ON THE FALSITY OF RIEMANN'S HYPOTHESIS

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ABSTRACT

In 1859, Riemann conjectured that all the nontrivial zeros $\rho_n = \sigma_n + i\gamma_n$ of the zeta-function $\zeta(s)$ should lie on the critical line $\text{Re} s = 1/2$. This hypothesis is the most celebrated open problem in mathematics which resists the attacks from many people. Several calculations indicate that this hypothesis is true for the first millions of zeros and therefore most people believe it to be true. However, I disprove this hypothesis as follows:

**Theorem 1.** Let $\zeta^*(s)$ be the function generated from $\zeta(s)$ by replacing each $\rho_n$ by $\rho_n^* = \frac{1}{2} + i\gamma_n$, then we have $\zeta^*(s) > \zeta(s)$, for each $s > 1$.

Clearly, Riemann's hypothesis means that $\zeta^*(s) \equiv \zeta(s)$ which is false due to Theorem 1. In fact, the number $N(t)$ of nontrivial zeros in $\{0 \leq \sigma \leq 1, 0 < u \leq t\}$ satisfies

$$N(t) = \frac{t}{2\pi} \left( \log \frac{t}{2\pi} - 1 \right) + 1 + \frac{t}{4\pi} \log(1 + \frac{1}{4t^2}) - \frac{1}{4\pi} \tan^{-1} 2t$$

$$+ \frac{t}{2\pi} \int_0^\infty \frac{x - [x] - \frac{1}{2}}{(x + \frac{1}{4})^2 + \frac{1}{4}} dx + \frac{1}{\pi} \arg \zeta(\frac{1}{2} + it).$$

This implies the symmetrization $\zeta^*(s) = \zeta^*(1-s)$ and $\zeta^*(s)$ has no trivial zeros ($\zeta^*(-2n) > 0 = \zeta(-2n)$) which yields Theorem 1. Furthermore, we prove

**Theorem 2.** Let $N^*(t)$ be the number of $\rho_n$ with $\sigma_n \neq \frac{1}{2}$, then we have $N^*(t) = \Omega(t)$. 

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1. The key idea and proof

The Riemann zeta-function $\zeta(s)$ has its origin in the identity expressed by the Dirichlet series (see Titchmarsh [8, p.1])

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for} \quad \text{Re} s > 1,
\]

and it has the trivial zeros at $-2n$, that is $\zeta(-2n) = 0$, for $n = 1, 2, \ldots$ [8, p.19].

Let $\zeta^*(s)$ be the function defined in Theorem 1 and call it the zeta-like function. Then the Riemann hypothesis simply means that the zeta and zeta-like function are the same, that is $\zeta(s) = \zeta^*(s)$ identically. Therefore, to disprove this, it suffices to show $\zeta^*(s) \neq \zeta(s)$ at some points. It is nontrivial to show that the function $\zeta^*(s)$ has no trivial zeros, as will be seen from the following key result:

**Theorem 3.** The function $\zeta^*(s)$ has no trivial zeros and satisfies $\zeta^*(s) > \zeta(s)$, for each $s > 1$, and in fact,

\[
\log \zeta^*(s) = S^*(s) - \log s(s - 1) > \log \zeta(s), \quad \zeta^*(s) = \zeta^*(1 - s) \quad \text{and} \quad \zeta^*(-2n) > 1,
\]

where $S^*(s) = \int_0^\infty \log(1 + \frac{s(s-1)}{4+t^2})dS(t)$, $S(t) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + it)$ and the function $S^*(s)$ is analytic everywhere except at $\rho^*_n = \frac{1}{2} + i\gamma_n$, for $n = 1, 2, \ldots$.

We now sketch the proof of Theorem 3. First of all, we express the two functions as follows (see Lemma 1 and Theorem 6):

\[
\zeta(s) = \frac{\pi^{\frac{s}{2}} \xi(s)}{(s-1)\Gamma(\frac{s}{2} + 1)} \quad \text{and} \quad \zeta^*(s) = \frac{\pi^{\frac{s}{2}} \xi^*(s)}{(s-1)\Gamma(\frac{s}{2} + 1)},
\]

\[
\log \xi(s) = \sum_{n=1}^{\infty} \log(1 - \frac{s}{\rho_n})(1 - \frac{s}{\rho_n}) - \log 2
\]

and

\[
\log \xi^*(s) = \sum_{n=1}^{\infty} \log(1 + \frac{s(s-1)}{\frac{1}{4} + \gamma_n^2}) - \log 2 = \int_0^\infty \log(1 + \frac{s(s-1)}{\frac{1}{4} + t^2})dN(t) - \log 2
\]

\[
= \frac{s}{2} \log \frac{s}{2\pi} - \frac{s}{2} - \frac{1}{2} \log s + \frac{1}{2}(1 + \log \pi - \log 2) + J^*(s) + S^*(s),
\]
where $\rho_n = \sigma_n + i\gamma_n$ and $\rho^*_n = \frac{1}{2} + i\gamma_n$ are the nontrivial zeros of $\zeta(s)$ and $\zeta^*(s)$ respectively, and

$$J^*(s) = \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + \frac{s}{2}} dx - \frac{1}{2} (1 - \log 2).$$

