetry breaking phenomenon: a snowflake has much more symmetry (it has crystal structure) than a drop of water (which consists of a random collection of molecules).

The Bost-Connes system $(\mathcal{A}, \sigma_t)$ also exhibits a phase transition phenomenon with symmetry breaking at $\beta = 1$. For $\beta < 1$, i.e., at high temperature, there is enough disorder so that the symmetry is trivial. For $\beta > 1$, the set of equilibrium states "freezes" and has as symmetry the Galois group of the maximal abelian extension of $\mathbb{Q}$. Bost and Connes also defined explicitly a rational subalgebra of $A_\mathbb{Q} \subset \mathcal{A}$ such that the evaluation of KMS states on $A_\mathbb{Q}$, at small temperature, generate $\mathbb{Q}^{\mathrm{ab}}$. This system is related to $\mathrm{GL}_{1,\mathbb{Q}}$.

Much more recently, Connes and Marcolli [CM04] defined an analogous system for $\mathrm{GL}_{2,\mathbb{Q}}$, and overcame extreme technicalities to give in this case a meaning to all prominent features of the Bost-Connes system (symmetries, rational subalgebra, zeta function as partition function, relation to the Galois group of the modular field and its modular reciprocity law). One of the key points in their new approach is that their system is related to the study of the "noncommutative space" of $\mathbb{Q}$-lattices up to commensurability:

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{M}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm};$$

and that the set of KMS_β states at small temperature is in natural bijection with the Shimura variety

$$\mathrm{Sh}(\mathrm{GL}_2, \mathbb{H}^{\pm}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}.$$

The C*-algebra corresponding to the "noncommutative space" of $\mathbb{Q}$-lattices up to commensurability is a groupoid C*-algebra. This also gives a nice explanation for the origin of the Bost-Connes system.

We choose one direction of generalizing this work of Connes and Marcolli by replacing their basic Shimura datum $(\mathrm{GL}_2, \mathbb{H}^{\pm})$ by a general Shimura datum (G, X). In order to deal with the technical issue of defining the partition function, the construction of the Connes-Marcolli system involves a groupoid $\mathcal{U}$, corresponding to the commensurability relation on $\mathbb{Q}$-lattices, and the quotient of $\mathcal{U}$ by the arithmetic subgroup $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Q})$. We start by defining an algebra in more adelic terms, meaning that we use a quotient by the compact open subgroup $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \subset \mathrm{GL}_2(\mathbb{A}_f)$. The motivation for this construction comes from the fact that for number fields, one wants the partition function to be the Dedekind zeta function, and this is easier to obtain in the adelic language (as pointed out by Paula Cohen [Coh99]).

The Connes-Marcolli algebra is not exactly a groupoid algebra, because the quotient of the groupoid $\mathcal{U}$ by $\mathrm{SL}_2(\mathbb{Z})$ is no longer a groupoid, since $\mathrm{SL}_2(\mathbb{Z})$ does not act freely on $\mathbb{H}$. In fact, if we use the stacky quotient, then this is a groupoid, but one cannot define an associated convolution C*-algebra because there is no good notion of functions on stacks. There are two solutions to this problem, corresponding to two resolutions of the stack's singularities. The first is to choose a smaller $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ that acts freely on $\mathbb{H}$. This gives a finite resolution of the stack singularities. However, this first method works only for classical Shimura varieties, which does not include the case of a general number field. The second solution is to identify functions on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R})/\mathbb{C}^\times$ to functions on $\mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R})$ (which is another infinite resolution of stack singularities) which are invariant for the scaling action of $\mathbb{C}^\times$. This allows one to define a convolution algebra. This second method was the one chosen by Connes and Marcolli.

The role of $\mathrm{M}_{2,\mathbb{Q}}$ in the $\mathrm{GL}_{2,\mathbb{Q}}$ case is, in the case of general Shimura data (G, X), played by a multiplicative semigroup M such that $\mathrm{M}^\times = G$. We learned a lot about such semigroups from N. Ramachandran and L. Lafforgue. Their main properties are given in the appendix.

This article describes the first steps in our work on these Bost-Connes-Marcolli systems for general Shimura data.

We solve along the way the problem of defining a Bost-Connes system for general number fields, which has the Dedekind zeta function as partition function *and* the connected component of idele class group as symmetry group. For imaginary quadratic fields, this problem was very recently solved by Connes-Marcolli-Ramachandran [CMR05]. Previous works were either restricted to class number one, or did not have the right symmetry, or did not have the right partition function. Van Frankenhuijsen and Laca [LvF04] defined a system with Galois group as symmetry group for totally imaginary fields of class number one. P. Cohen [Coh99] constructed a system with the right partition function. See also: Arledge-Laca-Raeburn [ALR97], Harari-Leichtnam [HL97], Laca [Lac98]. For a nice survey of known results, see [CM04], Section 1.4.

We also study the explicit example of the Hilbert-Blumenthal modular varieties.

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After writing this paper, we learned from V.Lafforgue another construction of the Bost-Connes algebra for number fields that will certainly be useful to study finer aspects of those systems in dimension 1.

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2 Another take on the Bost-Connes system

Before describing the general setting, we would like to present the illustrating example of the Bost-Connes system, which illuminates our general constructions.

2.1 The classical system

The *Bost-Connes groupoid* is given by the partially defined action of $\mathbb{Q}_+^\times$ on $\widehat{\mathbb{Z}}$. More precisely, it is given by

$$Z_{BC} = \{(g,\rho) \in \mathbb{Q}_+^\times \times \widehat{\mathbb{Z}} \mid g\rho \in \widehat{\mathbb{Z}}\}.$$

The unit space is $\widehat{\mathbb{Z}}$, and the source and target maps are given by $s(g,\rho) = \rho$ and $t(g,\rho) = g\rho$. Composition is given by $(g_2,\rho_2) \circ (g_1,\rho_1) = (g_2g_1,\rho_1)$ if $g_1\rho_1 = \rho_2$.

The Bost-Connes Hecke algebra is simply $\mathcal{H} := C_c(Z_{BC})$, equiped with the convolution product

$$(f_1 * f_2)(g,\rho) = \sum_{h\in\mathbb{Q}_+^\times, h\rho\in\widehat{\mathbb{Z}}} f_1(gh^{-1}, h\rho) f_2(h,\rho).$$

The time evolution on this algebra is given by

$$\sigma_t(f)(g,\rho) = g^{it} f(g,\rho). \tag{2.1}$$

For each $\rho_0 \in \widehat{\mathbb{Z}}^\times$, we define a representation $\pi_0 : \mathcal{H} \to \mathcal{B}(\ell^2(\mathbb{N}^\times))$ by

$$(\pi_0(f)(\xi))(n) = \sum_{h\in\mathbb{N}^\times} f(nh^{-1}, h\rho_0)\xi(h).$$

To finish, the Hamiltonian of this system is given by

$$H : \ell^2(\mathbb{N}^\times) \to \ell^2(\mathbb{N}^\times), f(n) \mapsto log(n)f(n).$$

By definition, the partition function

$$\zeta_{BC}(s) = \mathrm{Tr}(e^{-sH}) = \sum_{n\in\mathbb{N}^\times} n^{-s} = \zeta(s)$$

is exactly Riemann's zeta function.

2.2 The same system in adelic terms

We will now give a more complicated description of the Bost-Connes groupoid, that has the advantage of admitting a direct generalization to other number fields, whose partition function is the Dedekind zeta function. This is the first step to be carried out in constructing Bost-Connes systems for number fields. Moreover, the advantage of this adelic formulation is that it also makes sense for general Shimura varieties.

We first remark that the quotient of $\widehat{\mathbb{Z}}$ by the partially defined action of $\mathbb{Q}_+^\times$ is the same as the quotient of $\widehat{\mathbb{Z}} \times \{\pm 1\}$ by the partially defined action of $\mathbb{Q}^\times$. In fact, this equality of quotient spaces can be described at the level of groupoids.

Let $U^{\text{princ}} \subset \mathbb{Q}^\times \times \widehat{\mathbb{Z}} \times \{\pm 1\}$ be the groupoid of elements (g, ρ, z) such that $g\rho \in \widehat{\mathbb{Z}}$. This groupoid encodes the partially defined action of $\mathbb{Q}^\times$ on $\widehat{\mathbb{Z}} \times \{\pm 1\}$.

Now consider the quotient Z^{princ} of U^{princ} by the action of $(\mathbb{Z}^\times)^2 = \{\pm 1\}^2$ given by

$$(\gamma_1, \gamma_2).(g, \rho, z) := (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z).$$

This is also a groupoid.

There is a natural morphism of groupoids $Z_{BC} \to Z^{\text{princ}}$ given by $(r, \rho) \mapsto (r, \rho, 1)$, which is in fact an isomorphism (cf. 5.1.2).

This new descrition of the Bost-Connes groupoid is nicer because it clearly relates the Bost-Connes system with the pair $(\mathbb{G}_m, \{\pm 1\})$, which is called the *multiplicative Shimura datum*.

To make this connexion clearer, it is natural to seek a fully adelic description of the Bost-Connes groupoid. This is because Shimura varieties are defined adelically. The adelic framework also facilitates the definition of Bost-Connes systems for number fields with Dedekind zeta function as partition function.

Recall that $\mathbb{A}_f := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. The strong approximation theorem for the multiplicative group $\mathbb{G}_m$ tells us that

$$\mathbb{A}_f^\times = \mathbb{Q}_+^\times . \widehat{\mathbb{Z}}^\times = \widehat{\mathbb{Z}}^\times . \mathbb{Q}_+^\times .$$

We will now denote

$$\text{Sh}(\mathbb{G}_m, \{\pm 1\}) := \mathbb{Q}^\times \backslash \{\pm 1\} \times \mathbb{A}_f^\times$$

and

$$Y := \widehat{\mathbb{Z}} \times \text{Sh}(\mathbb{G}_m, \{\pm 1\}).$$

Consider the partially defined action of $\mathbb{A}_f^\times$ on Y given by

$$g.(\rho, [z, l]) := (g\rho, [z, lg^{-1}])$$

and let

$$U \subset \mathbb{A}_f^\times \times Y$$

be the corresponding groupoid of elements (g, y) such that $gy \in Y$.

Now consider the quotient Z of U by the action of $(\widehat{\mathbb{Z}}^\times)^2$ given by

$$(\gamma_1, \gamma_2).(g, y) := (\gamma_1 g \gamma_2^{-1}, \gamma_2 y).$$

The strong approximation theorem for $\mathbb{G}_m$ implies that the natural map

$$\begin{array}{ccc} \mathbb{Q}^\times \times \widehat{\mathbb{Z}} \times \{\pm 1\} & \to & \mathbb{A}_f^\times \times \widehat{\mathbb{Z}} \times \text{Sh}(\mathbb{G}_m, \{\pm 1\}) \\ (g, \rho, z) & \to & (g, \rho, [z, 1]) \end{array}$$

induces an isomorphism of groupoids $Z^{\text{princ}} \to Z$ (cf. 5.1.3).

The Bost-Connes algebra can thus be described as the algebra $\mathcal{H} = C_c(Z)$ with the convolution product

$$(f_1 * f_2)(g, y) = \sum_{h \in \widehat{\mathbb{Z}}^\times \backslash \mathbb{A}_f^\times, hy \in \mathcal{Y}} f_1(gh^{-1}, hy) f_2(h, y).$$

Using the isomorphism $d : \widehat{\mathbb{Z}}^\times \backslash \mathbb{A}_f^\times \to \mathbb{Q}_+^\times$, we can define the time evolution by

$$\sigma_t(f)(g, y) = d(g)^{it} f(g, y).$$

This is exactly the time evolution we defined in 2.1.

Let $\widehat{\mathbb{Z}}^\natural := \mathbb{A}_f^\times \cap \widehat{\mathbb{Z}}$ and $\mathbb{Z}^\natural := \mathbb{Z} - \{0\}$. The strong approximation theorem gives us that

$$\widehat{\mathbb{Z}}^\times \backslash \widehat{\mathbb{Z}}^\natural \cong \mathbb{Z}^\times \backslash \mathbb{Z}^\natural \cong \mathbb{N}^\times.$$

Let $\mathcal{H}_0 := \ell^2(\widehat{\mathbb{Z}}^\times \backslash \widehat{\mathbb{Z}}^\natural) \cong \ell^2(\mathbb{N}^\times)$. For each $\rho_0 \in \widehat{\mathbb{Z}}^\times$, we define a representation $\pi_0 : \mathcal{H} \to \mathcal{B}(\mathcal{H}_0)$ by

$$(\pi_0(f)(\xi))(n) = \sum_{h \in \mathbb{N}^\times} f(nh^{-1}, h\rho_0)\xi(h).$$

To finish, the Hamiltonian of this system is given by

$$H : \mathcal{H}_0 \to \mathcal{H}_0, f(n) \mapsto log(d(n))f(n).$$

This adelic system is perfectly identical to the original Bost-Connes system. We will however see in the sequel that essentially the same definitions of algebra, time evolution, representations and Hamiltonian now work for general Shimura data.

3 Background material

In this paper we draw upon the theory of Shimura varieties and operator algebras. Since these fields have traditionally had little to do with each other, we review for the convenience of the reader some of the basic (well-known) results that we shall need. This also allows us to establish notation. We stress that our definition of a Shimura variety is a slight variation on the usual one given by Deligne [Del79], 2.1.

3.1 Shimura varieties

First recall briefly the definition of Shimura data. We will use a mix of Deligne's definition (see [Del79], 2.1) and Pink's definition (see [Pin90], 2.1). Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$.

Definition 3.1.1. A *Shimura datum* is a triple (G, X, h), with G a connected reductive group over $\mathbb{Q}$, X a left homogeneous space under $G(\mathbb{R})$ and $h : X \to \mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$ a $G(\mathbb{R})$-equivariant map [1] with finite fibres such that:

1. For $h_x \in h(X)$, $\mathrm{Lie}(G_\mathbb{R})$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$;
2. The involution $\mathrm{int}\, h_x(i)$ is a Cartan involution of the adjoint group $G_\mathbb{R}^{ad}$;
3. The adjoint group has no factor G' defined over $\mathbb{Q}$ on which the projection of h_x is trivial.

