Zeta Functions and Associated RH Hypotheses
Report on the conference in New York, 29 May — 1 June 2002
(What does the extra H stand for?)
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1. What is a zeta function?
2. Size of a zeta function and the location of zeros
3. Big series
4. Big groups
5. Big objects
6. None of these

1. What is a zeta function (or an L-function)?

We know one when we see one. It has
D: An ordinary Dirichlet series $\sum a(n)/n^s$.
E: An Euler product over primes.
F: A continuation and a functional equation.

The definition of the Selberg class (Perelli's talk) takes an expanded version:
E splits into
B: Bounds for $a(p)$ (Ramanujan Hypothesis).
E: Euler product,
and F splits into
C: Continuation for all s.
F: Functional equation.
G: Gamma factors.

Put like this, it's obvious that there's a missing property
A: Arithmetic? Algebraic? Arakelov?
Is A part of the definition, or is A implied by BCDEFG?
And do ABCDEFG only occur when
H: A Hermitian operator on a Hilbert space.

Do these eight axioms belong together? If you number them in binary as
000 A, 001 B, 010 D, 011 E, 100 H, 101 F, 110 C, 111 G,
then the left-hand bit flags arithmetic (0), analytic (1),
the middle bit flags hard (0), easy (1),
and the right-hand bit flags additive (0), multiplicative (1).
There is a vector space structure over $GF(2)$, or three commuting involutions, each flipping one of the three digits, generating a Weyl group of order 8.

More seriously, which subsets of these properties are possible?
The L-series of an oldform has DE, but not F.
The Epstein zeta function of a quadratic form has DF but not E.
The Selberg zeta function has EF but not D.

2. Size of zeta-function and location of zeros

The size of an L-function $\sum a(n)/n^s$ tells you about the distribution of $a(n)n^{-it}$ on the integers. We had a 'grand Lindelöf Hypothesis' formulated: the size of an L-function is at most the 'analytic conductor' to the power epsilon.

The position of the zeros tells you about the distribution of $a(p)p^{-it}$ on the primes. The Grand Riemann Hypothesis is that all the zeros either correspond to poles of the gamma factors, or they lie on the line $\Re s = 1/2$. There are two cases.

1. Zeros with non-zero imaginary part.
2. Zeros on the positive real axis.

There is a possibility that the Riemann Hypothesis holds in Case 1 but not in Case 2.

Partial results. For the size of an L-function there is a 'convexity bound' which corresponds to simple or even to trivial information about the distribution of $a(n)n^{-it}$ on the integers. Anything better means that you know something more about the coefficients.

For the zeros there are three sorts of results.

Zero-free regions: no zeros very close to $\Re s = 1$.

Zero-density results: at most few zeros in $\alpha \leq \Re s < 1$.

Zeros on-line results: the number of zeros on $\Re s = 1/2$ is at least some proportion δ of the total number of zeros.

Connections. Littlewood showed that the Lindelöf Hypothesis is true if there are no zeros nearby in t-aspect (except those on the critical line $\Re s = 1/2$).

Estimating the number of zeros does not come from knowing the distribution of $a(p)p^{-it}$ on primes, but from ingenious arguments using the Euler product and bounds for the L-function, or for the Rankin-Selberg product $\sum a(n)b(n)/n^s$ of two L-functions (Lapid and Gonek's talks).

Bounds for L-functions use an idea that Montgomery calls 'mean-to-max' and Iwaniec calls 'amplification'. An estimate for one sum comes from a mean square or mean $2k$-th power over a 'family' of related (but shorter?) sums. The 'family' of sums often corresponds to a family of L-functions in Sarnak's sense, perhaps with shifts in t-aspect. The $2k$-th power of sums may have to be interpreted in terms of tensor powers of representations of some arithmetic group. Bounds of the form

$$\zeta\left(\frac{1}{2} + it\right) = O\left(t^{\theta+\epsilon}\right)$$

use mean fourth or fifth powers of sums (Huxley's talk), and the group $SL(2, Z)$ turns up, just as it does in Motohashi's work on the mean fourth power of $\left|\zeta\left(\frac{1}{2} + it\right)\right|$.

Stark talked about exceptional zeros on the real axis close to 1. Lapid showed that $L(1/2)$ is non-negative for certain L-functions; this implies that the number of zeros between 1/2 and 1 is even, probably zero. Rojas used zero-free regions and zero-density theorems to give algorithms to find whether certain Diophantine equations have solutions. Iwaniec has ingenious ideas for getting to $1/\zeta(s)$ by way of amplification and families.

Sarnak and Ram Murty talked about 'subconvexity' bounds for L-functions. Michel was persuaded to give a very detailed account, including a sketch of the proof of one of his own theorems, which shows how much information has to be used about the coefficients $a(n)$.

3. Big Series

There are series more complicated than an ordinary Dirichlet series, which may contain the zeta function as a special value or as a mean value (a Fourier coefficient). Sarnak mentioned two of these.

