from garry.herrington@gmail.com , 6 Jan 2019

Problem with a Claimed Proof of the Riemann Hypothesis
> Eswaran 2018

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Response to R.Raghavan’s Note

§ 1 Preliminaries

For brevity, let:

D1. $\mathbb{N}_n \equiv$ the sequence 1,2,3, …, n.

D2. $\lambda n \equiv \lambda_n \equiv \lambda(n)$

D3. $\lambda\mathbb{N}_n \equiv \lambda1, \lambda2, \lambda3, \ldots, \lambda n$

D4. $\Sigma\, \lambda\mathbb{N}_n \equiv \lambda1 + \lambda2 + \lambda3 \ \ldots + \lambda n$

D5. $\mathbb{N}$ and $\lambda\mathbb{N}$ and $\Sigma\,\lambda\mathbb{N}$ are defined by D(1,3,4) respectively when $n \longrightarrow \infty$

D6. "random walk" ≡ 1-D random walk $s_1, s_2, s_3, \ldots, s_n$ with $s_n = \pm1$ and $p = ½$

D7. L(N) in Eswaran 2018 $\equiv \Sigma\, \lambda\mathbb{N}_N$

§ 2 The Primary Problem with Eswaran 2018

1. I am aware of the claims in Eswaran 2018 that are described in R.Raghavan’s Note. There are problems with these claims but it is not necessary to consider these secondary problems because the primary problem described in Note 1 is alone sufficient to invalidate the claimed proof of the Riemann Hypothesis in Eswaran 2018.

2. This primary problem is that Eswaran 2018 uses Theorems (1,2) described in Note 1 to claim a proof of the Riemann Hypothesis. For example, Eswaran 2018, page 2 says,

"Therefore, if it could be shown that the L(N) series is a random walk , and that $|L(N)| \sim N^{½}$ as $N \longrightarrow \infty$, the RH would be proved. This is the approach taken here."

3. This approach requires, in particular, Theorem 1 in Note 1 and the use of that theorem for the sequence $\lambda\mathbb{N}$ requires that sequence to be a random walk. However, as explained in Note 1, the sequence $\lambda\mathbb{N}$ is not a random walk because λn is not independent of n and therefore the sequence $\lambda\mathbb{N}$ lacks an essential defining property of a random walk.

§ 3 Contradictory Claims in Eswaran 2018 about whether $\lambda\mathbb{N}$ is a random walk

1. The claims in Eswaran 2018 about whether the sequence $\lambda\mathbb{N}$ is a random walk appear to be contradictory or are at least unclear.

2. Eswaran 2018, page 3 claims, "With Theorem 3B and Appendices III and IV, we have proved that the L(N) series is a random walk of infinite length", but Eswaran 2018, page 17 claims, "We first show that the λ's in the natural sequence, far from being random, are actually perfectly predictable and therefore deterministic."

3. A lack of precision pervades Eswaran 2018 and means the reader frequently has to guess what is meant. For example, the page 3 quote above includes the statement, "we have proved that the L(N) series is a random walk of infinite length". This statement taken at face value is nonsensical. The L(N) series is the finite series, $\Sigma\ \lambda\mathbb{N}_N$, which is a different entity from an infinite sequence and, in particular, it is a different entity from an infinite random walk which is a special type of infinite sequence.

4. My guess is that the page 3 quote above is claiming the sequence $\lambda\mathbb{N}$ is a random walk. However the page 17 quote above appears to contradict this and seems to say the sequence $\lambda\mathbb{N}$ is not a random walk.

§ 4 Absence of Any Evidence in Eswaran 2018 of Independent Expert Review

1. A media report from "The Hindu" (26 Nov 2018), provided by R.Raghavan, says the author of Eswaran 2018, Dr Kumar Eswaran, is "Professor of Computer Science and Engineering, Sreenidhi Institute of Science and Technology, Hyderabad" and he is quoted as saying "Many senior scientists from Indian institutions have approved the proof…".