A Shimura datum is called *classical* if it moreover fulfils the axiom

4 Let $Z_0(G)$ be the maximal split subtorus of the center of G; then $\mathrm{int}\, h_x(i)$ is a Cartan involution of $G/Z_0(G)$.

[1] for the natural conjugation action of $G(\mathbb{R})$ on $\mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$

Example 3.1.2. Let F be a number field, $T = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ and $X_F = T(\mathbb{R})/T(\mathbb{R})^+$. We have $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^i \times \mathbb{R}^j$. We put on $F \otimes_{\mathbb{Q}} \mathbb{R}$ the Hodge structure that is trivial on $\mathbb{R}^j$ and given by the complex structure on $\mathbb{C}^i$. This gives a morphism $h_1 : \mathbb{S} \to T_{\mathbb{R}}$. The triple $(\mathbb{G}_{m,F}, X_F, h_1)$ is called the *multiplicative Shimura datum of the field* F. This Shimura datum is classical if and only if $F = \mathbb{Q}$ of F is imaginary quadratic.

We will often denote a Shimura data just by a couple (G, X) when the morphism h is clear from the situation.

Example 3.1.3. Let $h : \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$ be the morphism given by $h(a+ib) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. Let $\mathbb{H}^{\pm}$ be the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of h. It identifies with the Poincaré double half plane with action of $\mathrm{GL}_2(\mathbb{R})$ by homographies. Then $(\mathrm{GL}_2, \mathbb{H}^{\pm})$ is called the *modular Shimura datum.*

Definition 3.1.4. Let (G, X) be a Shimura datum. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. The *level K Shimura variety* is

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

and the *Shimura variety* is the projective limit

$$\mathrm{Sh}(G, X) := \varprojlim_K \mathrm{Sh}_K(G, X)$$

over all compact open subgroups $K \subset G(\mathbb{A}_f)$.

We first remark that, from the modular viewpoint, it is more natural to study the *level K Shimura stack*, given by the stacky quotient

$$\mathfrak{Sh}_K(G, X) := [G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K].$$

In the case of the multiplicative Shimura datum of a number field, i.e., $G = \mathbb{G}_{m,F}$, this stack can have infinite isotropy groups given by $G(\mathbb{Q})^+ \cap K$. These isotropy groups are given by generalized congruence relations on the group of units $\mathcal{O}_F^\times$. However, if we pass to the limit, the isotropy groups are given by the intersection $\bigcap_K \ (G(\mathbb{Q})^+ \cap K)$ over all compact open subgroups K in $G(\mathbb{A}_f)$, i.e., are then trivial. This gives an isomorphism

$$\mathrm{Sh}(G, X) \cong \varprojlim_K \mathfrak{Sh}_K(G, X),$$

which shows that we do not loose information by considering only coarse spaces in the study of the full Shimura variety.

There is a natural right action of $G(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X)$ given, for each $g \in G(\mathbb{A}_f)$ and $K \subset G(\mathbb{A}_f)$ compact open, by an isomorphism

$$\begin{array}{cccc} (.g): & \mathfrak{Sh}_K(G, X) & \to & \mathfrak{Sh}_{g^{-1}Kg}(G, X) \\ & [x, l] & \mapsto & [x, lg]. \end{array}$$

Under the hypothesis that (G, X) is classical (see also [Del79], 2.1.1.4, 2.1.1.5), there is an easier description of the Shimura variety (see [Del79], Corollaire 2.1.11):

$$\mathrm{Sh}(G, X) \cong G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

Unfortunately, these hypothesis are not always fulfilled in the case of the multiplicative Shimura datum of a general number field F. In fact, the quotient $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$ is not always Hausdorff in this case.

For example, if $F = \mathbb{Q}(\sqrt{2})$,

$$\mathrm{Sh}(\mathbb{G}_{m,F}, X_F) \ncong F^\times\backslash X_F \times \mathbb{A}_{f,F}^\times$$

(this is essentially due to the fact that the group of units $\mathcal{O}_F^\times$ is infinite).

Points in a Shimura variety will be denoted by pairs $[z, l]$. If the Shimura datum is classical, this means that $z \in X$ and $l \in G(\mathbb{A}_f)$. Otherwise, $[z, l] = [z_K, l_K]_{K \subset G(\mathbb{A}_f)}$ is a family of points in $\mathrm{Sh}_K(G, X)$ indexed by the set of compact open subgroups in $G(\mathbb{A}_f)$.

Definition 3.1.5. Let (G, X) be a Shimura datum. A compact open subgroup $K \subset G(\mathbb{A}_f)$ is called *fine* if it acts freely on $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$.

We would now like to be able to define *natural* algebras of continuous "functions" on the finite Shimura varieties in play. In order to do that, we have to resolve their stack singularities.

We remark that if K is fine, then the quotient analytic stack

$$\mathfrak{Sh}_K(G, X) = [G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K]$$

is a usual analytic space, but otherwise it is worthwhile from the moduli viewpoint to keep track of the nontrivial stack structure. For classical Shimura data, one can resolve the stack singularities by choosing a smaller compact open subgroup $K' \subset G(\mathbb{A}_f)$ that acts freely. This is what we will usually do in order to be able to define continuous "functions" on the stack $\mathfrak{Sh}_K(G, X)$.

However, this finite resolution of the stack singularities is usually not possible for nonclassical Shimura data, as we can see on the following example. Let $F = \mathbb{Q}(\sqrt{2})$ and $(\mathbb{G}_{m,F}, X_F)$ be the corresponding Shimura datum (here $X_F \cong \{\pm 1\}^2$). Let $K = \widehat{\mathcal{O}}_F^\times$ and consider the stack $\mathfrak{Sh}_K(\mathbb{G}_{m,F}, X_F)$. Its coarse quotient is the ideal class group of F, i.e., the trivial group $\{1\}$. Since F has class number one, this coarse quotient can also be described as $\mathcal{O}_F^\times\backslash X_F$. In this case, $\mathcal{O}_F^\times$ is infinite, so that we can not choose a smaller $K' \subset K$ that acts freely on $F^\times\backslash X_F \times \mathbb{A}_{f,F}^\times$. In fact, the unit group is finite if and only if $F = \mathbb{Q}$ or F is imaginary quadratic, i.e., if and only if $(\mathbb{G}_{m,F}, X_F)$ is classical in our language.

If we want to resolve the stack singularities, we can use the quotient map

$$F^\times\backslash\mathbb{A}_F^\times/K \to \mathfrak{Sh}_K(\mathbb{G}_{m,F}, X_F)$$

for the scaling action of the connected component of identity D_F in the full idele class group $C_F := F^\times\backslash\mathbb{A}_F^\times$.

Remark 3.1.6. From the viewpoint of moduli spaces, it is important that the coarse space $\mathrm{Sh}_K(\mathbb{G}_{m,F}, X_F)$, i.e., the big ideal class group, be replaced by the corresponding group stack with infinite stabilizers (given by groups of units with congruence conditions):

$$\mathfrak{Sh}_K(\mathbb{G}_{m,F}, X_F).$$

This "equivariant viewpoint" of the finite level Shimura variety could also be important to understand geometrically the definition of Stark's zeta functions, and also for the understanding of Manin's real multiplication program [Man].

3.2 C*-algebras and quantum statistical mechanics

We review here basic definitions from the theory of C*-algebras, emphasising those parts relevant to quantum statistical mechanics. Good references for the material in this section are [BR87] and [BR97]. For an overview of the grand physical picture, see [Haa96].

Definition 3.2.1. A *C*-algebra* is a (not necessarily unital) complex algebra A endowed with a conjugate-linear involutive anti-automorphism $^*\colon A \to A$, and a norm $\|\cdot\|$, satisfying the following conditions: For every a, $b \in A$ we have

1. A is complete with respect to the norm, and $\|ab\| \leq \|a\|\|b\|$ (i.e., A is a *Banach algebra*); and
2. $\|a^*a\| = \|a\|^2$, the crucial *C*-condition.*

Actually a C*-algebra is not as abstract as it may seem, because every C*-algebra can be realized as a norm-closed subalgebra of the algebra of bounded operators on a Hilbert space (Theorem of Gelfand-Naimark [BR87], Theorem 2.1.10), and every such subalgebra is a C*-algebra.

The operator algebraic formulation of quantum statistical mechanics (see the introduction to [BR87]) consists of a C*-algebra A together with a 1-parameter group of automorphism $\sigma_t\colon A \to A$, which is continuous in the sense that $t \mapsto \sigma_t(a)$ is continous for every $a \in A$. The algebra A is then the algebra of quantum observables, while σ_t is the time evolution. The pair (A, σ_t) is an example of a *C*-dynamical system.* The *states* of the C*-algebra A are the continuous complex-linear functionals Φ of norm 1 which are positive, i.e., $\Phi(a^*a) \geq 0$ for every $a \in A$. The number $\Phi(a)$ is then the expectation value of the observable a in the physical state Φ.

To regard the pair (A, σ_t) as a statistical mechanical system we need an appropriate notion of an "equilibrium state" at temperature $T = 1/\beta$. This is provided by the KMS condition.

Definition 3.2.2. The *KMS-β condition* $(0 < \beta < \infty)$ for a state Φ is the condition: For every pair of elements $a, b \in A$, there is a complex-valued function F on the closed strip $\Omega = \{\, z \in \mathbb{C} \mid 0 \leq \mathrm{im} z \leq \beta \,\}$ such that

$$F(t) = \Phi\big(a\sigma_t(b)\big), \quad F(t + i\beta) = \Phi\big(\sigma_t(b)a\big);$$

furthermore, the function F is required to be bounded and continuous on Ω, and analytic on its interior.

This is the definition one often sees in the literature, although in practice it is easier to use the following equivalent characterization.

Proposition 3.2.3. *Let (A, σ_t) be a C*-dynamical system, and let Φ be a state of A.*

1. *([BR87], Corollary 2.5.23) There is a norm-dense *-subalgebra A^{an} of A on which the time evolution has an analytic continuation to an entire function $z \mapsto \sigma_z(a)$.*
2. *([BR87], Definition 5.3.1 and Corollary 5.3.7) The state Φ is a KMS-β state if and only if*
$$\Phi\big(a\sigma_{i\beta}(b)\big) = \Phi(ba)$$
*for all a, b in a norm-dense σ_t-invariant *-subalgebra of A^{an}.*

We now proceed to a description of the structure of the set of KMS-β states. But before doing so, we need to explain the GNS construction, which is a method of getting representations of a C*-algebra from its states; it is a basic, widely used result in the theory of operator algebras. We also need to define the notion of a factor state. We shall use standard notation: given a Hilbert space $\mathcal{H}$, we denote the C*-algebra of all bounded operators on $\mathcal{H}$ by $B(\mathcal{H})$, and the inner product on $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$.

Proposition 3.2.4 (GNS construction; [BR87], 2.3.16). *Let Φ be a state of a C*-algebra A. Then there is a triple $(\mathcal{H}_\Phi, \pi_\Phi, \xi_\Phi)$ consisting of a representation π_Φ of A on a Hilbert space $\mathcal{H}_\Phi$ and a unit vector $\xi_\Phi \in \mathcal{H}_\Phi$ such that:*

1. *$\Phi(a) = \langle \pi_\Phi(a)\xi_\Phi, \xi_\Phi \rangle$ for all $a \in A$; and*
2. *The orbit $\pi_\Phi(A)\xi_\Phi$ is norm-dense in $B(\mathcal{H}_\Phi)$.*

The triple $(\mathcal{H}_\Phi, \pi_\Phi, \xi_\Phi)$ is unique up to unitary equivalence.

The states of particular relevance to the KMS theory are the *factor states*. These are the states Φ for which the corresponding GNS representation π_Φ generates a *factor*, which is to say that the weak closure of $\pi_\Phi(A)$ in $B(\mathcal{H}_\Phi)$ has centre consisting of the scalar operators. (This weak closure is an example of a *Von Neumann algebra.*)

We can now state the main structure theorem for the set of KMS-β states.

Proposition 3.2.5 (Structure of KMS states; [BR97], Theorem 5.3.30). *The set $\mathcal{E}_\beta$ of KMS-β states is a convex, weak*-compact simplex. The extreme points of $\mathcal{E}_\beta$ are precisely those KMS-β states that are factor states.*

4 Abstract Bost-Connes-Marcolli systems

The aim of this section is to define Bost-Connes-Marcolli systems for general Shimura data (G, X) and study their basic formal properties. A better understanding of the general setup might be gained by looking at section 7 where we specialise to the case of multiplicative Shimura datum (the case relevant for number fields).

4.1 BCM data

In order to define a generalization of the Connes-Marcolli algebra to general Shimura data, we want to make clear the separation between algebraic and level structure data, which is already implicit in the construction of Connes and Marcolli.

Algebraic data. We first need to consider a semigroup M which plays the role for general reductive groups that $\mathrm{M}_{2,\mathbb{Q}}$ plays for $\mathrm{GL}_{2,\mathbb{Q}}$.

Definition 4.1.1. Let G be reductive group over a field. An *enveloping semigroup* for G is a multiplicative semigroup M which is irreducible and normal, and such that $M^\times = G$.

Definition 4.1.2. A *BCM datum* is a tuple $\mathcal{D} = (G, X, V, M)$ with (G, X) a Shimura datum, V a faithful representation of G and M an enveloping semigroup for G contained in $\mathrm{End}(V)$.

The faithful representation will often be denoted $\phi : G \to \mathrm{GL}(V)$.

Level structure data. Every Shimura datum (G, X) comes implicitly with a family of level structures given by the family of compact open subgroups $K \subset G(\mathbb{A}_f)$. Connes and Marcolli fixed the full level structure $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \subset \mathrm{GL}_2(\mathbb{A}_f)$ as starting datum for their construction. To avoid the problem they had with stack singularities of their groupoid, we will fix a finer level structure as part of the datum.

The level structure also plays a role in defining the partition function of our system. Consideration of maximal level structures then yields standard zeta functions as partition functions, for example, the Dedekind zeta function of a number field. A technical requirement in the definition of the partition function is a choice of a lattice in the representation of G, to be able to define a rational determinant for the adelic matrices in play.

