On the Real part of non-trivial zeros of Riemann´s Zeta function.

A.M.S. (Subject Classification, 1980): 30E20, 45E05, 45M05, 12A70.

To The Salvadorian People
That Revolt in spite of
Their Misery and Ignorance.

The Mathematician thinks that a problem
Is easy to solve
After it has been solved

Preface:

The following notes summarize five years of effort. The research process has not been linear, suffering moments of advance and moments of stagnation, and radical changes in the tactic used to solve the problem. I tried with Differential Equations (ordinaries and with partial derivatives), Classical Analysis and Harmonic Functions, until I found in an almost fortuitous way the Boundary Value Problems and the Singular Integral Equations Theory, which I realized it could be of considerable use in solving the problem. The results are surprising.

Naturally, the difference of standpoint can be observed as regards the Riemann´s zeta function, between Titchmarsh or Iviĉ (see Bibliography) and that presented here. In the first case, a mathematical tool is used that, although it has developed and advanced greatly in respect to Riemann´s Zeta Function Theory, in the specific case of finding the real part of non-trivial zeros, is particularly obscure . I believe that the main reason the Boundary Value Problems and Singular Integral Equations Theory have not been used for trying to prove the "Riemann´s Hypothesis" is the following: in the first place, the Boundary Value Problems was originated in Physics problems, in Quantum Mechanics, Fluids Mechanics and the Mathematical Theory of Elasticity, and it was here that it was developed and strengthened. On the other hand, the problem as regards the mentioned hypothesis has been taken up, in general, from the point of view of the relationship, between the zeta function and the distribution of prime numbers using Classical Analysis: Integral Transformations, bounds, asymptotic approximations, etc. Even more, in all textbooks and journal revised till now, the problem o determining the real part of non trivial zeros of the Riemann´s Zeta function has been worked around the asymptotic formula for the number of zeros in the critical strip, and in particular with the formula established by Riemann and afterwards proved by Von Mangolt:

$$N(T) = \left(\frac{T}{2\pi}\right)\left(\ln\frac{T}{2} - 1\right) + O(\ln T)$$

Where $N(T)$ is the number of zeros of the form $\beta + i\alpha$ with $0 < \alpha \leq T$

As is known, Hardy was the first in proving that the ζ function has infinity number of zeros on the line $\operatorname{Re} z = \frac{1}{2}$ and afterwards Selberg proved that the number of zeros $N_0(T)$ on this line satisfies the inequality $N_0(T) > A \cdot T \cdot \ln T$ for some $A > 0$.

This is approximately the state of the game in the research of the zeros of ζ function in the critical strip, that can truly be considerer, a part of analytic number theory.

The main idea we propose is to change the point of view of the problem, using the theory of Cauchy type singular integral, the boundary value problems and the singular integrals equations theory, for the horizontal study of the problem.

If these notes stand the test of analysis and opinions of specialists, two important subjects will be highlighted in Analytic Number Theory:

a- Signals of unity of the Boundary Value Problems and Analytic Number Theory

b- Proof of Riemann´s Hypothesis.

As the reader will appreciate it has been sufficient to examine the problem with this new standpoint instead of the usual one. This and the development of the boundary value problems and singular integral equations have made these notes possible.

I wish to extend my gratitude to William Lambert Martin (R.I.P) and Héctor Figueroa for the rigor, the enthusiasm and the curiosity for Mathematics they once taught me. I particularly wish to thank Professor Edwin Castro Fernández in whose classes the three mentioned subjects were synthesized. Without the advice of Professor Castro these notes wouldn't exist. He was the first in read the manuscript and made the respective observations and notes.

Finally, I wish to thank Professor Manuel Barahona Droguett, for having enough historical perspective to stimulate me to undertake the effort, which the publishing of a paper like this involves.

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Abstract:

The Riemann´s zeta function has been subject of amount of research since its introduction in Mathematics, as well as the Riemann´s Hypothesis(RH).
Till now, the basic direction on research respect to RH has been "climbing" the critical line $x = \operatorname{Re} s = \frac{1}{2}$, as well as the region named "critical strip": $0 < \operatorname{Re} s < 1$. We propose to change the point of view in order to approach the zeta function in the critical strip but in horizontal sense in order to attempt to prove the RH:
If s is a zero of the ζ function on the critical strip then $x = \operatorname{Re} s = \frac{1}{2}$.
This will be done by means of the theory of Singular Integral Equations, mainly trying to prove that non- trivial zeros of ζ are associated to the asymptotic behavior of certain kind of solutions of the singular integral equation:
$\int_L \varphi_s(\tau) \cdot \frac{d\tau}{\tau - t} = 0, \; t \in L$ For some φ_s Hölder function. All integrals that appear should be understood in the sense of VP (Cauchy).