The Epstein zeta function of a binary quadratic form gives the non-holomorphic Eisenstein series for $PSL(2, Z)$, which is connected to $\zeta(s)$ in at least three different ways. The occurrence of $\zeta(s)$ in the zero order Fourier coefficient leads to the functional equation and the non-vanishing of $\zeta(1 + it)$. Sarnak announced that he could get a zero-free region this way.

The Selberg zeta function is a Dirichlet series not over the natural numbers, but (in the simplest case of $PSL(2, Z)$), over the units of real quadratic fields. It was left well alone at this conference.

Goldfeld has Dirichlet series in several complex variables constructed from products of the values of given L-functions. The principal part at a pole where each s variable is $1/2$ should give the asymptotics for a moment problem, but the pole is usually at a vertex of the domain of analyticity.

Lagarias has extracted a function of two variables from the promising work of Schoof and Van der Geer on effectivity as a weight function on divisors. It is a Dirichlet series in the s variable, but not in the other one.

4. Big Groups.

One view of Mathematics is that it is the study of symmetry. The whole of elementary number theory is based on Euclid's algorithm, which writes a rational number P/Q in its lowest terms form a/c, and also produces a rational b/d with

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1,$$

so the group $SL(2, Z)$ underlies all number theory.

Conrey and his team noticed that since

$$|\zeta(s)|^{2k} = (\zeta(s))^k (\zeta(\bar{s}))^k,$$

the group of permutations of $\{1, \ldots, 2k\}$ which preserves $\{1, \ldots, k\}$ and $\{k+1, \ldots, 2k\}$ as sets must be important. Keating and Snaith modelled the zeta function by the average behaviour of the characteristic polynomial of an N by N unitary matrix. They found wonderful numerical agreement (Conrey's talk). This seems to be a sort of universality property. You can probably approximate the zeros of an N by N unitary matrix by the (scaled) zeros of the zeta function in a unit interval $t \le \Im\rho \le t + 1$ at height something like e^N, and the set of such t should have a positive density (of course the density tends to zero as you demand better and better approximations).

If $SL(2, Z)$ tells you about elementary number theory, do bigger arithmetic groups tell you more? The group $\mathrm{Gal}(\bar{Q}/Q)$ must know everything. Katz described Deligne's approach to the Ramanujan Hypothesis, which Weil calls the "Riemann Hypothesis for finite fields", using $SL(N, Q_\ell)$ and $\mathrm{Gal}(\bar{F}_p/F_p)$ with p and ℓ distinct primes. The dreaded monodromy group was also mentioned.

Zirnbauer explained how averages over Haar measure on (say) the unitary group U_N can often be calculated using the "dual pairs" constuction and a character on a group of $2N$ by $2N$ matrices. Julia gave a rapid survey of interesting groups which appear in the real world, or at any rate in field theories in Physics.

5. Big objects.

The idea is to find some complicated but natural construction in which the zeta function, or the Weil explicit formula, will appear, preferably in a context which throws light on the Riemann Hypothesis. Both addition and multiplication should play non-trivial roles. Perhaps we are like the seven blind men who set out to describe an elephant. One feels the trunk, one the tusks, one the ears,..., and their descriptions of the same object are quite different.

Haran's approach by analogy with quantum groups looks rather like the Cohen-Connes space of adèles modulo multiplication by non-zero rational numbers. Deninger wants a topological space which will give the Weil formula as an alternating sum; the real difficulties may begin with the Archimedean valuations. Fesenko is developing a theory of two-dimensional local fields with enough measurable sets for an integration theory. Deitmar dreams of a compactification of Spec Z which allows a hyperbolic flow with orbits mod rationals of length $\log p$.

Lin Weng constructs moduli spaces of lattices of higher rank, which should lead to zeta functions with functional equations, but not necessarily Euler products.

Another type of big object is a space of functions which contains or is generated by certain functions constructed arithmetically. If the space is big enough to contain some useful function, then the Riemann Hypothesis must hold. Balazard and Burnol spoke about constructions of this type. There is usually an interesting operator on the space of functions.

Something which is not quite a function space is the class of probability distributions obtained from Ising models of lattice interactions. It looks as though it should be a convex set. Newman has been investigating one-dimensional lattices with long-range interactions. The molecules have spin up or down independently and randomly. Inspired by Polya, Lee and Yang showed that the Laplace transform, which corresponds to the zeta function, has pure imaginary zeros. The proof does not require a Hermitian operator. It can be done using the linear fractional action of $GL(2, R)$. Sets you thinking, eh?

6. None of these.

Cappell's talk on lattice points and toric varieties produced an interesting operator in an algebra of differential operators, which can be used to count the lattice points in a polyhedron. There is no obvious connection with the zeta function — so far.

Leboeuf talked about how a physicist would expect the distribution of zeros to affect the distribution of primes. His notation was hard to follow. He seemed to be suggesting that $(p_{n+1} - p_n)/\log p_n$ should have a distribution which was not Poisson, but showed fluctuations.