Definition 4.1.3. Let $\mathcal{D} = (G, X, V, M)$ be a BCM datum. A *level structure on $\mathcal{D}$* is a triple $\mathcal{L} = (L, K, K_M)$, with $L \subset V$ a lattice, $K \subset G(\mathbb{A}_f)$ a compact open subgroup and $K_M \subset M(\mathbb{A}_f)$ a compact open subsemigroup, such that

- K_M stabilizes $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$,
- $\phi(K)$ is contained in K_M.

The pair $(\mathcal{D}, \mathcal{L})$ will be called *a BCM pair*.

We can summarize the relation between L, K and K_M by the following diagram:

$$\begin{array}{ccccc} K & \xrightarrow{\phi} & K_M & \hookrightarrow & \mathrm{End}(L)(\widehat{\mathbb{Z}}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G(\mathbb{A}_f) & \xrightarrow{\phi} & M(\mathbb{A}_f) & \hookrightarrow & \mathrm{End}(V)(\mathbb{A}_f) \end{array}$$

Definition 4.1.4. The *maximal level structure* $\mathcal{L}_0 = (L, K_0, K_{M,0}^{\times})$ associated with a datum $\mathcal{D} = (G, X, V, M)$ and a lattice $L \subset V$ is defined by setting

$$\begin{array}{rcl} K_{M,0} & := & M(\mathbb{A}_f) \cap \mathrm{End}(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}), \\ K_0 & := & \phi^{-1}(K_{M,0}^{\times}). \end{array}$$

Definition 4.1.5. The level structure $\mathcal{L}$ on $\mathcal{D}$ is called *fine* if K acts freely on $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$.

The maximal level structure is usually not fine enough to avoid stack singularity problems in the generalization of the Connes-Marcolli algebra. This is why we introduce the additional data of a compact open subgroup $K \subset K_M$. For example, for the Connes-Marcolli case, one takes $K = \mathrm{GL}_2(\widehat{\mathbb{Z}})$, $K_M = \mathrm{M}_2(\widehat{\mathbb{Z}})$, but the fact that this choice of K is not fine implies that the groupoid we introduce in the next section has stack singularities. Thus we instead choose a smaller $K = K(N) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ given by the kernel of the mod N reduction of matrices.

Symmetries and zeta function. The symmetries of the Connes-Marcolli system play an important role in its relations with arithmetic. The analogous symmetry in our generalization is the following (which will be justified in Subsection 4.5).

Definition 4.1.6. The semigroup $\mathrm{Sym}_f(\mathcal{D}, \mathcal{L}) := \phi^{-1}(K_M)$ is called the *finite symmetry semigroup of the BCM pair* $(\mathcal{D}, \mathcal{L})$. We will denote by $\mathrm{Sym}_f^{\times}(\mathcal{D}, \mathcal{L})$ the group of invertible elements in $\mathrm{Sym}_f(\mathcal{D}, \mathcal{L})$.

We included in $\mathcal{L}$ the datum of a lattice in the representation ϕ in order to define a determinant map.

Lemma 4.1.7. *The determinant* $\det : \mathrm{GL}(L) \to \mathbb{G}_m$ *induces a natural map,*

$$(\det \circ \phi) : K\backslash G(\mathbb{A}_f)/K \to \mathbb{Q}_+^{\times}.$$

The image of $\mathrm{Sym}_f(\mathcal{D}, \mathcal{L})$ *under this map is contained in* $\mathbb{N}^{\times}$.

Proof. Since $\phi(K) = K_M^{\times} \subset \mathrm{GL}(L)(\widehat{\mathbb{Z}})$, the representation $\phi : G \to \mathrm{GL}(L \otimes_{\mathbb{Z}} \mathbb{Q})$ induces a map

$$\phi : K\backslash G(\mathbb{A}_f)/K \to \mathrm{GL}(L)(\widehat{\mathbb{Z}})\backslash \mathrm{GL}(L)(\mathbb{A}_f)/\mathrm{GL}(L)(\widehat{\mathbb{Z}}).$$

The determinant map $\mathrm{GL}(L) \to \mathbb{G}_m$ induces a natural map

$$\det : \mathrm{GL}(L)(\widehat{\mathbb{Z}})\backslash \mathrm{GL}(L)(\mathbb{A}_f)/\mathrm{GL}(L)(\widehat{\mathbb{Z}}) \to \widehat{\mathbb{Z}}^{\times}\backslash \mathbb{A}_f^{\times}/\widehat{\mathbb{Z}}^{\times} \cong \widehat{\mathbb{Z}}^{\times}\backslash \mathbb{A}_f^{\times} \cong \mathbb{Z}^{\times}\backslash \mathbb{Q}^{\times} \cong \mathbb{Q}_+^{\times}.$$

The composition $\det \circ \phi$ gives us the desired map. The image of Sym_f under this map is contained in the image of $\mathrm{GL}(L)(\mathbb{A}_f) \cap (\mathrm{End}(L) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ under the determinant map, which is exactly $\widehat{\mathbb{Z}}^{\natural} := \mathbb{A}_f^{\times} \cap \widehat{\mathbb{Z}}$. The quotient $\widehat{\mathbb{Z}}^{\times}\backslash \widehat{\mathbb{Z}}^{\natural}$ is identified with $\mathbb{Z}^{\times}\backslash \mathbb{Z} \cong \mathbb{N}^{\times} \subset \mathbb{Q}^{\times}$. □

Definition 4.1.8. The *zeta function* of the BCM pair $(\mathcal{D}, \mathcal{L})$ is the complex valued series

$$\zeta_{\mathcal{D},\mathcal{L}}(\beta) := \sum_{g \in \mathrm{Sym}_f^\times \backslash \mathrm{Sym}_f} \det(\phi(g))^{-\beta}.$$

The BCM pair $(\mathcal{D}, \mathcal{L})$ is called *summable* if there exists $\beta_0 \in \mathbb{R}$ such that $\zeta_{\mathcal{D},\mathcal{L}}(\beta)$ converges in the right plane $\{\beta \in \mathbb{C} \mid \mathrm{Re}(\beta) > \beta_0\}$ and extends to a meromorphic function on the full complex plane.

4.2 The BCM groupoid

Let $(\mathcal{D}, \mathcal{L}) = ((G, X, V, M), (L, K, K_M))$ be a BCM pair. There are left and right action of $G(\mathbb{A}_f)$ on $M(\mathbb{A}_f)$.

4.2.1 Definition

Connes and Marcolli remarked in [CM04] that, if we want to take a quotient of a groupoid by a group action, it is essential that the action is free on the unit space of the groupoid. If we take the usual quotient set of a groupoid by an action that is not free on the unit space, this will not give a groupoid. We are thus obliged to use unit spaces that are in fact stacks. Some of them have nice (i.e., with finite stabilizers) singularities. Others don't, but the language of stacks allows one to work in full generality without bothering about the freeness of actions in play.

We will denote the stacks by german letters; the corresponding coarse spaces will be denoted by right letters.

Let $Y_{\mathcal{D},\mathcal{L}} = K_M \times \mathrm{Sh}(G, X)$. We denote points of $Y_{\mathcal{D},\mathcal{L}}$ by triples $y = (\rho, [z, l])$ with $\rho \in K_M$, $[z, l] \in \mathrm{Sh}(G, X)$.

We want to study the equivalence relation on $Y_{\mathcal{D},\mathcal{L}}$ given by the following partially defined action of $G(\mathbb{A}_f)$:

$$g.y = (g\rho, [z, lg^{-1}]), \quad \text{where } y = (\rho, [z, l]).$$

This equivalence relation will be called the *commensurability relation*. This terminology is derived from the notion of commensurability for $\mathbb{Q}$-lattices, cf. [CM04].

Consider the subspace

$$U_{\mathcal{D},\mathcal{L}} \subset G(\mathbb{A}_f) \times Y_{\mathcal{D},\mathcal{L}}$$

of pairs (g, y) such that $gy \in Y_{\mathcal{D},\mathcal{L}}$, i.e. $g\rho \in K_M$.

The space $U_{\mathcal{D},\mathcal{L}}$ is a groupoid with unit space $Y_{\mathcal{D},\mathcal{L}}$. The source and target maps $s : U_{\mathcal{D},\mathcal{L}} \to Y_{\mathcal{D},\mathcal{L}}$ and $t : U_{\mathcal{D},\mathcal{L}} \to Y_{\mathcal{D},\mathcal{L}}$ are given by $s(g, y) = y$ and $t(g, y) = gy$. The composition is given, for $y_1 = g_2 y_2$, by $(g_1, y_1) \circ (g_2, y_2) = (g_1 g_2, y_2)$. Notice that the groupoid obtained by restricting this groupoid to the $(g, (\rho, [z, l]))$ such that ρ is invertible is free, i.e., the equality $t(g, y) = s(g, y)$ implies $g = 1$.

There is a natural action of K^2 on the groupoid $U_{\mathcal{D},\mathcal{L}}$, given by

$$(g, y) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 y),$$

and the induced action on $Y_{\mathcal{D},\mathcal{L}}$ is given by

$$y \mapsto \gamma_2 y.$$

There are two motivations for quotienting $U_{\mathcal{D},\mathcal{L}}$ by this action. The first one is physical: it is necessary to obtain a reasonable partition function for our system. The second is moduli theoretic: $U_{\mathcal{D},\mathcal{L}}$ is only a pro-analytic groupoid and the quotient by K^2 is fibered over the Shimura variety $\mathfrak{Sh}_K(G, X)$ which is an algebraic moduli stack of finite type whose definition could be made over $\overline{\mathbb{Q}}$, at least when (G, X) is classical and the Shimura variety has a canonical model.

Let $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ be the quotient stack $[K^2\backslash U_{\mathcal{D},\mathcal{L}}]$ and $\mathfrak{S}_{\mathcal{D},\mathcal{L}}$ be the quotient stack $[K\backslash Y_{\mathcal{D},\mathcal{L}}]$. The natural maps

$$s, t : \mathfrak{Z}_{\mathcal{D},\mathcal{L}} \to \mathfrak{S}_{\mathcal{D},\mathcal{L}}$$

define a stack-groupoid structure on $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ with unit stack $\mathfrak{S}_{\mathcal{D},\mathcal{L}}$.

Definition 4.2.2. The stack-groupoid $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ is called the *Bost-Connes-Marcolli*[2] groupoid.

Let $Z_{\mathcal{D},\mathcal{L}} := K^2\backslash U_{\mathcal{D},\mathcal{L}}$ be the (classical, i.e., coarse) quotient of $U_{\mathcal{D},\mathcal{L}}$ by the action of K^2. If K is small enough, i.e., if K acts freely on $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$, then $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ is equal to the classical quotient $Z_{\mathcal{D},\mathcal{L}}$, which is a groupoid in the usual sense, with units $S = K\backslash Y_{\mathcal{D},\mathcal{L}}$. Otherwise, suppose that there exists a compact open subgroup $K' \subset K$ that acts freely on $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$ and choose on $\mathcal{D}$ the level structure $\mathcal{L}' = (L, K', K_M)$. The stack $\mathfrak{Z}_{\mathcal{D},\mathcal{L}'}$ is a usual topological space that is a finite covering of the coarse space $Z_{\mathcal{D},\mathcal{L}}$ and such that the stack $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ is the stacky quotient of $Z_{\mathcal{D},\mathcal{L}'}$ by the projection equivalence relation to $Z_{\mathcal{D},\mathcal{L}}$.

The reader who prefers to work with usual analytic spaces will thus suppose that K is small enough, but as we remarked before, our basic examples (number fields) do not fulfil this hypothesis. We have also to recall that for nonclassical Shimura data (G, X) in the sense of definition 3.1.1, there exists no such small enough $K \subset G(\mathbb{A}_f)$. This is essentially due to the fact that the "unit group" $C(\mathbb{Q}) \cap K$ (where C denotes the center of G) can be infinite.

4.2.3 The commensurability class map

Recall that ϕ is the representation of G, that we will see as the natural map $G \to M$.

For classical Shimura data. We want to give an explicit description of the quotient of $Y_{\mathcal{D},\mathcal{L}}$ by the commensurability equivalence relation, in the case where (G, X) is classical, i.e., when

$$\mathrm{Sh}(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

Let $K_{\mathbb{A}}^M = \phi(G)(\mathbb{A}_f).K_M \subset M(\mathbb{A}_f)$. There is a natural surjective map of sets

$$\pi : Y_{\mathcal{D},\mathcal{L}} \to G(\mathbb{Q})\backslash X \times K_{\mathbb{A}}^M$$

given by $\pi(\rho, [z, l]) = [z, l\rho]$.

Let $Y_{\mathcal{D},\mathcal{L}}^\times = K_M^\times \times (G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f))$ be the invertible part of $Y_{\mathcal{D},\mathcal{L}}$ and $Z_{\mathcal{D},\mathcal{L}}^\times \subset \mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ be the corresponding subspace (which is a groupoid in the usual sense because K acts freely on $K_M^\times$). Let $S_{\mathcal{D},\mathcal{L}}^\times := K\backslash Y_{\mathcal{D},\mathcal{L}}^\times$ be the unit space of $Z_{\mathcal{D},\mathcal{L}}^\times$. Since $K_M^\times \subset M^\times(\mathbb{A}_f) = \phi(G(\mathbb{A}_f))$, the map π induces a natural map

$$\begin{array}{rccc} \pi^\times : & Y_{\mathcal{D},\mathcal{L}}^\times & \to & G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f) \\ & (\rho, [z, l]) & \mapsto & [z, l\phi^{-1}(\rho)], \end{array}$$

which is complex analytic and surjective. Both π and $\pi^\times$ pass to the quotient of their sources by the left action of K. We will continue to denote this factorisation by π and $\pi^\times$.

Definition 4.2.4. The maps π and $\pi^\times$ are called the *commensurability class maps.*

This definition is justified by the following lemma.

Lemma 4.2.5. *The maps π and $\pi^\times$ are in fact the coarse quotient maps for the groupoids $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ and $Z_{\mathcal{D},\mathcal{L}}^\times$ acting on their unit spaces $\mathfrak{S}_{\mathcal{D},\mathcal{L}}$ and $S_{\mathcal{D},\mathcal{L}}^\times$.*

[2]We will often call it the BCM groupoid, for short.