Key words: zeta function, singular integral equation.

A.M.S. Subject Classification 1980: 45E05, 45M05, 12A70, 30E20.

The following sequence of propositions has as a main objective the Proof of Riemann´s Hypothesis:

If $\zeta(s)=0$ on the critical strip, then $x=\operatorname{Re} s=\dfrac{1}{2}$

Proposition 1: For Re s >1:

$$\zeta(s)=\frac{1}{\Gamma(s)}\cdot\int_1^{+\infty}\frac{(\ln\tau)^{s-1}}{\tau}\cdot\frac{d\tau}{\tau-1} \qquad (1)$$

Proof: Let's expand the kernel of the integral:

$$\frac{1}{\tau-1}=\frac{1}{\tau}\cdot\frac{1}{1-\frac{1}{\tau}}=\frac{1}{\tau}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{\tau}\right)^n$$, That converges uniformly for $|\tau|>1$

In consequence:

$$\int_1^{+\infty}\frac{(\ln\tau)^{s-1}}{\tau}\cdot\frac{d\tau}{\tau-1}=\int_1^{+\infty}\frac{(\ln\tau)^{s-1}}{\tau^2}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{\tau}\right)^n d\tau=\sum_{n=0}^{\infty}\int_1^{+\infty}\frac{(\ln\tau)^{s-1}}{\tau^2}\cdot\frac{d\tau}{\tau^n} \qquad (1^*)$$

And the interchange of the sum and the integral is justified by Levi's dominated convergence theorem.

If now we put $x=\ln\tau$ the integral under the sum sign is :

$$\int_1^{+\infty}\frac{(\ln\tau)^{s-1}}{\tau^2}\cdot\frac{d\tau}{\tau^n}=\int_0^{+\infty}x^{s-1}\cdot e^{-x(n+1)}dx$$

and by putting: $t=x(n+1)$ in the last integral of the right hand we will have:

$$\int_0^{+\infty}x^{s-1}\cdot e^{-x(n+1)}dx=\frac{1}{(n+1)^s}\cdot\int_0^{+\infty}t^{s-1}\cdot e^{-t}dt=\frac{1}{(n+1)^s}\cdot\Gamma(s).$$

Putting this result in (1*) will follows:

$$\int_1^{+\infty}\frac{(\ln\tau)^{s-1}}{\tau}\cdot\frac{d\tau}{\tau-1}=\sum_{n=0}^{\infty}\frac{1}{(n+1)^s}\cdot\Gamma(s)=\zeta(s)\cdot\Gamma(s)$$, that is the Proposition ■

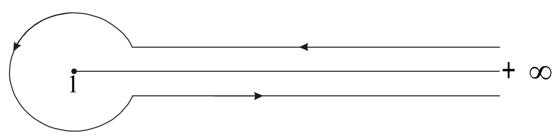
The Formula (1) enables us to extend the ζ function to the whole plane as in the following proposition.

Proposition 2:

For $s \in \mathbb{C}, \quad s \neq 1$:

$$\zeta(s) = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \cdot \int_L \frac{(-\ln\tau)^{s-1}}{\tau} \cdot \frac{d\tau}{\tau - 1} \qquad (2)$$

Where L stand to be the contour depicted below:



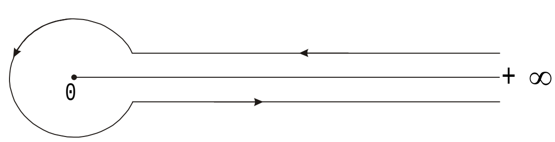
And $(-\ln\tau)^{s-1} = e^{(s-1)\ln(-\ln\tau)}$; the logarithm of $-\ln\tau$ is determined in such a way that it is real for negatives values of $\ln\tau$.