2. Assuming this media report is accurate then it raises an issue mentioned in Note 1 which is that Eswaran 2018 contains no evidence, such as an acknowledgement, of independent expert review. As a professor, presumably with contacts in academia, and also with many senior scientists approving the proof, it seems reasonable to assume that Professor Eswaran could have found at least one suitably qualified person to review the proof.

§ 5 Conclusions About Eswaran 2018

1. The claimed proof of the Riemann Hypothesis in Eswaran 2018 is invalid because it depends on a premise that the sequence $\lambda\mathbb{N}$ is a random walk. As explained in Note 1, this premise is false because λn is not independent of n and so the sequence $\lambda\mathbb{N}$ does not have an essential defining property of a random walk.

2. It seems from the absence of any evidence in Eswaran 2018 that the claimed proof of the Riemann Hypothesis has not had an independent expert review.

Appendix 1: "R.Raghavan's Note"

Answer to Herrington's Objection
November 24, 2018

Dear Dr Garry Herrington,

I had very recently attended and studied Dr. Eswaran's Lecture.

I also read your comment of Eswaran's Paper. (Your comments are pasted at the end for your convenience)

Your main objection is that Eswaran treats the sequence λ1, λ2, λ3, … λn as a Random walk, where in actuality it is a perfectly deterministic series.

My Comment:

The point is: Eswaran knows that the λ1, λ2, λ3, … λn is a fixed sequence which is unalterable because each λn (= λ(n)) has been obtained by actually factorizing the integer n. What he means to say is that the sequence can be treated as an instance of a random walk provided for large and arbitrary n the, λ(n) satisfy the following three criteria viz. (i) equal probabilities of being +1 or -1 , (ii) the λ-sequence has no cycle and (iii) unpredictability that is there is no fixed integer k, such that for an arbitrary large n, λ(n) is predictable from only its previous k values of λ i.e. from {λ(n-1), λ(n-2), λ(n-3), … λ(n-k) }. If the three criteria is satisfied by λ(n), then it is reasonable to expect that L(N) satisfy the same bounds as s(N)- the distance travelled by a random walker in N steps - and then since s(N) is bounded by $N^{½+ε}$ one can then deduce L(N) will be similarly bounded by $N^{½+ε}$ from this result RH can be deduced as a consequence of Littlewoods Theorem which is proved in Eswaran's paper.

In his paper Eswaran goes about proving the above three criteria and thus proving RH.

End of My Comment
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I had the opportunity to attend Eswaran's Endowment Lecture see the news item:

https://www.thehansindia.com/posts/index/Young-Hans/2018-11-20/Hyderabad-basedmathematicians-endeavour-generates-interest/444502

His Lecture Slides can be read, it has some more information regarding Borwein's condition:

https://www.researchgate.net/publication/329072432_Memorial_Lecture_On_the_Final_and_Exhaustive_Proof_of_the_Riemann_Hypothesis_at_the_Institute_of_Engineers_Khairatabad_Hyderabad_India
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Regards
R. Raghavan

Appendix 2: "Note 1"

from garry.herrington@gmail.com , 1 Nov 2018

Problem with a Claimed Proof of the Riemann Hypothesis
> Eswaran 2018

Eswaran 2018 claims a proof of the Riemann Hypothesis under the title "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles". This is a bold claim given the history of attempts to prove the Riemann Hypothesis and the absence in Eswaran 2018 of any evidence of independent expert review.

§ 1 Theorems on which the claimed proof is based

Theorem 1

If $s_1, s_2, s_3, \ldots, s_n$ is a 1-D random walk with steps $s_n = \pm 1$ and a probability p that $s_n = +1$ then $E(|s_1 + s_2 + s_3 + \ldots + s_n|) \underset{1}{\leq} n^{½}$ where E(x) is the expected value of x (Stillwell 2016, p.285).

Theorem 2

The Riemann Hypothesis is equivalent to $\lim_{n\to\infty} (\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n) / n^{½ + \varepsilon} \underset{2}{=} 0$ $\forall \varepsilon > 0$ where ε is independent of n (Borwein et al 2008, Theorem 1.2, p.6).