Proof. Let us now prove that π and $\pi^\times$ are the coarse quotient maps. If $(g,\rho,[z,l]) \in U_{\mathcal{D},\mathcal{L}}$, then $\pi(g\rho,[z,lg^{-1}]) = [z,l\rho] = \pi(\rho,[z,l])$ which proves that π factors through

$$|\mathfrak{S}_{\mathcal{D},\mathcal{L}}/\mathfrak{Z}_{\mathcal{D},\mathcal{L}}| \to G(\mathbb{Q})\backslash X \times K_{\mathbb{A}}^M.$$

This surjective map is in fact an isomorphism. Indeed, if $(\rho,[z,l]),(\rho',[z',l']) \in Y_{\mathcal{D},\mathcal{L}}$ have same image under π, then there exists $g \in G(\mathbb{Q})$ such that $gl\rho = l'\rho'$ and $gz = z'$. We then know that in the quotient space $|\mathfrak{S}_{\mathcal{D},\mathcal{L}}/\mathfrak{Z}_{\mathcal{D},\mathcal{L}}|$,

$$\begin{array}{rcl}(\rho,[z,l]) & = & (l^{-1}g^{-1}gl\rho,[z,l]) \\ & = & (l^{-1}g^{-1}l'\rho',[z,l]) \\ & \sim & (l'^{-1}gll^{-1}g^{-1}l'\rho',[z,ll^{-1}g^{-1}l']) \\ & = & (\rho',[z,g^{-1}l']) \\ & = & (\rho',[gz,l']) \\ & = & (\rho',[z',l']).\end{array}$$

This proves injectivity of π and surjectivity was already known. □

For commutative Shimura data. Commutative Shimura data form another family of examples for which we can construct the commensurability class map in simple terms. The multiplicative datum of a number field is in this familly. Thus we now suppose that $\mathcal{D} = (G,X,V,M)$ is a BCM datum such that G and M are commutative and let $\mathcal{L}$ be a level structure on $\mathcal{D}$. For each $K', K \subset G(\mathbb{A}_f)$ compact open, there is a natural map

$$\mathfrak{Y}_{K'} := K_M \times \mathfrak{Sh}_{K'}(G,X) \to [G(\mathbb{Q})\backslash X \times M(\mathbb{A}_f)/K']$$

given by $(\rho,[z,l]) \mapsto [z,l\rho]$. This map is K-equivariant for the trivial action of K on the right hand side because the image of $k.(\rho,[z,l]) = (k\rho,[z,lk^{-1}])$ is equal to the image of $(\rho,[z,l])$. Recall that $\mathfrak{S}_{\mathcal{D},\mathcal{L}} := [K\backslash Y_{\mathcal{D},\mathcal{L}}]$ and $S^\times_{\mathcal{D},\mathcal{L}} = K\backslash Y^\times_{\mathcal{D},\mathcal{L}}$. If we pass to the limit on $K' \subset G(\mathbb{A}_f)$, and then to the quotient by K, we obtain natural maps

$$\pi : \mathfrak{S}_{\mathcal{D},\mathcal{L}} \to \varprojlim_{K'} [G(\mathbb{Q})\backslash X \times M(\mathbb{A}_f)/K']$$

and

$$\pi^\times : S^\times_{\mathcal{D},\mathcal{L}} \to \mathrm{Sh}(G,X)$$

that will be called as before the *commensurability class maps.*

The image of the map π is as before the coarse quotient for the action of the groupoid $\mathfrak{Z}$ on its unit space $\mathfrak{S}$.

Denote $\mathfrak{S}_{K'} := K\backslash(K_M \times \mathfrak{Sh}_{K'}(G,X))$. We should remark here that in this commutative case, the space $\mathfrak{S}_{K'}$ is the unit space of a well defined groupoid $\mathfrak{Z}_{K'}$ because the $G(\mathbb{A}_f)$ action on $\mathfrak{Sh}_{K'}(G,X)$ is well defined. This shows that

$$\mathfrak{Z} = \varprojlim_{K'} \mathfrak{Z}_{K'}, \tag{4.1}$$

which will be useful for the description of the symmetries of Bost-Connes systems for number fields.

4.3 Defining BCM algebras

4.3.1 Functions on BCM stacks?

Let $\mathcal{D} = (G,X,V,M)$ be a BCM datum and V be a representation of G. We would like to define the BCM algebra of $(\mathcal{D},\mathcal{L}_0)$ as a groupoid algebra. Unfortunately, the corresponding groupoid is usually only a stack and there is no canonical notion of functions on such a space.

If (G, X) is classical, there is a very natural way to resolve the stack singularities of $\mathfrak{Z}_{\mathcal{D},\mathcal{L}_0}$ by choosing a fine level structure $\mathcal{L}$, for which the projection map

$$Z_{\mathcal{D},\mathcal{L}} \to \mathfrak{Z}_{\mathcal{D},\mathcal{L}_0}$$

is such a resolution. The corresponding algebra is a completely natural replacement for the groupoid algebra of the stack-groupoid $\mathfrak{Z}_{\mathcal{D},\mathcal{L}_0}$.

If (G, X) is nonclassical, there is no nice resolution of the stack singularities of $\mathfrak{Z}_{\mathcal{D},\mathcal{L}_0}$. We thus have to work with the algebra of functions on the coarse quotient $Z_{\mathcal{D},\mathcal{L}_0}$. However $Z_{\mathcal{D},\mathcal{L}_0}$ is *not* a groupoid, and so to define a convolution algebra from the function algebra $C_c(Z_{\mathcal{D},\mathcal{L}_0})$, we use the trick used by Connes and Marcolli in [CM04], 1.83. Namely, we introduce the groupoid

$$\mathcal{R}_{\mathcal{D},\mathcal{L}_0} \subset K\backslash G(\mathbb{A}_f) \underset{K}{\times} (K_M \times \lim_{\longleftarrow K'} G(\mathbb{Q})\backslash G(\mathbb{A})/K').$$

and identify $C_c(Z_{\mathcal{D},\mathcal{L}_0})$ with the subalgebra of $C_c(\mathcal{R}_{\mathcal{D},\mathcal{L}})$ obtained by composing by the projection map $\mathcal{R}_{\mathcal{D},\mathcal{L}_0} \to Z_{\mathcal{D},\mathcal{L}_0}$. Since $\mathcal{R}_{\mathcal{D},\mathcal{L}_0}$ is a groupoid, convolution can be defined on $C_c(Z_{\mathcal{D},\mathcal{L}})$.

Remark that this solution, even if not completely satisfactory from the geometrical viewpoint (because we work on coarse quotients), suffices (and seems to be necessary) for the physical interpretation, i.e., analysis of KMS states.

4.3.2 BCM algebras

Now we give the precise definition of the algebra alluded to in the previous paragraph. Let $(\mathcal{D}, \mathcal{L}) = ((G, X, V, M)(L, K, K_M))$ be a BCM pair.

Let

$$\mathcal{H}(\mathcal{D}, \mathcal{L}) := C_c(Z_{\mathcal{D},\mathcal{L}})$$

be the algebra of compactly supported continuous functions on $Z_{\mathcal{D},\mathcal{L}}$.

We define the convolution product on $\mathcal{H}(\mathcal{D}, \mathcal{L})$ by

$$(f_1 * f_2)(g, y) := \sum_{h \in K\backslash G(\mathbb{A}_f),\, hy \in Y_{\mathcal{D},\mathcal{L}}} f_1(gh^{-1}, hy) f_2(h, y),$$

and the adjoint by

$$f^*(g, y) := \overline{f(g^{-1}, gy)}.$$

The fact that we consider functions with compact support implies that the sum defining the convolution product is finite.

Definition 4.3.3. The algebra $\mathcal{H}(\mathcal{D}, \mathcal{L})$ is called the *Bost-Connes-Marcolli algebra* of the pair $(\mathcal{D}, \mathcal{L})$.

Remark 4.3.4. We proved in Lemma 4.2.5 that, if (G, X) is classical, the quotient of $Y_{\mathcal{D},\mathcal{L}}$ by the commensurability equivalence relation (encoded by the action of the groupoid $Z_{\mathcal{D},\mathcal{L}}$) does not depend on the choice of K. This implies that in the classical case, the Morita equivalence class of $\mathcal{H}(\mathcal{D}, \mathcal{L})$ is independent of the choice of fine level structure K. More precisely, all these algebras are in fact Morita equivalent to the algebra corresponding to the "noncommutative quotient"

$$G(\mathbb{Q})\backslash X \times K_{\mathbb{A}}^M.$$

4.4 Time evolution, Hamiltonian and partition function

Let $(\mathcal{D},\mathcal{L}) = ((G,X,V,M),(L,K,K_M))$ be a BCM pair with fine level. Denote $K_0 = \mathrm{Sym}_f^\times(\mathcal{D},\mathcal{L}) := \phi^{-1}(K_M^\times)$.

Definition 4.4.1. The *time evolution* on $\mathcal{H}(\mathcal{D},\mathcal{L})$ is defined by

$$\sigma_t(f)(g,y) = \det(\phi(g))^{it} f(g,y). \tag{4.2}$$

Let $y = (\rho,[z,l])$ be in $Y_{\mathcal{D},\mathcal{L}}$ and denote $G_y = \{g \in G(\mathbb{A}_f) \mid g\rho \in K_M\}$. Let $\mathcal{H}_y$ be the Hilbert space $\ell^2(K\backslash G_y)$.

Definition 4.4.2. The *representation* $\pi_y : \mathcal{H}(\mathcal{D},\mathcal{L}) \to \mathcal{B}(\mathcal{H}_y)$ *of the Hecke algebra on* $\mathcal{H}_y$ is defined by

$$(\pi_y(f)\xi)(g) := \sum_{h\in K\backslash G_y} f(gh^{-1},hy)\xi(h),\ \forall g \in G_y,$$

for $f \in \mathcal{H}(\mathcal{D},\mathcal{L})$ and $\xi \in \mathcal{H}_y$.

Lemma 4.4.3. *The representation π_y is well defined, i.e., $\pi_y(f)$ is bounded for each $f \in \mathcal{H}(\mathcal{D},\mathcal{L})$.*

Proof. For $f \in \mathcal{H}(\mathcal{D},\mathcal{L})$, We want to prove that the norm

$$\|\pi_y(f)\| := \sup_{\|\xi\|=1} \|\pi_y(f)\xi\|_2$$

is bounded. This follows from the fact that the functions we consider are with compact support. More precisely, Denote $Z := Z_{\mathcal{D},\mathcal{L}}$. Given $f \in \mathcal{H}(\mathcal{D},\mathcal{L}) = C_c(Z)$, we need to show that there is a bound $C > 0$, such that for every pair of vectors $\xi,\eta \in \mathcal{H}_y$ we have

$$|\langle \pi_y(f)\xi,\eta\rangle| \le C\|\xi\|\|\eta\|.$$

To this end, we introduce the following notation. We set

$$S_y = \{\, [gh^{-1},hy] \in Z \mid g,h \in K\backslash G_y \,\},$$

and for each $\gamma \in S_y$ we set

$$R_y(\gamma) = \{\, \gamma' \in Z_y \mid s(\gamma') = t(\gamma) \,\}.$$

These are discrete sets. Here we use the usual notation for groupoids, namely $Z_y = t^{-1}\{y\}$, which we shall identify with $K\backslash G_y$.

Using the Cauchy-Schwarz inequality, we now get a bound on $|\langle \pi_y(f)\xi,\eta\rangle|$ as follows:

$$\begin{aligned}
|\langle \pi_y(f)\xi,\eta\rangle| &\le \sum_{\gamma_1\in Z_y} \left|\big(\pi_y(f)\xi\big)(\gamma_1)\overline{\eta(\gamma_1)}\right| \\
&\le \sum_{\gamma_1,\gamma_2\in Z_y} \left|f(\gamma_1\gamma_2^{-1})\xi(\gamma_2)\overline{\eta(\gamma_1)}\right| \\
&= \sum_{\gamma\in S_y} |f(\gamma)| \sum_{\gamma'\in R_y(\gamma)} |\xi(\gamma')\eta(\gamma\gamma')| \\
&\le \sum_{\gamma\in S_y} |f(\gamma)| \Big(\sum_{\gamma'\in R_y(\gamma)} |\xi(\gamma')|^2\Big)^{\frac{1}{2}} \Big(\sum_{\gamma'\in R_y(\gamma)} |\eta(\gamma\gamma')|^2\Big)^{\frac{1}{2}} \\
&\le \|\xi\|\|\eta\| \sum_{\gamma\in S_y} |f(\gamma)|.
\end{aligned}$$

Because f has compact support, the sum $\sum_{\gamma\in S_y} |f(\gamma)|$ is finite, and we thereby get the desired bound. ∎

We view the Hamiltonian as a virtual operator on $\ell^2(K_0\backslash G_y)$. By this we mean that the Hamiltonian does not depend on the choice of K and there is a minimal space on which it is defined: the space $\ell^2(K_0\backslash G_y)$. Consequently, its trace must be computed as a virtual (i.e., equivariant) trace, i.e., must be divided by $\mathrm{card}(K\backslash K_0)$. These considerations are related to the fact that, if (G, X) is classical, we prefer to define BCM algebras using fine level structures to resolve the stack singularities of $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}$.

Proposition 4.4.4. *The operator on $\mathcal{H}_y$ given by*

$$(H_y\xi)(g) = \log\det(\phi(g)) \cdot \xi(g)$$

is the Hamiltonian*, i.e., the infinitesimal generator of the time evolution, meaning that we have the equality*

$$\pi_y(\sigma_t(f)) = e^{itH_y}\pi_y(f)e^{-itH_y} \tag{4.3}$$

for all $f \in \mathcal{H}(\mathcal{D}, \mathcal{L})$.