Proof:

Putting: $z = \ln\tau$ we get to:

$$\zeta(s) = \frac{e^{-i\pi s} \cdot \Gamma(1-s)}{2\pi i} \cdot \int_C \frac{(-z)^{s-1}}{e^z - 1} dz \qquad (3)$$

Where C stand to be the contour of below:



The functional relationship (3) is essentially the starting point of Riemann´s own analytical extension of ζ function (see: Edwards, H.M.: “The Riemann´s zeta function”. Academic Press, N.Y.1974 the translation of Riemann´s paper “Ueber die Anzahl der Primzahlen…”). ■

It is important to note, at this point, that in [5], Titchmarsh, E.C, that has been maybe the most important historical reference to Riemann´s ζ

function, the analytical extension was wrong, dropping on it the minus sign of the numerator under the integral sign in (3).(see: page18 of 15], Chap 2,§2.4) From (3), most of Mathematicians extent the contour to the punctured plane ℂ−{ 1 } after avoiding the singularities: $z = 2k\pi\imath$, for $k\in\mathbb{Z}$, in such a manner that from that point of view, it can be deduced easily the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \cdot \Gamma(1-s)\zeta(1-s)$$

that characterize plenty the ζ function.

We will avoid this point of view, and enunciate rather the following:

Proposition 3

For $s \in \mathbb{C}$, $s \neq 1$:

$$\zeta(s) = \frac{e^{2\pi i s} - 1}{2\pi i e^{i\pi s}} \cdot \Gamma(1-s) \cdot \int_1^{+\infty} \frac{(\ln\tau)^{s-1}}{\tau} \frac{d\tau}{\tau - 1} \qquad (4)$$

Proof:

From Riemann´s point of view, we can write:

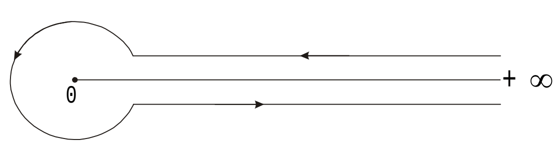
$$\Pi(s-1)\zeta(s) = \Gamma(s)\zeta(s) = \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx \quad \text{for Re } s > 1$$

(Riemann denoted $\Pi(s-1) = \Gamma(s)$; we will keep the notation Γ)

and also that:

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} dx = (e^{-i\pi s} - e^{i\pi s}) \int_0^{+\infty} \frac{x^{s-1}}{e^{x-1}} dx$$

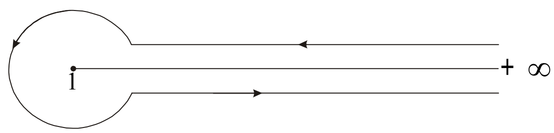
Here *C*:



But if we put $\tau = e^x$: we can write:

$$(*) \quad \int_C \frac{(-x)^{s-1}}{e^x - 1} dx = \int_L \frac{(-\ln \tau)^{s-1}}{\tau} \cdot \frac{d\tau}{\tau - 1} = (e^{-i\pi s} - e^{i\pi s}) \int_1^{+\infty} \frac{(\ln \tau)^{s-1}}{\tau} \cdot \frac{d\tau}{\tau - 1}$$

Where L is the contour depicted below:



Riemann himself wrote that equality:

$$2\sin(\pi s)\Gamma(s)\zeta(s) = i \int_C \frac{(-x)^{s-1}}{e^x - 1} dx \quad \text{remains valid for all } s \neq 1$$

and asserted also that this integral equals (*), therefore we can write:

$$2\sin(\pi s)\Gamma(s)\zeta(s) = i(e^{-i\pi s} - e^{i\pi s}) \cdot \int_1^{+\infty} \frac{(\ln \tau)^{s-1}}{\tau} \cdot \frac{d\tau}{\tau - 1}$$

or, what is the same:

$$\zeta(s) = \frac{i}{2\sin(\pi s)\Gamma(s)} \cdot \frac{1 - e^{2i\pi s}}{e^{i\pi s}} \cdot \int_1^{+\infty} \frac{(\ln \tau)^{s-1}}{\tau} \cdot \frac{d\tau}{\tau - 1} \quad \text{for } s \neq 1$$