λ_n is the "Liouville function" and is defined as -1 if Ω(n) is odd and +1 if Ω(n) is even where $\Omega(n) = m_1 + m_2 + m_3 + \ldots + m_k$ when $n = p_1^{m_1} p_2^{m_2} p_3^{m_3} \ldots p_k^{m_k}$ (Borwein et al 2008, Definition 1.1, p.6).

§ 2 A problem with the claimed proof

If p = ½ in Theorem 1 then relation(1) may be written, $-2n^{½} < s_1 + s_2 + s_3 + \ldots + s_n < 2n^{½}$ and so $\lim_{n\to\infty} (s_1 + s_2 + s_3 + \ldots + s_n) / n^{½ + \varepsilon} \underset{3}{=} 0$ $\forall \varepsilon > 0$ where ε is independent of n.

If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ was a 1-D random walk with p= ½ ,as claimed in Eswaran 2018, then relations(2,3) would imply the Riemann Hypothesis. However $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ is not a 1-D random walk because the value of λ_n is determined by n whereas in a 1-D random walk $s_1, s_2, s_3, \ldots, s_n$ the value of s_n is independent of n. Therefore the claim $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ is a 1-D random walk is false because λ_n does not satisfy an essential defining property of a 1-D random walk.

§ 3 Some History

An argument similar to the one in Eswaran 2018 which uses the Liouville function λ_n is made in Good and Churchhouse 1968 using the Möbius function μ(n) but those authors do not claim a proof of the Riemann Hypothesis.

Quote from Good and Churchhouse 1968, p. 857

"The aim of the present note is to suggest a "reason" for believing Riemann's hypothesis.

The Möbius function is defined by $\mu(n) = (-)^k$ if the positive integer n is the product of k different primes, $\mu(1) = 1$, and $\mu(n) = 0$ if n has any repeated factor.

It is known (see, for example, Titchmarsh [9, p. 315]) that a necessary and sufficient condition for the truth of the Riemann hypothesis is that $M(x) = O(x^{½ + \varepsilon})$, for all $\varepsilon > 0$, where $M(x) = \Sigma\ \mu(n)\ (n \leq x)$. The condition $M(x) = O(x^{½ + \varepsilon})$ would be true if the Möbius sequence $\{\mu(n)\}$ were a random sequence, taking the values -1, 0, and 1, with specified probabilities, those of -1 and 1 being equal.

More generally, if we first select a subsequence from $\{\mu(n)\}$ by striking out all the terms for which $\mu(n) = 0$, and if this subsequence were 'equiprobably random', i.e. if the value -1 and 1 each had (conditional) probability ½ , then the condition $M(x) = O(x^{½ + \varepsilon})$ would still be true. Of course a deterministic sequence can at best be 'pseudorandom' in the usual incompletely defined sense in which the term is used, and of course all our probability arguments are put forward in a purely heuristic spirit without any claim that they are mathematical proofs."

End quote

The reference to Titchmarsh in the quote is given in the references below.

Good and Churchhouse 1968 is discussed in Davis and Hersh 1988 pp. 363 - 369 which mentions that an earlier paper, Denjoy 1931, “uses similar but less detailed probabilistic arguments”. Denjoy 1931 is one of the references in Eswaran 2018 but Good and Churchhouse 1968 is not.

References

1. Borwein, P., Choi,S., Rooney,B., and Weirathmueller,A., "The Riemann Hypothesis: A Resource for the Aficionado and Virtuoso Alike", Springer, 2008

2. Davis,P. and Hersh, R., “The Mathematical Experience”, Pelican Books 1988

3. Denjoy, A., “L’Hypothèse de Riemann sur la distribution des zeros de ζ(s), reliée à la théorie des probabilités”, Comptes Rendus Acad. Sci. Paris 192, 656-658, (1931).

4. Eswaran, K., "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles", www.researchgate.net, May 2018

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6. Stillwell, J., “Elements of Mathematics: From Euclid to Gödel”, Princeton University Press, 2016

7. Titchmarsh,E., “The Theory of the Riemann Zeta-Function”, Clarendon Press, Oxford,1951. MR 13, 741.

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