Proof. This is just a matter of unwinding the definitions. Let $\xi \in \mathcal{H}_y$, and let $g \in G_y$. On the one hand we have

$$\begin{aligned}\big(\pi_y(\sigma_t f)\xi\big)(g) &= \sum_{h\in K\backslash G_y} (\sigma_t f)(gh^{-1}, hy)\xi(h) \\ &= \sum_{h\in K\backslash G_y} \det(\phi(g))^{it}\det(\phi(h))^{-it} f(gh^{-1}, hy)\xi(h),\end{aligned}$$

while on the other hand we have

$$\begin{aligned}\big(e^{itH_y}(\pi_y f)e^{-itH_y}\xi\big)(g) &= \det(\phi(g))^{it}\big((\pi_y f)e^{-itH_y}\xi\big)(g) \\ &= \det(\phi(g))^{it}\sum_{h\in K\backslash G_y} f(gh^{-1}, hy)(e^{-itH_y}\xi)(h) \\ &= \det(\phi(g))^{it}\sum_{h\in K\backslash G_y} f(gh^{-1}, hy)\det(\phi(h))^{-it}\xi(h).\end{aligned}$$

We thereby obtain the desired equality. □

Definition 4.4.5. Let $y \in Y_{\mathcal{D},\mathcal{L}}$ and $\beta > 0$. The *partition function* of the system $(\mathcal{H}(\mathcal{D}, \mathcal{L}), \sigma_t, \mathcal{H}_y, H_y)$, is

$$\zeta_y(\beta) := \frac{1}{\mathrm{card}(K\backslash K_0)}\mathrm{Trace}(e^{-\beta H_y}).$$

Let $Y^{\times}_{\mathcal{D},\mathcal{L}} \subset Y_{\mathcal{D},\mathcal{L}}$ be the set of invertible $y = (\rho, [z, l])$, i.e., $\rho \in K^{\times}_M$.

Proposition 4.4.6. *Suppose that $y \in Y^{\times}_{\mathcal{D},\mathcal{L}}$. Then $G_y = \mathrm{Sym}_f := \phi^{-1}(K_M)$. The partition function of the system $(\mathcal{H}(\mathcal{D}, \mathcal{L}), \sigma_t, \mathcal{H}_y, H_y)$, coincides with the zeta function $\zeta_{\mathcal{D},\mathcal{L}}(\beta)$ of $(\mathcal{D}, \mathcal{L})$ (see Definition 4.1.8).*

Moreover, it follows from 4.1.7 that the Hamiltonian has positive energy in the representation π_y.

4.5 Symmetries

Let $(\mathcal{D}, \mathcal{L}) = ((G, X, V, M), (L, K, K_M))$ be a BCM pair with fine level. We will denote C the center of G

Recall that Sym_f is the semigroup $\phi^{-1}(K_M)$. For $m \in \mathrm{Sym}_f$ and $c \in C(\mathbb{R})$, we define

$$\theta_{(m,c)}(f)(g, \rho, [z, l]) := f(g, \rho\phi(m), [cz, l]).$$

Lemma 4.5.1. *This gives a well defined right action of*

$$\mathrm{Sym}(\mathcal{D}, \mathcal{L}) := \mathrm{Sym}_f(\mathcal{D}, \mathcal{L}) \times C(\mathbb{R})$$

on $\mathcal{H}(\mathcal{D}, \mathcal{L})$ which moreover commutes with the time evolution.

Proof. The commutation with time evolution is clear because Sym_f acts on K_M on the right side, and $C(\mathbb{R})$ commutes with the $G(\mathbb{R})$-action on X. Since the action of $G(\mathbb{A}_f)$ on the right on $M(\mathbb{A}_f)$ is continuous, the action of Sym_f on K_M on the right is also continuous. This finishes the proof. □

Let CK_M be the center of K_M.

Definition 4.5.2. Let $\mathrm{Inn}(\mathcal{D}, \mathcal{L})$ be the subsemigroup of Sym defined by

$$\mathrm{Inn}(\mathcal{D}, \mathcal{L}) := C(\mathbb{Q}) \cap \phi^{-1}(CK_M).$$

Remark 4.5.3. There is an inclusions of semigroups

$$\mathrm{Inn}(\mathcal{D}, \mathcal{L}) \subset \mathrm{Sym}(\mathcal{D}, \mathcal{L}).$$

This gives a natural action of $\mathrm{Inn}(\mathcal{D}, \mathcal{L})$ on $\mathcal{H}(\mathcal{D}, \mathcal{L})$.

Definition 4.5.4. The semigroup $\mathrm{Out}(\mathcal{D}, \mathcal{L}) := \mathrm{Inn}(\mathcal{D}, \mathcal{L})\backslash\mathrm{Sym}(\mathcal{D}, \mathcal{L})$ is called the *outer symmetry semigroup of the BCM system* $(\mathcal{H}(\mathcal{D}, \mathcal{L}), \sigma_t)$.

In practical situations, the following hypotheses will often be fulfilled (see Propositions 7.3.3 and 9.2.1).

Definition 4.5.5. The level structure $\mathcal{L} = (L, K, K_M)$ is called *faithful* if the image $\phi(C(\mathbb{Q}))$ of the center of G commutes with K_M, i.e., $\phi(C(\mathbb{Q})) \subset CK_M$. The level structure $\mathcal{L}$ is called *full* if the natural morphism $\mathrm{Out} \to C(\mathbb{Q})\backslash G(\mathbb{A}_f)$ is surjective. The level structure $\mathcal{L}$ is called *fully faithful* if the natural morphism $\mathrm{Out} \to C(\mathbb{Q})\backslash G(\mathbb{A}_f)$ is an isomorphism.

These symmetries are symmetries up to inner automorphisms.

Proposition 4.5.6. *There is a morphism*

$$\mathrm{Out}(\mathcal{D}, \mathcal{L}) \to \mathrm{Out}(\mathcal{H}(\mathcal{D}, \mathcal{L}), \sigma_t)$$

to the quotient of the automorphism group of the BCM system by inner automorphisms of the algebra.

Proof. We have to prove that Inn acts by inner automorphisms. For $n \in \mathrm{Inn}$, we let μ_n be

$$\mu_n(g, y) = 1 \text{ if } g \in K.n^{-1},\ \mu_n(g, y) = 0 \text{ if } g \notin K.n^{-1}.$$

We will show that

$$\theta_{(n,n)}(f) = \mu_n f \mu_n^*,$$

i.e., the action of $\theta_{(n,n)}$ is given by the inner automorphism corresponding to μ_n.

We have, for all $y \in Y_{\mathcal{D},\mathcal{L}}$,

$$\begin{aligned}(\mu_n f \mu_n^*)(g,y) &= \sum_{h \in K\backslash G(\mathbb{A}_f), hy \in Y} \mu_n(gh^{-1}, hy)(f\mu_n^*)(h,y), \\ &= \sum_{h \in K\backslash G(\mathbb{A}_f), hy \in Y} \mu_n(gh^{-1}, hy) \sum_{k \in K\backslash G(\mathbb{A}_f), ky \in Y} f(hk^{-1}, ky)\mu_n^*(k,y), \\ &= \sum_{h,k \in K\backslash G(\mathbb{A}_f), hy, ky \in Y} \mu_n(gh^{-1}, hy) f(hk^{-1}, ky)\mu_n(k^{-1}, ky).\end{aligned}$$

Now, by definition of μ_n, the only nontrivial term of this sum is obtained when $k^{-1} = n^{-1}$ and $gh^{-1} = n^{-1}$, i.e., $k = n$ and $h = ng$. Since n is central,

$$\begin{aligned}(\mu_n f \mu_n^*)(g,y) &= f(ngn^{-1}, ny), \\ &= f(g, n\rho, [z, ln^{-1}]), \\ &= f(g, \rho n, [nz, l]), \\ &= \theta_{(n,n)}(f)(g,y).\end{aligned}$$

□

5 Comparison with the original Bost-Connes-Marcolli systems

We want to understand how our systems are related with the usual Bost-Connes-Marcolli systems in the class number one case. These class number one systems are called principal BCM systems. They are directly related to Connes-Marcolli systems defined in [CM04].

5.1 Principal BCM systems

Let $(\mathcal{D}, \mathcal{L}) = ((G, X, V, M), (L, K, K_M))$ be a BCM pair with (G, X) classical.

Let $\Gamma := G(\mathbb{Q}) \cap K$ and

$$U^{\text{princ}} := \{(g, \rho, z) \in G(\mathbb{Q}) \times K_M \times X \mid g\rho \in K_M\}.$$

Let X^+ be a connected component of X, $G(\mathbb{Q})^+$ be $G(\mathbb{Q}) \cap G(\mathbb{R})^+$ (where $G(\mathbb{R})^+$ is the identity component of $G(\mathbb{R})$) and $\Gamma_+ = G(\mathbb{Q})^+ \cap K$. Let

$$U^+ := \{(g, \rho, z) \in G(\mathbb{Q})^+ \times K_M \times X^+ \mid g\rho \in K_M\}.$$

We have a natural action of Γ^2 (resp. Γ_+^2) on U^{princ} (resp. U^+) given by $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z)$. Let $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}^{\text{princ}}$ (resp. $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}^{+}$) be the stacky quotient of U^{princ} (resp. U^+) by Γ^2 (resp. Γ_+^2).

Definition 5.1.1. The groupoid $\mathfrak{Z}_{\mathcal{D},\mathcal{L}}^{\text{princ}}$ is called the *principal BCM groupoid* for the pair $(\mathcal{D}, \mathcal{L})$.

Proposition 5.1.2. *Suppose that the natural map $\Gamma \to G(\mathbb{Q})/G(\mathbb{Q})^+$ is surjective. The natural map*

$$\mathfrak{Z}_{\mathcal{D},\mathcal{L}}^{+} \to \mathfrak{Z}_{\mathcal{D},\mathcal{L}}^{\text{princ}}$$

is an isomorphism.

Proof. Surjectivity: Let $u = (g, \rho, z) \in U^{\text{princ}}$. We want to show that there exists $\gamma_1, \gamma_2 \in \Gamma$ such that $(\gamma_1, \gamma_2).u = (\gamma_1 g \gamma_2^{-1}, \rho, \gamma_2 z) \in U^+$. There exists $\gamma_2 \in \Gamma$ with $\gamma_2 z \in X^+$ because: 1) the definition 3.1.1 of a Shimura datum implies that $\pi_0(X)$ is a $\pi_0(G(\mathbb{R}))$-homogeneous space; and 2) from our hypothesis and the theorem of real approximation, we get a surjection $\Gamma \to G(\mathbb{Q})/G(\mathbb{Q})^+ \cong G(\mathbb{R})/G(\mathbb{R})^+$. Our hypothesis now implies that there exists $\gamma_1 \in \Gamma$ such that $\gamma_1 g \gamma_2^{-1} \in G(\mathbb{Q})^+$. This proves surjectivity.

Injectivity: Now suppose that two points (g_1, ρ_1, z_1) and (g_2, ρ_2, z_2) have the same image in the quotient. Then there exists $\gamma_1, \gamma_2 \in \Gamma$ such that $(g_1, \rho_1, z_1) = (\gamma_1 g_2 \gamma_2^{-1}, \gamma_2 \rho_2, \gamma_2 z_2)$. Since γ_2 stabilizes X^+, it is in $G(\mathbb{R})^+$, and therefore also in Γ_+. This implies that γ_1 is in Γ_+. This proves injectivity. □

We denote by $h(G, K)$ the cardinality of the finite set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$.

Proposition 5.1.3. *If $h(G, K) = 1$ then the principal and the full BCM groupoids are the same, i.e., the natural map*

$$\mathfrak{Z}^{\text{princ}}_{\mathcal{D},\mathcal{L}} \to \mathfrak{Z}_{\mathcal{D},\mathcal{L}}$$

is an isomorphism.

Proof. There is a natural map

$$\begin{array}{cccc} \psi: & (\Gamma\backslash G(\mathbb{Q})) \times K_M \times X & \to & (K\backslash G(\mathbb{A}_f)) \times K_M \times G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)) \\ & (g, \rho, z) & \mapsto & (g, \rho, [z, 1]) \end{array}.$$

The action of $\gamma_2 \in \Gamma$ on each side is given respectively by $(g, \rho, z) \mapsto (g\gamma_2^{-1}, \gamma_2\rho, \gamma_2 z)$ and $(g, \rho, [z, l]) \mapsto (g\gamma_2^{-1}, \gamma_2\rho, [z, l\gamma_2^{-1}])$. Since $\Gamma = K \cap G(\mathbb{Q})$, we have

$$\begin{aligned} \psi(\gamma_2 \cdot (g, \rho, z)) &= (g\gamma_2^{-1}, \gamma_2\rho, [\gamma_2 z, 1]), \\ &= (g\gamma_2^{-1}, \gamma_2\rho, [z, \gamma_2^{-1}]), \\ &= \gamma_2 \cdot \psi(g, \rho, z). \end{aligned}$$

This proves that ψ, being Γ-equivariant, induces a well defined map

$$\overline{\psi} : (\Gamma\backslash G(\mathbb{Q})) \underset{\Gamma}{\times} [K_M \times X] \to (K\backslash G(\mathbb{A}_f)) \underset{K}{\times} [K_M \times G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))].$$

Let us first prove that $\overline{\psi}$ is surjective. This essentially follows from the equalities $G(\mathbb{A}_f) = K.G(\mathbb{Q}) = G(\mathbb{Q}).K$ (the class number one hypothesis $h(G, K) = 1$).

For $(g, \rho, [z, l]) \in (K\backslash G(\mathbb{A}_f)) \underset{K}{\times} [K_M \times G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))]$, there exists $\gamma_2 \in K$ and $l_2 \in G(\mathbb{Q})$ such that $l = l_2\gamma_2$. Then, we have the equalities in our quotient space

$$\begin{aligned} (g, \rho, [z, l]) &= (g, \rho, [z, l_2\gamma_2]), \\ &= (g\gamma_2^{-1}, \gamma_2\rho, [z, l_2]), \\ &= (g\gamma_2^{-1}, \gamma_2\rho, [l_2^{-1} z, 1]). \end{aligned}$$

There exists $\gamma_1 \in K$ and $g_1 \in G(\mathbb{Q})$ such that $\gamma_1 g_1 = g\gamma_2^{-1}$ and we have the following equalities in our quotient space

$$\begin{aligned} (g, \rho, [z, l]) &= (g\gamma_2^{-1}, \gamma_2\rho, [l_2^{-1} z, 1]) \\ &= (\gamma_1 g_1, \gamma_2\rho, [l_2^{-1} z, 1]), \\ &= \psi(g_1, \gamma_2\rho, l_2^{-1} z). \end{aligned}$$

Thus $\overline{\psi}$ is surjective.