But since: $\sin(\pi s)\Gamma(s) = \dfrac{\pi}{\Gamma(1-s)}$, will follows:

$$\zeta(s) = \frac{i\Gamma(1-s)}{2\pi \cdot e^{i\pi s}} \cdot (1 - e^{2\pi i s}) \int_1^{+\infty} \frac{(\ln \tau)^{s-1}}{\tau} \frac{d\tau}{\tau - 1},$$ and multiplying and

dividing by i , we will have:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i \cdot e^{i\pi s}} \cdot (e^{2\pi i s} - 1) \int_1^{+\infty} \frac{(\ln \tau)^{s-1}}{\tau} \frac{d\tau}{\tau - 1},$$ that proves the Proposition■

From this last proposition, easily follows that $\zeta(-2n) = 0$, by the simply evaluation of the representation, since $e^{2\pi i s} - 1 = \cos 2\pi s + i \sin 2\pi s - 1$ equals zero for $s = -2n$, $n \in \mathbb{N}$, and the integral is bounded for those values of s. These zeros are named trivial zeros of the ζ function. It is to note the difference between this standpoint and the one in the traditional deduction of the trivial zeros from the functional equation of ζ, mentioned before.
From here we get the fundamental theoretical turn in the solution of the problem of finding the real part of non-trivial zeros: the improper (singular) integral that appears in the functional equation (4) should be understood as a limit point of the principal value of a integral of Cauchy type over a contour that extend to infinity:

$$\int_1^{+\infty} \frac{(\ln \tau)^{s-1}}{\tau} \frac{d\tau}{\tau - 1} = \lim_{t \to 1^+} \int_1^{+\infty} \frac{(\ln \tau)^{s-1}}{\tau} \frac{d\tau}{\tau - t}$$

In order to confirm this last equality, we give the following

<u>*Proposition 4*</u>:

Let φ be an arbitrary Hölder function on $]1, +\infty[$, then:

$$\lim_{t \to 1^+} \int_1^{+\infty} \varphi(\tau) \cdot \frac{d\tau}{\tau - t} = \int_1^{+\infty} \varphi(\tau) \cdot \frac{d\tau}{\tau - 1}$$, and the limit is uniform.

<u>*Proof:*</u>

This is an standard result of the theory of Singular Integrals,(see: Muskhelishvili, N,I. “Singular Integral Equations”. Wolter-Noordhoof Publishing Co. Groningen, The Netherlands, 1972, page 38 and ff.) ■

Definition5:

We define a function: Φ_s on the set $]1\ ,+\infty\ [$ by means of one singular integral of Cauchy type over a contour that extent to infinity:

$$\Phi_s(t) = \frac{1}{2\pi i} \cdot \int_1^{+\infty} \varphi_s(\tau) \cdot \frac{d\tau}{\tau - t} \quad (\ t \in \left]\, 1,\ +\infty\ \right[\ ) \qquad (5)$$

with density function: $$\varphi_s(\tau) = \frac{(\ln\tau)^{s-1}}{\tau} \quad (\ \tau \in \left]1, +\infty\right[\ ) \qquad (5^*)$$

Where $s = x + iy$, and $x \in]\,0,\ 1\,[$, the real part of s , is fixed but unknown and y is an arbitrary given real number.

Proposition 6:

The integral (5) with density (5*) is a Cauchy type integral over a contour that extend to infinity.

Proof:

It is enough to verify that the two following conditions are satisfied:
a-) The density function φ_s is a Lipschitz function on $]1\ ,+\infty\ [$.

b-) For enough greater values of t there holds: $\left|\varphi_s(t) - \varphi_s(\infty)\right| \le \dfrac{A}{t^{\mu}}$

for some $A > 0$, $\mu > 0$

a-) Is an easy matter : φ_s is differentiable respect to t and the derivatives are finite on $]1\ ,+\infty\ [$. The matter that $\varphi'(1) = \infty$ doesn't make difference regards to the conclusion.

b-) Let's first prove that $\lim\limits_{\tau \to +\infty} \varphi_s(\tau) = 0$:

Indeed: $\left|\varphi_s(\tau)\right| = \dfrac{(\ln\tau)^{x-1}}{\tau}$. Since $x \in]\,0,\ 1\,[$, $x - 1 < 0$, we get thus:

$$|\varphi_s(\tau)| = \frac{1}{\tau\cdot(\ln\tau)^{1-x}} \to 0 \text{ if } \tau\to+\infty, \text{ that is: } |\varphi_s(\infty)| = 0 \qquad (6)$$

