Problem with an Implicit Claim of a Proof of the Riemann Hypothesis

> Choie et al 2007

§ 1 The Implicit Claim

1. Proposition 5.1 (“5.1”) in Choie et al 2007, p.367, states, "If Robin’s inequality holds for all Hardy–Ramanujan integers $5041 \leq n \leq x$ then it holds for all integers $5041 \leq n \leq x$.

2. “Robin’s Inequality” is $\sigma(n) < e^\gamma n \log \log n$ and Hardy–Ramanujan integers are defined in Choie et al 2007, p.367, "Recall that $p_1, p_2, \ldots$ denote the consecutive primes. An integer of the form $\prod_{i=1}^{s} p_i^{e_i}$ with $e_1 \geq e_2 \geq e_3 \ldots \geq e_s \geq 0$ we will call a Hardy-Raumanujan integer".

3. Although Choie et al 2007 does not explicitly claim a proof of the Riemann Hypothesis, on p.368 the paper does explicitly claim a proof of 5.1, and 5.1 is equivalent to the Riemann Hypothesis via Theorem 1 in Robin 1984, so Choie et al 2007 is implicitly claiming a proof of the Riemann Hypothesis.

§ 2 Preliminaries

4. Define the following sets,

- $H \equiv$ set of all Hardy-Ramanujan numbers in $\mathbb{N}$
- $X \equiv$ set of integers $n$ such that $5041 \leq n \leq x$
- $V \equiv$ set of numbers in $\mathbb{N}$ that violate Robin’s Inequality
- $\emptyset \equiv$ empty set

§ 3 5.1 is equivalent to the Riemann Hypothesis

5. Choie et al 2007, 5.1, can be rewritten using the sets defined in §2 as,

$$(H \cap X) \cap V = \emptyset \implies X \cap V = \emptyset.$$ 

6. The associativity of $\cap$ implies relation(1) is true iff $H \cap (X \cap V) = \emptyset \implies X \cap V = \emptyset$ and relation(2) is true iff $X \cap V \neq \emptyset$ because $H \neq \emptyset$. 
7. The \( x \) in the definition of \( X \) may be arbitrarily large, so \( X \cap V = \emptyset \) iff \( \mathbb{N}(\geq 5041) \cap V \neq \emptyset \).

Relation(4) is equivalent to the Riemann Hypothesis by Theorem 1 of Robin 1984 which states, "The hypothesis of Riemann is equivalent to: \( \forall n \geq 5041, \sigma(n) < e^\gamma n \log \log n. \"

8. The transitivity of the "iffs" connecting 5.1 via relations(1,2,3,4) with the Riemann Hypothesis imply 5.1 is equivalent to the Riemann Hypothesis.

§ 4 A Problem with the Proof of 5.1

9. There appears to be a problem with the proof of 5.1 due to its dependence on Lemma 5.5. Choie et al 2007, p.368 states, "On invoking the second part of Lemma 5.4 and Lemma 5.5, the proof of Proposition 5.1 is completed." This is all that Choie et al 2007 says about how 5.1 is proved and, in particular, no details are given of how Lemma 5.5 is invoked.

10. A problem with invoking Lemma 5.5 is there are an infinitude of numbers in \( \mathbb{N} \) to which Lemma 5.5 does not apply because these numbers do not satisfy the premise of Lemma 5.5. The proof of 5.1 in Choie et al 2007 does not consider this case.

11. Lemma 5.5 states, "Let \( \tilde{e} \) denote the factorization pattern of \( n \). If \( n \geq 5041 \) and \( m(\tilde{e}) \leq 5040 \), then \( n \) is in \( R \)."

12. In Lemma 5.5, \( \tilde{e} \equiv (e_1, e_2, e_3, \ldots, e_s) \), and \( \tilde{e} \) is called the "factorization pattern", and \( m(\tilde{e}) \) is the smallest number having the factorization pattern \( \tilde{e} \), and \( R \) is the complement of \( V \) in \( \mathbb{N} \).

13. The sequence, \( \tilde{e} \), is defined in Choie et al 2007, p.368 as the sequence of the powers of the primes in the factorization of \( n \) when these powers are ordered as a non-strictly decreasing sequence. Choie et al 2007 refers to \( \tilde{e} \) sometimes as the "exponent pattern" but more frequently as the "factorization pattern".

14. The premise of Lemma 5.5 is, \( n \geq 5041 \) and \( m(\tilde{e}) \leq 5040 \). There are an infinitude of numbers in \( \mathbb{N} \) which do not satisfy the premise of Lemma 5.5.
An example of such a number is $n = 80,805,879 = 3^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19^3$. Its factorization pattern is $(3,2,1,1,1)$ so $m(\tilde{e}) = 2^3 \cdot 3^2 \cdot 5 \cdot 7^3 \cdot 11 \\leq 27,720 > 5040$. Therefore, $n = 80,805,879$ does not satisfy the premise of Lemma 5.5.

§5 Conclusion

Choie et al 2007, 5.1 is equivalent to $X \cap V = \emptyset$ and this equality is equivalent to the Riemann Hypothesis via Theorem 1 in Robin 1984.

There is a problem with the proof of 5.1 because it depends on Lemma 5.5 and there is an infinitude of numbers in $\mathbb{N}$ to which Lemma 5.5 does not apply. The proof of 5.1 in Choie et al 2007 does not consider this case.

References


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