Now we prove that $\overline{\psi}$ is injective. Suppose that

$$\overline{\psi}(g_1, \rho_1, z_1) = \overline{\psi}(g_2, \rho_2, z_2).$$

Then there exists $\gamma_1 \in K$, $\gamma_2 \in K$, $\gamma_3 \in G(\mathbb{Q})$ such that

$$(\gamma_1 g_1 \gamma_2^{-1}, \gamma_2 \rho_1, [\gamma_3 z_1, \gamma_3 \gamma_2^{-1}]) = (g_2, \rho_2, [z_2, 1]).$$

This implies $\gamma_3 = \gamma_2$ and then $\gamma_2 \in K \cap G(\mathbb{Q}) = \Gamma$. But we also have $\gamma_1 = g_2 \gamma_2 g_1^{-1} \in G(\mathbb{Q}) \cap K = \Gamma$. This shows that

$$(g_2, \rho_2, z_2) = (\gamma_1 g_1 \gamma_2^{-1}, \gamma_2 \rho_1, \gamma_2 z_1)$$

with $\gamma_1, \gamma_2 \in \Gamma$, i.e., the two points are the same in $(\Gamma \backslash G(\mathbb{Q})) \underset{\Gamma}{\times} [K_M \times X]$. This proves injectivity.

To finish, we prove that the natural isomorphism $\overline{\psi} : \mathfrak{Z}^{\text{princ}}_{\mathcal{D},Lc} \to \mathfrak{Z}_{\mathcal{D},\mathcal{L}}$ is compatible with the groupoid structures. Let $Y^{\text{princ}} = K_M \times X$. If $(g, \rho, z) \in \mathfrak{Z}^{\text{princ}}$, the image of $(\rho, z) \in Y^{\text{princ}}$ by $g \in G(\mathbb{Q})$ is given by $(g\rho, gz) \in Y^{\text{princ}}$. The image of $(\rho, [z, 1]) \in Y$ by g is given by $(g\rho, [z, g^{-1}]) \in Y$, which is equal to $(g\rho, [gz, 1])$. This finishes the proof. □

Definition 5.1.4. Let $(\mathcal{D}, \mathcal{L})$ be a BCM pair with fine level. The algebra $\mathcal{H}_{\text{princ}}(\mathcal{D}, \mathcal{L}) = C_c(Z^{\text{princ}}_{\mathcal{D},\mathcal{L}})$ is called the *principal BCM algebra* for $(\mathcal{D}, \mathcal{L})$.

5.2 The Bost-Connes system

Let $F/\mathbb{Q}$ be a number field. Let $G = \mathbb{G}_{m,F}$, $X_F = G(\mathbb{R})/G(\mathbb{R})^+ \cong \{\pm 1\}^{\text{Hom}(F,\mathbb{R})}$, $V = F$ and $M = \text{Res}_{F/\mathbb{Q}} \text{M}_{1,F}$. Let $K = \widehat{\mathcal{O}}_F^\times$, $L = \mathcal{O}_F$ and $K_M = \widehat{\mathcal{O}}_F = \text{M}_1(\widehat{\mathcal{O}}_F)$.

Definition 5.2.1. The pair

$$\mathcal{P}(\mathbb{G}_{m,F}, X_F) = ((\mathbb{G}_{m,F}, X_F, F, \text{Res}_{F/\mathbb{Q}} \text{M}_{1,F}), (\mathcal{O}_F, \widehat{\mathcal{O}}_F^\times, \widehat{\mathcal{O}}_F))$$

is called the *Bost-Connes pair* for F. The corresponding algebra $\mathcal{H}(\mathbb{G}_{m,F}, X)$ is called the *Bost-Connes algebra* for F.

Proposition 5.2.2. *In case $F = \mathbb{Q}$, $\mathcal{H}(\mathbb{G}_{m,\mathbb{Q}}, \{\pm 1\})$ is the original Bost-Connes algebra.*

Proof. Recall that the original Bost-Connes algebra can be written as the algebra of continuous functions with compact support on the subset $Z^+ \subset \mathbb{Q}_+^\times \times \widehat{\mathbb{Z}}$ of pairs (g, ρ) with $g\rho \in \widehat{\mathbb{Z}}$. This is the algebra of continuous functions with compact support on the space $Z^{+,\text{princ}}_{\mathcal{D},\mathcal{L}}$ described in section 5.1. The map $\mathbb{Z}^\times = \{\pm 1\} \to \mathbb{R}^\times / \mathbb{R}^{+,\times}$ is an isomorphism, so that we can apply proposition 5.1.2, which tells us that $Z^{+,\text{princ}}_{\mathcal{D},\mathcal{L}} \cong Z^{\text{princ}}_{\mathcal{D},\mathcal{L}}$. Since $h(\mathbb{G}_{m,\mathbb{Q}}, \widehat{\mathbb{Z}}^\times)$ is the usual class number of $\mathbb{Q}$, i.e., 1, proposition 5.1.3 moreover implies that $Z^{\text{princ}}_{\mathcal{D},\mathcal{L}} \cong Z_{\mathcal{D},\mathcal{L}}$. □

5.3 The Connes-Marcolli system

We now show that in the $\text{GL}_{2,\mathbb{Q}}$ case, we obtain exactly the same groupoid as Connes and Marcolli [CM04]. This groupoid is only a stack-groupoid, not a usual groupoid. This restriction was circumvented by Connes and Marcolli using functions of weight 0 for the scaling action (see [CM04], remark shortly preceeding 1.83). Such a scaling action is not canonically defined in the general case we consider. As explained before, we deliberately chose to view this groupoid as a stack-groupoid in order to define a natural groupoid algebra for it that depends on the resolution of stack singularities given by the choice of K.

Consider the Shimura datum $(\mathrm{GL}_{2,\mathbb{Q}}, \mathbb{H}^\pm)$, $V = \mathbb{Q}^2$, and $M = \mathrm{M}_{2,\mathbb{Q}}$. let $L = \mathbb{Z}^2$, $K = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $K_M = \mathrm{M}_2(\widehat{\mathbb{Z}})$.

Definition 5.3.1. The pair

$$\mathcal{P}(\mathrm{GL}_2, \mathbb{H}^\pm) := ((\mathrm{GL}_2, \mathbb{H}^\pm, \mathbb{Q}^2, \mathrm{M}_{2,\mathbb{Q}}), (\mathbb{Z}^2, \mathrm{GL}_2(\widehat{\mathbb{Z}}), \mathrm{M}_2(\widehat{\mathbb{Z}})))$$

is called *the modular BCM pair.* The corresponding BCM stack-groupoid is denoted by $\mathfrak{Z}_{\mathrm{GL}_2,\mathbb{H}^\pm}$.

The stack-groupoid $\mathfrak{Z}^+_{\mathrm{GL}_2,\mathbb{H}^\pm}$ is defined as in Section 5.1. This is exactly the groupoid studied by Connes and Marcolli in [CM04].

Lemma 5.3.2. *Our BCM stack-groupoid is the same as Connes and Marcolli's one. In other words, the natural map*

$$\mathfrak{Z}^+_{\mathrm{GL}_2,\mathbb{H}} \to \mathfrak{Z}_{\mathrm{GL}_2,\mathbb{H}^\pm}$$

is an isomorphism.

Proof. We have in this case $h(G, K) = 1$ so by proposition 5.1.3, we have $[Z^{\mathrm{princ}}_{\mathrm{GL}_2,\mathbb{H}^\pm}] \cong [Z_{\mathrm{GL}_2,\mathbb{H}^\pm}]$. The map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2(\mathbb{R})^+$ is surjective, so that we can apply proposition 5.1.2, which tells us that $\mathfrak{Z}^+_{\mathrm{GL}_2,\mathbb{H}^{pm}} \cong \mathfrak{Z}^{\mathrm{princ}}_{\mathrm{GL}_2,\mathbb{H}^\pm}$. □

6 Operator theoretic results on BCM algebras

6.1 The C*-algebra associated to a BCM datum

Let $(\mathcal{D}, \mathcal{L}) = ((G, X, V, M), (L, K, K_M))$ be a BCM pair with fine level. On the algebra $\mathcal{H}(\mathcal{D}, \mathcal{L})$, we put the following norm: for every $f \in \mathcal{H}(\mathcal{D}, \mathcal{L})$,

$$\|f\| = \sup_{y \in Y_{\mathcal{D},\mathcal{L}}} \|\pi_y(f)\|.$$

Lemma 6.1.1. *This defines a C*-norm on $\mathcal{H}(\mathcal{D}, \mathcal{L})$, i.e., $\|f^*f\| = \|f\|^2$.*

Proof. Indeed, it is easy to check that this is a seminorm satisfying the C*-condition (Definition 3.2.1): observe that for arbitarily small $\epsilon > 0$ there is a y such that $\|f\|^2 - \epsilon = \|\pi_y(f)\|^2$. We then have

$$\|f^*f\| \geq \|\pi_y(f^*f)\| = \|\pi_y(f)\|^2 = \|f\|^2 - \epsilon,$$

which of course means that $\|f^*f\| \geq \|f\|^2$. This inequality is easily shown to imply the C*-condition.

That we get a norm (i.e., $\|f\| = 0$ only when $f = 0$), and not just a seminorm, follows from the fact that $f(g, y) \neq 0$ implies that $\pi_y(f) \neq 0$:

$$\langle \pi_y(f)\varepsilon_g, \varepsilon_g \rangle = f(1, gy) = f(g, y) \neq 0.$$

Here $\epsilon_g \in \mathcal{H}_y$ is the unit vector which takes value 1 at g, and 0 elsewhere. □

Definition 6.1.2. The completion of $\mathcal{H}(\mathcal{D}, \mathcal{L})$ under the norm $\|.\|$ is denoted $\mathcal{A}(\mathcal{D}, \mathcal{L})$ and called the *BCM C*-algebra.*

6.2 Construction of extreme KMS states at small temperature

Let $(\mathcal{D},\mathcal{L}) = ((G,X,V,M),(L,K,K_M))$ be a summable BCM pair. Recall that $Y^{\times}_{\mathcal{D},\mathcal{L}} = \{(g,\rho,[z,l]) \in Y_{\mathcal{D},\mathcal{L}} \mid \rho \text{ invertible}\}$.

Lemma 6.2.1. *Let $y \in Y^{\times}_{\mathcal{D},\mathcal{L}}$. Let β be such that the zeta function $\zeta_{\mathcal{D},\mathcal{L}}(\beta)$ converges. The state*

$$\Phi_{\beta,y}(f) := \frac{\mathrm{Trace}(\pi_y(f)e^{-\beta H_y})}{\zeta_{\mathcal{D},\mathcal{L}}(\beta)}$$

is a KMS_β *state for the system* $(\mathcal{A}(\mathcal{D},\mathcal{L}),\sigma_t)$.

Proof. By construction, the algebra $\mathcal{H}(\mathcal{D},\mathcal{L})$ is a norm-dense subalgebra of $\mathcal{A}(\mathcal{D},\mathcal{L})$, which is also σ_z-invariant. Thus, to verify the KMS_β condition, it is enough to show that

$$\Phi_{\beta,y}(f_1\sigma_{i\beta}(f_2)) = \Phi_{\beta,y}(f_2f_1)$$

for every pair of functions $f_1,f_2 \in \mathcal{H}(\mathcal{D},\mathcal{L})$; see Proposition 3.2.3. The convergence of the zeta function implies that the operator $e^{-\beta H_y}$ is trace class. The invariance of the trace under cyclic permutations implies that

$$\begin{aligned}
\zeta_{\mathcal{D},\mathcal{L}}(\beta).\Phi_{\beta,y}(f_1\sigma_{i\beta}(f_2)) &= \mathrm{Trace}(f_1e^{-\beta H_y}f_2e^{\beta H_y}e^{-\beta H_y}),\\
&= \mathrm{Trace}(f_1e^{-\beta H_y}f_2),\\
&= \mathrm{Trace}(f_2f_1e^{-\beta H_y}),\\
&= \zeta_{\mathcal{D},\mathcal{L}}(\beta).\Phi_{\beta,y}(f_2f_1),
\end{aligned}$$

which finishes the proof of the KMS condition. □

The commutant of a subset $S \in \mathcal{B}(\mathcal{H}_y)$ is by definition $S' = \{a \in \mathcal{B}(\mathcal{H}_y) \mid as = sa\}$.

Lemma 6.2.2. *If $y \in Y^{\times}_{\mathcal{D},\mathcal{L}}$, then the commutant $\pi_y(\mathcal{A}(\mathcal{D},\mathcal{L}))'$ consists only of scalar operators.*

Proof. In general, if $y \in Y_{\mathcal{D},\mathcal{L}}$, then the Von Neumann algebra $\pi_y(\mathcal{A})'$ is generated by the right regular representation of the isotropy group $Z_{y,y} := \{[g,y] \in Z \mid s[g,y] = [y] = [gy] = t[g,y]\}$ (cf. [Con79] Proposition VII.5).

If y is now in $Y^{\times}_{\mathcal{D},\mathcal{L}}$, then the isotropy group $Z_{y,y}$ is trivial. Therefore, the commutant $\pi_y(\mathcal{A})'$ consists only of scalar operators. □

Recall that the set of KMS_β states is a convex simplex (see Proposition 3.2.5), whose extreme points are called *extreme* KMS_β *states.*

Proposition 6.2.3. *Let $y \in Y^{\times}_{\mathcal{D},\mathcal{L}}$ be an invertible element of $Y_{\mathcal{D},\mathcal{L}}$. Let β be such that the zeta function $\zeta_{\mathcal{D},\mathcal{L}}(\beta)$ converges. The* KMS_β *state*

$$\Phi_{\beta,y}(f) := \frac{\mathrm{Trace}(\pi_y(f)e^{-\beta H_y})}{\zeta_{\mathcal{D},\mathcal{L}}(\beta)}$$

is extremal of type I_∞.

Proof. By Proposition 3.2.5, the property, for $\Phi_{\beta,y}$, of being extreme is equivalent to the property of being a factor state, i.e., the algebra $\mathcal{A}(\mathcal{D},\mathcal{L})$ generates a factor in the GNS representation of $\Phi_{\beta,y}$. Following Harari-Leichtnam, [HL97], proof of Theorem 5.3.1, the GNS representation is (up to unitary equivalence)

$$\tilde{\pi}_y = \pi_y \otimes \mathrm{id}_{\mathcal{H}_y} : \mathcal{A}(\mathcal{D},\mathcal{L}) \to \mathcal{B}(\mathcal{H}_y \otimes \mathcal{H}_y),$$

and the associated cyclic vector is

$$\Omega_{\beta,y} = \zeta_{\mathcal{D},\mathcal{L}}(\beta)^{-1/2} \sum_{h \in K\backslash \mathrm{Sym}_f} \det(\phi(h))^{-1/2} \epsilon_h \otimes \epsilon_h,$$

where ϵ_h is the basis factor of $\mathcal{H}_y$ that takes values 1 at h, and 0 elsewhere.