Besides: $$|\tau|\cdot|\varphi_s(\tau)| = \frac{1}{(\ln\tau)^{1-x}} \to 0 \text{ if } \tau\to+\infty$$

Then there exists *A > 0* and $\mu = 1$, such that the proposition remains valid□

The fact $|\varphi_s(\infty)| = 0$ in (6) guarantees the convergence of the integral. (*Gakhov, F.D. " Boundary Value Problems", Pergamon Press, Oxford,1966, § 4, Secc 4.6).*

If we consider now $\varphi_s(\tau)$ as an unknown function, since we don't know the value of $x = \operatorname{Re} s$ when the function (5) with density (5*) equals to zero when $t \to 1^+$, then we have the right to invoking the following proposition:

Proposition 7:

The solution of the singular integral equation:

$$\int_1^{+\infty} \varphi(\tau)\cdot\frac{d\tau}{\tau - t} = 0 \qquad (7)$$

in the class of Hölder functions unbounded at initial extreme of the contour of integration is:

$$\varphi(t) = \frac{a_0}{\sqrt{t-1}}, \qquad (7^*)$$

for some complex constant $a_0 = a_1 + ia_2$ that doesn't depend on *t*.

Proof:

This is an standard result of the theory of Singular Integral Equations. (See, for instance *: Gakhov, F.D. op. cit. § 42.3.*) ■

If we are now looking for the solution of the integral equation:

$$\int_1^{+\infty} \frac{(\ln\tau)^{s-1}}{\tau} \frac{d\tau}{\tau - t} = 0, \text{ for } t \to 1^+ \qquad (8)$$

in the class of Hölder functions unbounded at the initial extreme of the contour of integration, then the solution of (8) must behave in the same way than (7*) in a right side neighborhood of 1 , that is both densities must be asymptotically equal when $t \to 1^+$.
We have thus the following
Proposition 8:

$$\text{If: } \frac{(\ln t)^{x-1+iy}}{t} \sim \frac{a_1 + ia_2}{\sqrt{t-1}} \text{ for } t \to 1^+ \text{ then : } x = \frac{1}{2}$$

Proof: We have:

$$\lim_{t \to 1^+} \frac{\dfrac{(\ln t)^{x-1+iy}}{t}}{\dfrac{a_1 + ia_2}{\sqrt{t-1}}} = 1$$

then there must exists real function $v_a(t)$, with $\lim_{t \to a} v_a(t) = 0$ in such a manner that:

$$\frac{\dfrac{(\ln t)^{x-1+iy}}{t}}{\dfrac{a_1 + ia_2}{\sqrt{t-1}}} = 1 + v_a(t)$$. By taking logarithm in both sides of the equality we get to:

$$(x - 1 + iy)\ln\ln t = \ln[(a_1 + ia_2)t] + \ln(1 + v_a(t)) - \frac{1}{2}\ln(t-1)$$

But since :

$$\ln[(a_1 + ia_2)t] = \ln|(a_1 + ia_2)t| + i(\arg(a_1 + ia_2)t + 2k\pi)$$

We get to the pair of equations:

$$x-1=\frac{\ln|(a_1+ia_2)t|}{\ln\ln t}+\frac{\ln(1+v_a(t))}{\ln\ln t}-\frac{1}{2}\frac{\ln(t-1)}{\ln\ln t} \qquad (9)$$

$$y=\frac{[\arg((a_1+ia_2)t)]+2k\pi}{\ln\ln t}, \quad \text{for } k\in\mathbb{Z} \qquad (10)$$

By approaching $t\to 1^+$ from the first of these two equations we will have:

$$\lim_{t\to 1^+}\frac{\ln(|a_1+ia_2|t)}{\ln\ln t}=0, \quad \lim_{t\to 1^+}\frac{\ln(1+v_a(t))}{\ln\ln t}=0, \quad \lim_{t\to 1^+}\frac{\ln(t-1)}{\ln\ln t}=1$$

and in consequence : $x-1=-\frac{1}{2}$. This is the Riemann´s Hypothesis.■

It is to note that the method used here for approach the Riemann's Hypothesis does not gives information about the imaginary part of non-trivial zeros, since from (10) follows:

$$y\to\pm\infty \quad \text{if} \quad k\to\pm\infty \quad \text{and} \quad t\to 1^+$$

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