Problem with a Claimed Proof of the Riemann Hypothesis
> Bergstrom 2008

§ 1 The Claim

Bergstrom 2008 claims a proof of the Riemann Hypothesis in section 9, "Proof of Conjecture 1.1", page 8, and repeats this claim in FAQ #8 on page 39.

Quote from Bergstrom 2008, section 9, page 8

We then get

$$\lim_{N \to \infty} \frac{\pi^s (-4^s + 8^s) \cos(\frac{1}{2} s \pi) (-2 + 2^s) (s + 2) (s + 1) s \Gamma(s - 3)}{N^{(2s - 1)}} = 1$$

(20)

This equation can be true only if the modulus of $N^{(2s - 1)}$ is equal to $N^0 = 1$, which thus requires that,

$$\sigma = \frac{1}{2}$$

This thus proves Conjecture 1.1 that Re(s) must be equal to $\frac{1}{2}$ for all zeros of the Riemann zeta-function $\zeta(s)$ in the range $0 < \text{Re}(s) < 1$.

End of quote

The equation in line (20) is false because two of the possible values of the limit do not equal 1 and the third possible value does not necessarily equal 1.

§ 2 Reformulation of the limit in (20)

If $a \in \mathbb{R}(0<)$ and $z, z_1, z_2 \in \mathbb{C}$ then $|a^z| = a^{\text{Re}(z)}$, $a|z| = |az|$ and $|z_1 z_2| = |z_1| |z_2|$.

These relations can be used to reformulate (20) as $\lim_{N \to \infty} N^{(2\sigma - 1)} f(s) \equiv 1$ where,

$$f(s) \equiv \frac{\pi^s (-4^s + 8^s) \cos(\frac{1}{2} s \pi) (-2 + 2^s) (s + 2) (s + 1) s \Gamma(s - 3)}{2}$$

In (20), $N$ is an arbitrary member of $\mathbb{Z}(\geq 1)$ and $s$ an arbitrary member of $\mathbb{C}'$ which is $\mathbb{C}$ after excluding the singularities of $f(s)$.

As $N$ and $s$ are independent, relation(1) can be reformulated to $f(s) \lim_{N \to \infty} N^{(2\sigma - 1)} = 1$. 
§ 3 The Problem

The equation in (20) is equivalent to relation(2). The possible values of the limit in relation(2) are,

\begin{itemize}
  \item \(2\sigma - 1 > 0 \rightarrow f(s) \lim_{N \to \infty} N^{(2\sigma - 1)} = \infty\)
  \item \(2\sigma - 1 = 0 \rightarrow f(s) \lim_{N \to \infty} N^{(2\sigma - 1)} = f(s_1)\) where \(s_1 = \frac{1}{2} + it \in \mathbb{C}'\)
  \item \(2\sigma - 1 < 0 \rightarrow f(s) \lim_{N \to \infty} N^{(2\sigma - 1)} = 0\)
\end{itemize}

Hence, as stated in Bergstrom 2008 (see quote in § 1 above), if (20) is true then \(\sigma = \frac{1}{2}\). However if (20) is true then also \(f(s_1) = 1\) and this is not necessarily true, for example \(f(\frac{1}{2} + i) = 0.801\), from references (3,4).

Bergstrom 2008, FAQ #8, page 39 claims to prove \(f(s_1) = 1\) but there is an error at the beginning of the proof.

Quote from Bergstrom 2008, FAQ #8, page 39

FAQ #8 (pages 8-9)

“Can you please show me the details of how the rest of the expression on the left-hand side of (20) becomes equal to the middle expression (which is unity) in Remark 9.2 on pages 8-9 in your preprint.”

ANSWER: Step1.

First I need to show that

\[ |(s+2) (s+1) s \Gamma(s-3)| = |\Gamma(s)| \quad (1) \]

for \(s = \frac{1}{2} + it\) as I say on top of page 9 in the preprint.

The following recurrence relation is valid for the gamma function (choose \(n = 4\) in Abramowitz and Stegun, eq 6.1.16)

\[ (z+3) (z+2) (z+1) \Gamma(z+1) = \Gamma(4+z) \quad (2) \]

End of quote

The relation in line (2) is valid for \(z \in \mathbb{C}\) after excluding \(\mathbb{Z}(\leq 0)\) which is the set of singularities of \(\Gamma(z)\).

When \(s = 4+z\) then line (2) becomes, \((s-1) (s-2) (s-3) \Gamma(s-3) \neq \Gamma(s)\)
Relation(3) is valid for $s \in \mathbb{C}''$ which is $\mathbb{C}$ after excluding $\mathbb{Z}(\leq 0) \cup \{1,2,3\}$, the set on which $\Gamma(s-3)$ and/or $\Gamma(s)$ is undefined.

The left side of line(1) is defined when $s \in \mathbb{C}''$.

If $s \in \mathbb{C}''$ then equating the left sides of relation(3) and line (1), and after rearrangement,

$$\left( \frac{|s-1|}{|s+1|} \right) \left( \frac{|s-2|}{|s+2|} \right) \left( \frac{|s-3|}{|s|} \right) = 1$$

Relation(4) is false because each factor in brackets is $\in \mathbb{R}(0,1)$.

As relation(3) is true then the falsity of relation(4) implies the falsity of line (1) for $s \in \mathbb{C}''$ which includes all values of $s$ with $\text{Re}(s) = \frac{1}{2}$.

References

1. Abramowitz, M. and Stegun, I., "Handbook of Mathematical Functions", Dover, 7e
3. Calculation of functions of a complex variable, Microsoft Excel 2007 > Formulas > More Functions > Engineering
4. Online calculator of $\Gamma(z)$ for complex inputs, ttp://members.chello.nl/k.ijntema/gamma.html

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