The properties that characterize the triple $(\mathcal{H}_y \otimes \mathcal{H}_y, \tilde{\pi}_y, \Omega_{\beta,y})$ as the GNS representation of $\Phi_{\beta,y}$ are precisely:

1. $\Phi_{\beta,y}(f) = \langle \tilde{\pi}_y(f)\Omega_{\beta,y}, \Omega_{\beta,y}\rangle$, for every $f \in \mathcal{A}(\mathcal{D},\mathcal{L})$; and
2. The orbit $\tilde{\pi}_y(\mathcal{A}(\mathcal{D},\mathcal{L}))\Omega_{\beta,y}$ is dense in the Hilbert space $\mathcal{H}_y \otimes \mathcal{H}_y$.

These two properties are verified by direct calculation. For example, to verify the second condition first observe that

$$\pi_y(f)\epsilon_h = \sum_{g \in K\backslash G_y} f(gh^{-1}, hy)\, \epsilon_g,$$

and so

$$\tilde{\pi}_y(f)\Omega_{\beta,y} = \zeta_{\mathcal{D},\mathcal{L}}(\beta)^{-1/2} \sum_{g,\, h \in K\backslash G_y} \det(\phi(h))^{-\beta/2} f(gh^{-1}, hy)\, \epsilon_g \otimes \epsilon_h.$$

But since $G_y = \mathrm{Sym}_f$, every $\det(\phi(h))$ is positive, and we can choose f to have sufficiently small support about (gh^{-1}, hy) to see that the basis vector $\epsilon_g \otimes \epsilon_h$ lies in the closure of $\tilde{\pi}_y(\mathcal{A}(\mathcal{D},\mathcal{L}))\xi_{\beta,y}$.

By Lemma 6.2.2, we know that the commutant $\pi_y(\mathcal{A}(\mathcal{D},\mathcal{L}))'$ consists of scalar operators. It is then clear that $\tilde{\pi}_y(\mathcal{A}(\mathcal{D},\mathcal{L}))' = \pi_y(\mathcal{A}(\mathcal{D},\mathcal{L}))' \otimes \mathcal{B}(\mathcal{H}_y) = \mathbb{C} \otimes \mathcal{B}(\mathcal{H}_y)$, and so

$$\tilde{\pi}_y(\mathcal{A}(\mathcal{D},\mathcal{L}))'' = \mathcal{B}(\mathcal{H}_y) \otimes \mathbb{C}\mathrm{id}_{\mathcal{H}_y} \cong B(\mathcal{H}_y).$$

This proves that $\Phi_{\beta,y}$ is a Type I_∞ factor state. □

Question 6.2.4. *Let $(\mathcal{D},\mathcal{L}) = ((G,X,V,M),(L,K,K_M))$ be a BCM pair. Is is true that for $\beta \gg 0$, the map $y \mapsto \Phi_{\beta,y}$ induces a bijection from the Shimura variety $\mathrm{Sh}(G,X)$ to the space $\mathcal{E}_\beta$ of extremal KMS_β states on $(\mathcal{H}(\mathcal{D},\mathcal{L}),\sigma_t)$?*

7 A Bost-Connes system for number fields

7.1 Reminder of Dedekind zeta functions

A first step in understanding what a good analogue of the Bost-Connes algebra may be, is to find good candidates for partition functions. This was done by Paula Cohen in [Coh99] for number fields, where she constructed a Bost-Connes analogue with Dedekind zeta function as partition function.

Let F be a number field. There is a multiplicative semigroup injection $\mathcal{O}_F^\times \to \mathcal{O}_F$. Let $\widehat{\mathcal{O}}_F^\natural := \widehat{\mathcal{O}}_F \cap \mathbb{A}_{f,F}^\times$ and $\widehat{\mathbb{Z}}^\natural := \widehat{\mathbb{Z}} \cap \mathbb{A}_{f,\mathbb{Q}}$. Then the space $\widehat{\mathcal{O}}_F^\times \backslash \widehat{\mathcal{O}}_F^\natural$ is identified with the multiplicative semigroup I_F of integral ideals in F. The norm map induces a natural map

$$\mathrm{Nm} : \widehat{\mathcal{O}}_F^\times \backslash \widehat{\mathcal{O}}_F^\natural \to \widehat{\mathbb{Z}}^\times \backslash \widehat{\mathbb{Z}}^\natural \cong \mathbb{Z}^\times \backslash \mathbb{Z}^\natural \cong \mathbb{N}^\times.$$

This is the usual norm on ideal classes.

Now the Dedekind zeta function of F can be expressed as

$$\zeta_F(s) = \sum_{n \in \widehat{\mathcal{O}}_F^\times \backslash \widehat{\mathcal{O}}_F^\natural} \frac{1}{\mathrm{Nm}(n)^s}.$$

7.2 The adelic Bost-Connes algebra

We recall from Subsection 5.2 the definition of the Bost-Connes datum for number fields. Let $F/\mathbb{Q}$ be a number field. Let $G = \mathbb{G}_{m,F}$ and $X_F = G(\mathbb{R})/G(\mathbb{R})^+ \cong \{\pm 1\}^{\mathrm{Hom}(F,\mathbb{R})}$.

Following Definition 5.2.1, the Bost-Connes pair for F is

$$\mathcal{P}_F := \mathcal{P}(\mathbb{G}_{m,F}, X_F) = ((\mathbb{G}_{m,F}, X, F, F = \mathrm{End}_F(F)), (\mathcal{O}_F, \widehat{\mathcal{O}}_F^\times, \widehat{\mathcal{O}}_F)).$$

Remark that in this case, we have

$$\mathrm{Sh}(\mathbb{G}_{m,F}, X_F) \cong \pi_0(C_F).$$

Then $Y_F = \widehat{\mathcal{O}}_F \times \mathrm{Sh}(\mathbb{G}_{m,F}, X_F)$ and

$$U_F \subset \mathbb{A}_{f,F}^\times \times Y_F$$

is the subspace of tuples $(g, \rho, [z, l])$ such that $g\rho \in \widehat{\mathcal{O}}_F$. We let $\gamma_1, \gamma_2 \in \widehat{\mathcal{O}}_F^\times \times \widehat{\mathcal{O}}_F^\times$ act on $(g, y = (\rho, [z, l])) \in U_F$ by

$$(g, \rho, [z, l]) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, [z, l\gamma_2^{-1}]).$$

Let $Z_F := U_F/(\widehat{\mathcal{O}}_F^\times \times \widehat{\mathcal{O}}_F^\times)$ and $\mathcal{H}_F = \mathcal{H}(\mathbb{G}_{m,F}, X_F) := C_c(Z_F)$ be the corresponding Bost-Connes algebra for F.

7.3 Partition function and symmetries

Lemma 7.3.1. *Let $y \in Y_F$ and $\mathcal{H}_y = \ell^2(K\backslash G_y)$. The time evolution on $\mathcal{H}_F$ is given by*

$$\sigma_t(f)(g, y) = \mathrm{Nm}(g)^{it} f(g, y).$$

The Hamiltonian H_y in $\mathcal{H}_y$ is given by

$$(H_y \xi)(g) = \log(\mathrm{Nm}(g)) \cdot \xi(g).$$

Proof. Notice that for $a \in F$, the determinant of the $\mathbb{Q}$-linear map $x \mapsto a.x$ on F is the norm $\mathrm{Nm}(a)$. The lemma then follows from Definitions 4.4.1 and 4.4.4. □

Lemma 7.3.2. *The finite symmetry semigroup $\mathrm{Sym}_f(\mathbb{G}_{m,F}, X_F)$ of $\mathcal{P}_F$ is $\widehat{\mathcal{O}}_F^\natural$. Its zeta function $\zeta_{\mathcal{P}_F}$ is the Dedekind zeta function ζ_F of F.*

Proof. The description of the symmetry semigroup follows from its Definition 4.1.6. The description of the zeta function follows from Definition 4.1.8 and Subsection 7.1. □

We also use the full symmetry semigroup $\mathrm{Sym}(\mathbb{G}_{m,F}, X_F) = \widehat{\mathcal{O}}_F^\natural \times \mathbb{G}_{m,F}(\mathbb{R})$, which contains archimedean information.

Proposition 7.3.3. *We have $\mathrm{Inn}(\mathbb{G}_{m,F}, X_F) = \mathcal{O}_F^\natural := \mathcal{O}_F - \{0\}$ and the outer symmetry semigroup $\mathrm{Out}(\mathbb{G}_{m,F}, X_F)$ acts through*

$$\pi_0(F^\times \backslash \mathbb{A}_F^\times).$$

Proof. Recall from 4.1 that $\mathfrak{Z}_F$ can be written as a projective limit of groupoids $\mathfrak{Z}_{K'}$ for $K' \subset G(\mathbb{A}_f)$ compact open. The Sym-action can thus be enhanced to an action of the projective limit semigroup $\varprojlim_{K'} \mathrm{Sym}/K'$ over all compact open $K' \subset G(\mathbb{A}_f)$. We know from [Del79], 2.2.3, that

$$\varprojlim_{K'} F^\times \backslash (\mathbb{A}_{f,F}^\times / K' \times \pi_0(\mathbb{G}_{m,F}(\mathbb{R}))) := \mathrm{Sh}(\mathbb{G}_{m,F}, X_F) \cong \pi_0(F^\times \backslash \mathbb{A}_F^\times).$$

It thus remains to prove that the natural map

$$\mathcal{O}_F^\natural \backslash (\widehat{\mathcal{O}}_F^\natural \times \pi_0(\mathbb{G}_{m,F}(\mathbb{R}))) \to F^\times \backslash (\mathbb{A}_{f,F}^\times \times \pi_0(\mathbb{G}_{m,F}(\mathbb{R})))$$

is an isomorphism. The injectivity of this map is clear because $\mathcal{O}_F^\natural := \mathcal{O}_F - \{0\} = \widehat{\mathcal{O}}_F^\natural \cap F^\times$. Since $F^\times$ acts transitively on $\pi_0(\mathbb{G}_{m,F}(\mathbb{R}))$, to prove surjectivity it suffices to prove surjectivity of the upper map of the following diagram:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_F^\natural & \longrightarrow & F^\times \backslash \mathbb{A}_{f,F}^\times \\ \downarrow & & \downarrow \\ \mathcal{O}_F^\natural \backslash \widehat{\mathcal{O}}_F^\natural / \widehat{\mathcal{O}}_F^\times & \xrightarrow{\sim} & F^\times \backslash \mathbb{A}_{f,F}^\times / \widehat{\mathcal{O}}_F^\times \end{array}$$

The lower arrow is an isomorphism because these two groups are equal to the ideal class group of F. Let $g \in \mathbb{A}_{f,F}^\times$ be a finite idele. Then its image by the vertical projection gives an ideal class, which is the image of some $m \in \widehat{\mathcal{O}}_F^\natural$. We have $[m] = [g]$ in the right quotient so that there exists $k \in \widehat{\mathcal{O}}_F^\times$ such that $g = mk \mod F^\times$. Then $mk \in \widehat{\mathcal{O}}_F^\natural$ is in the preimage of the upper arrow of the diagram, which proves surjectivity. □

Remark 7.3.4. Analogous results were already obtained for F imaginary quadratic by Connes-Marcolli-Ramachandran (see [CMR05]). In this case, the datum $(\mathbb{G}_{m,F}, X_F)$ is classical so that the system is simpler.

8 A Bost-Connes system for Dirichlet characters

8.1 Reminder of zeta functions of Dirichlet characters

We here recall from Neukirch's book [Neu92], p. 501, some facts about characters.

Definition 8.1.1. A *Hecke character* is a character of the idele class group $C_F := A_F^\times / F^\times$, i.e., a continuous homomorphism $\chi : C_F \to S^1$ to the group S^1 of complex numbers of norm 1. A *Dirichlet character* is a Hecke character that factors through the quotient group $(F_\mathbb{R})_+^\times \backslash \mathbb{A}_F^\times / F^\times$ where $+$ denotes the connected component for the real topology.

Let $\mathfrak{m} = \prod_\mathfrak{p} \mathfrak{p}_\mathfrak{p}^n$ be a full ideal of $\mathcal{O}_F$ and let $K(\mathfrak{m})$ be the kernel of the natural map

$$\widehat{\mathcal{O}}_F^\times \to (\widehat{\mathcal{O}}_F / \mathfrak{m})^\times.$$

We say that $\mathfrak{m}$ is *a module of definition* for the Dirichlet character χ if $\chi(K(\mathfrak{m})) = 1$. We then call $K(\mathfrak{m})$ *a subgroup of definition* for χ.

Each Dirichlet character has a module of definition and for such an $\mathfrak{m}$, we have a factorisation $\chi : C(\mathfrak{m}) \to S^1$ where $C(\mathfrak{m}) = ((F_\mathbb{R})_+^\times \times K(\mathfrak{m})) \backslash \mathbb{A}_F^\times / F^\times$ is the big ray class group modulo $\mathfrak{m}$. Such an $\mathfrak{m}$ that is moreover minimal is called *the conductor of the Dirichlet character*.

Recall that $\widehat{\mathcal{O}}_F^\natural = \mathbb{A}_{f,F}^\times \cap \widehat{\mathcal{O}}_F$. If $\chi : \mathbb{A}_F^\times \to S^1$ is a Dirichlet character, we factor it through $(F_\mathbb{R})_+^\times \backslash \mathbb{A}_F^\times$, and thus restrict it to $\pi_0(F_\mathbb{R}^\times) \times \widehat{\mathcal{O}}_F^\natural$. Let $K(\mathfrak{m}) \subset \widehat{\mathcal{O}}_F^\times$ be a primitive subgroup of definition for χ and let $K^\natural(\mathfrak{m}) := \{n \in \widehat{\mathcal{O}}_F^\natural \mid \bar{n} = 1 \in \widehat{\mathcal{O}}_F/\mathfrak{m}\}$.

There is an injective map $K(\mathfrak{m})\backslash K^\natural(\mathfrak{m}) \hookrightarrow \widehat{\mathcal{O}}_F^\times \backslash \widehat{\mathcal{O}}_F^\natural$ whose image is the semigroup of all ideals of F prime to $\mathfrak{m}$.

At least if χ is trivial at infinity, it induces $\chi : K(\mathfrak{m})\backslash K^\natural(\mathfrak{m}) \to S^1$. Now, we can define the L-function of our Dirichlet character χ as

$$L_F(s, \chi) = \sum_{n \in K(\mathfrak{m})\backslash K^\natural(\mathfrak{m})} \frac{\chi(n)}{\mathrm{Nm}(n)^s},$$

where Nm was defined in section 7.1. In the particular case of a class character, we have

$$L_F(s, \chi) = \sum_{n \in \widehat{\mathcal{O}}_F^\times \backslash \widehat{\mathcal{O}}_F^\natural} \frac{\chi(n)}{\mathrm{Nm}(n)^s}.$$

8.2 A Bost-Connes algebra for Dirichlet characters

Let $\chi : \mathbb{A}_F \to S^1$ be a Dirichlet character that is supposed to be trivial at infinity. Let $G = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ and $X := G(\mathbb{R})/G(\mathbb{R})^+ \cong \{\pm 1\}^{[F:\mathbb{Q}]}$. Let $\mathfrak{m}$ be the conductor of χ and $K_M(\mathfrak{m}) \subset \widehat{\mathcal{O}}_F$ be the multiplicative semigroup defined by

$$K_M(\mathfrak{m}) = \mathrm{Ker}_{\mathrm{mult}}(\widehat{\mathcal{O}}_F \to \widehat{\mathcal{O}}_F/\mathfrak{m}) := \{n \in \widehat{\mathcal{O}}_F \mid \bar{n} = 1 \in\subset \widehat{\mathcal{O}}_F/\mathfrak{m}\}.$$

Recall that we denoted $K(\mathfrak{m}) \subset \widehat{\mathcal{O}}_F^\times$ the subgroup $K(\mathfrak{m}) = \mathrm{Ker}(\widehat{\mathcal{O}}_F^\times \to (\widehat{\mathcal{O}}_F/\mathfrak{m})^\times)$. Let $L = \mathcal{O}_F$ and $\phi : G \to \mathrm{GL}_\mathbb{Q}(F)$ be the regular representation.

Definition 8.2.1. The tuple $\mathcal{D}_{F,\chi} := ((\mathbb{G}_{m,F}, X, K(\mathfrak{m})), (K_M(\mathfrak{m}), \phi, L))$ is called the *Bost-Connes datum for the character* χ.

The time evolution and hamiltonian are the same as in the Bost-Connes case studied in Subsection 7.3.

Let a_χ be the operator on $\mathcal{H}_y$ defined by

$$(a_\chi \xi)(g) = \chi(g).\xi(g).$$

Definition 8.2.2. The χ-*twisted trace* Trace_χ on $\mathcal{B}(\mathcal{H}_y)$ is defined by

$$\mathrm{Trace}_\chi(D) = \mathrm{Trace}(a_\chi.D).$$

Definition 8.2.3. The χ-*twisted partition function* of $\mathcal{D}_{F,\chi}$ is defined as

$$\zeta_{\mathcal{D}_{F,\chi}}(s) = \mathrm{Trace}_\chi(e^{-\beta H_y}).$$

Lemma 8.2.4. *The χ-twisted partition function of $\mathcal{D}_{F,\chi}$ is equal to the Dirichlet L-function $L_F(s, \chi)$.*

Proof. This follows from the definition and Subsection 8.1. □

Notice that in this case, the symmetry semigroup is not full in the sense of Definition 4.5.5.

Remark 8.2.5. If we want to treat Dirichlet characters with nontrivial infinite component, it could be useful to construct the groupoid given by the partial action of $\mathbb{A}_F^\times$ on the space $\mathbb{A}_F \times \pi_0(C_F)$ where $C_F := F^\times \backslash \mathbb{A}_F^\times$. If we do the construction as before, using a quotient by $(\widehat{\mathcal{O}}_F^\times)^2$, the partition function will not be reasonable. It could be interesting to use Tate's thesis [Tat67], that expresses the Dedekind zeta function as an integral, to deal with this problem. It is not clear to us if a meaningful physical system can be constructed this way.

9 The Hilbert modular BCM system

We now use and translate the general formalism of Section 4 in the case of Hilbert modular Shimura data. This is a good training ground for the case of a general Shimura datum.

9.1 Construction

Let F be a totally real number field. Let $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, $X = (\mathbb{H}^{\pm})^{\mathrm{Hom}(F,\mathbb{R})}$. The Shimura datum (G, X) is the *Hilbert modular Shimura datum.* Let V be the $\mathbb{Q}$-vector space F^2 with the natural action ϕ of G. Let $M := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{M}_{2,F}$. Let $L \subset V$ be $\mathcal{O}_F^2$. Let $K_0 = \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \subset G(\mathbb{A}_f)$ and $K_M = \mathrm{M}_2(\widehat{\mathcal{O}}_F) \subset \mathrm{M}_2(\mathbb{A}_{f,F})$. Choose a fine subgroup $K \subset K_0$.

Definition 9.1.1. The pair $\mathcal{P}(\mathrm{GL}_{2,F}, X, K) := ((G, X, V, M), (L, K, K_M))$ is called a *Hilbert modular BCM pair* for F. The BCM algebra $\mathcal{H}(\mathcal{P})$ is called a *Hilbert modular BCM algebra.*

Lemma 9.1.2. *If we suppose that F has class number one, then the natural morphism*

$$\mathcal{H}(\mathrm{GL}_{2,F}, X, K) \to \mathcal{H}_{\mathrm{princ}}(\mathrm{GL}_{2,F}, X, K)$$

from the principal to the full Bost-Connes-Marcolli algebra is an isomorphism.

Proof. The hypothesis implies (in fact is equivalent to) $h(G, K) = 1$. The results then follows from proposition 5.1.3. □

We now describe more explicitly the time evolution whose construction was made in Subsection 4.4.

Let $C := \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$, which is the center of $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$. The natural determinant map $\det : G \to C$ induces $\det : K\backslash G(\mathbb{A}_f) \to C(\widehat{\mathbb{Z}})\backslash C(\mathbb{A}_f)$. The norm map $\mathrm{Nm} : C \to \mathbb{G}_{m,\mathbb{Q}}$ induces

$$\mathrm{Nm} : C(\widehat{\mathbb{Z}})\backslash C(\mathbb{A}_f) \to \widehat{\mathbb{Z}}^\times\backslash\mathbb{A}_f^\times \cong \mathbb{Z}^\times\backslash\mathbb{Q}^\times \cong \mathbb{Q}_+^\times \subset \mathbb{R}_+^\times.$$

Lemma 9.1.3. *The time evolution on the Hilbert modular BCM algebra $\mathcal{H}(\mathrm{GL}_{2,F}, X, K)$ is equal to*

$$\sigma_t(f)(g, y) = \mathrm{Nm}(\det(g))^{it} f(g, y).$$

9.2 Symmetries

We apply the general definitions of Subsection 4.5 to this case. We see that

$$\mathrm{Sym}_f = \mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural := \mathrm{GL}_2(\mathbb{A}_{f,F}) \cap \mathrm{M}_2(\widehat{\mathcal{O}}_F).$$

The center of $G = \mathrm{GL}_{2,F}$ is $C = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ and the center of $\mathrm{M}_2(\widehat{\mathcal{O}}_F)$ is $\widehat{\mathcal{O}}_F$ as a diagonal subsemigroup. We also have $\mathrm{Inn} = \mathcal{O}_F^\natural := \mathcal{O}_F \cap F^\times$ and an inclusion of semigroups $\mathcal{O}_F^\natural \subset \mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural$.

The following lemma explains what the symmetries are in the case of Hilbert modular BCM systems.

Proposition 9.2.1. *The outer symmetry semigroup* Out *of the Hilbert modular BCM system is isomorphic to $F^\times\backslash\mathrm{GL}_2(\mathbb{A}_{f,F}) \times \mathbb{G}_{m,F}(\mathbb{R})$, more precisely, the natural map*

$$\mathrm{Sym}_f := \mathcal{O}_F^\natural\backslash\mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural \to F^\times\backslash\mathrm{GL}_2(\mathbb{A}_{f,F})$$

is an isomorphism.

Proof. The injectivity of this map is clear because,

$$\mathcal{O}_F^\natural = F^\times \cap \mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural \subset \mathrm{GL}_2(\mathbb{A}_{f,F}).$$

Let $(\mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural)^{-1} = \{m \in G(\mathbb{A}_f) \mid m^{-1} \in \mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural\}$ be the semigroup of inverses of elements in $\mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural$. We then have

$$\mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural.(\mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural)^{-1} = \mathrm{GL}_2(\mathbb{A}_{f,F}).$$

Let $m \in \mathcal{O}_F^\natural \backslash \mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural$. We only need to prove that $m^{-1} \in \mathcal{O}_F^\natural \backslash \mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural$. Moreover, to invert a matrix it is enough to prove that its determinant is invertible. We have $\det(m) \in \mathcal{O}_F^\natural \backslash \widehat{\mathcal{O}}_F^\natural$. The nonarchimedean part of Proposition 7.3.3 gives $\mathcal{O}_F^\natural \backslash \widehat{\mathcal{O}}_F^\natural \cong F^\times \backslash \mathbb{A}_{f,F}^\times$, which implies that $\det(m)^{-1} \in \mathcal{O}_F^\natural \backslash \widehat{\mathcal{O}}_F^\natural \subset \mathcal{O}_F^\natural \backslash \mathrm{M}_2(\widehat{\mathcal{O}}_F)^\natural$. This finishes the proof. □

A Enveloping semigroups

In this appendix, we explain what enveloping semigroups are as they are a key ingredient in our formalism (cf. 4.1).

A.1 Drinfeld's classification

All groups will be over a field of characteristic 0. Recall from Subsection 4.1 the following definition.

Definition A.1.1. Let G be reductive group over a field. An *enveloping semigroup* for G is a multiplicative semigroup M such that $M^\times = G$, M is irreducible, and M is normal.

Such semigroups were classified by V. Drinfeld in private notes [Dri] that were kindly given to us by L. Lafforgue. Suppose that the base field is algebraically closed. Choose a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$. Let W denote the Weyl group for (G, B, T), $X = \mathrm{Hom}(T, \mathbb{G}_m)$.

Theorem A.1.2 (Drinfeld). *There exists a bijection between*

1. *the set of normal affine irreducible semigroups M containing G as their group of units, and*
2. *the set of W-invariant rational polyhedral convex cones $K \subset X \otimes_{\mathbb{Z}} \mathbb{R}$ which contain zero and are non-degenerate, i.e., not contained in a hyperplane.*

This classification implies that a semisimple group G has only one enveloping semigroup, namely G itself. This case is for us not very interesting (because a BCM system with such an envelopping semigroup has a trivial zeta function) and we would like to construct more interesting semigroups, in particular, we would like to construct cartesian diagrams

$$\begin{array}{ccc} G & \xrightarrow{\phi} & \mathrm{GL}(V) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathrm{End}(V) \end{array}$$

for some fixed representation $\phi : G \to \mathrm{GL}(V)$.

For example, for an adjoint Shimura datum (G, X) (i.e. $ZG = \{1\}$), the triviality of the enveloping semigroup implies that the BCM systems we construct have a trivial partition function. It is then interesting to construct another Shimura datum with adjoint datum (G, X) and such that the enveloping semigroup is not trivial anymore.

There is a natural method due to Vinberg to "enlarge the center" of a given semisimple simply connected group G in order to have an enveloping semigroup that is universal in a certain sense. For the sake of brevity, we will not discuss this construction here.

A.2 Ramachandran's construction of Chevalley semigroups

There is another way to construct enveloping semigroups quite explicitly, which was communicated to us by N. Ramachandran. The construction of N. Ramachandran uses the following theorem of Chevalley (see [Spr98], 5.1).

Theorem A.2.1 (Chevalley). *Let G be an algebraic group and $\phi : G \to \mathrm{GL}(V)$ be a faithful representation of G. There is a tensor construction $T^{i,j} := V^{\otimes i} \otimes V^{\vee, \otimes j}$ and a line $D \subset T^{i,j}$ such that $\phi(G) \subset \mathrm{GL}(V)$ is the stabilizer of this line.*

Definition A.2.2. Let G be an algebraic group over $\mathbb{Q}$, $\phi : G \to \mathrm{GL}(V)$ be a faithful representation of G. Let T and D be as in Chevalley's theorem. Suppose that $T = V^{\otimes i}$ (resp. $T = V^{\vee, \otimes i}$) contains no (resp. only) dual tensor factors. The multiplicative semigroup

$$M(G, V, T, D) := \{m \in \mathrm{End}(V) \mid m.D \subset D\}$$
$$(\text{resp. } M(G, V, T, D) := \{m \in \mathrm{End}(V) \mid {}^t m.D \subset D\})$$

is called a *Chevalley enveloping semigroup* for G in $\mathrm{End}(V)$.

Example A.2.3. If $G = \mathrm{GL}_2$ and V be the standard representation, then $D = \bigwedge^2 V \subset V^{\otimes 2}$ is a line as in Chevalley's theorem and M_2 is the corresponding Chevalley enveloping semigroup.

Example A.2.4. Let (V, ψ) be a $2g$-dimensional $\mathbb{Q}$-vector space equipped with an alternating form $\psi \in \bigwedge^2(V^\vee)$. Then the line $D = \mathbb{Q}\langle\psi\rangle \subset V^{\vee, \otimes 2}$ is a line as in Chevalley's theorem for the group GSp_{2g} with its standard representation and the points of the corresponding semigroup in a commutative $\mathbb{Q}$-algebra A are given by

$$\mathrm{MSp}_{2g}(A) := \{m \in \mathrm{End}(V)(A) \mid \exists\, \mu(m) \in A,\ \psi(m.x, m.y) = \mu(m)\psi(x, y),\ \forall\, x, y \in V_A\}.$$

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