Next, we need the expression of the gamma function (see Lemma 2),

$$\log \Gamma\left(\frac{s}{2} + 1\right) = \frac{s}{2} \log \frac{s}{2} - \frac{s}{2} + \frac{1}{2} \log s + \frac{1}{2} \log \pi + \frac{s+1}{2} \log(1 + \frac{2}{s}) - 1 + \Gamma^*(s)$$

where

$$\Gamma^*(s) = \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + \frac{s}{2} + 1} dx,$$

and

$$J^*(s) - \Gamma^*(s) = \frac{s+1}{2} \log(1 + \frac{2}{s}) - \frac{1}{2} (3 - \log 2),$$

due to Lemma 5. Combining (1.3), (1.4), (1.5) and (1.6), we obtain

$$\log \zeta^*(s) = \frac{s}{2} \log \pi + \log \zeta^*(s) - \log(s-1) - \log \Gamma\left(\frac{s}{2} + 1\right) = S^*(s) - \log s(s-1).$$

This proves the left side of (1.2). To prove the right side of (1.2), we need the following result:

**Theorem 4.** If $\rho$ and $1 - \bar{\rho}$ are a pair of nontrivial zeros not lying on the critical line, then the values $\zeta(s), s > 1$, increase and become maximum when $\rho$ and $1 - \bar{\rho}$ move towards the critical line. Consequently, we have either $\zeta^*(s) \equiv \zeta(s)$ or $\zeta^*(s) > \zeta(s)$, for each $s > 1$.

Clearly, we have the symmetrization $S^*(s) = S^*(1-s)$, so that $\zeta^*(s) = \zeta^*(1-s)$. Hence we get

$$\zeta^*(-2n) = \zeta^*(1+2n) > \zeta(1+2n) > 1 > 0 = \zeta(-2n), \text{ for } n = 1, 2, \ldots$$

This shows that the function $\zeta^*(s)$ has no trivial zeros. The reason why this can happen is because the cancellation of the same logarithmic poles between $J^*(s)$ and $\Gamma^*(s)$, see Lemma 5. This yields $\zeta^*(s) > \zeta(s), s > 1$, which proves Theorem 3. As a consequence, we obtain
Theorem 5. The function \( \log \zeta^*(s) \) cannot be represented as a Dirichlet series converging uniformly in a half-plane.

Turning to Theorem 2, we show that if \( N^*(t) = o(t) \), then (see [8, p.64])

\[
\log \zeta^*(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} = \log \zeta(s), \quad \text{for } \Re s > 1,
\]

a Dirichlet series contradicting Theorem 5. Hence \( N^*(t) = \Omega(t) \) which proves Theorem 2. This gives at once the following

Corollary 1. If we assume that \( N^*(t) = o(t) \), then Riemann's hypothesis would be true.

2. The expression of \( \zeta(s) \) and \( \zeta^*(s) \)

According to Riemann's theorem (see [8, p.22]), the function \( \zeta(s) \) satisfies the functional equation

\[
\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),
\]

and further its value \( \zeta(s) \) is real when \( s \) is. It follows from (2.1) that if \( \rho \) is a nontrivial zero of \( \zeta(s) \), so are \( \bar{\rho} \), \( 1 - \rho \) and \( 1 - \bar{\rho} \).

We now denote all the nontrivial zeros of \( \zeta(s) \) and \( \zeta^*(s) \) respectively by \( \rho_n = \sigma_n + i\gamma_n, \, 0 < \sigma_n < 1 \), and

\[
\rho_n^* = 1/2 + i\gamma_n, \quad \text{where } 0 < \gamma_1 \leq \gamma_2 \leq ....
\]

Then by Hadamard's factorization theorem, we have the expression (see [8, p.30] or Edwards [2, p.25])

\[
(2.3) \quad \zeta(s) = \frac{\pi^{s/2} \xi(s)}{(s-1)\Gamma(s/2 + 1)} \quad \text{and} \quad \zeta^*(s) = \frac{\pi^{s/2} \xi^*(s)}{(s-1)\Gamma(s/2 + 1)},
\]

where \( \Gamma(s/2 + 1) = \Pi(\frac{s}{2}) \) in [2], and the products

\[
(2.4) \quad \xi(s) = \frac{1}{2} \Pi_{\rho_n}(1 - \frac{s}{\rho_n}) \quad \text{and} \quad \xi^*(s) = \frac{1}{2} \Pi_{\rho_n^*}(1 - \frac{s}{\rho_n^*}).
\]

Using (2.2), we can write the second product in (2.4) as

\[
\xi^*(s) = \frac{1}{2} \prod_{n=1}^{\infty} \frac{(1 - \frac{s}{\rho_n})(1 - \frac{s}{\rho_n^*})}{\frac{1}{4} + \gamma_n^2} = \frac{1}{2} \prod_{n=1}^{\infty} \frac{s(s-1)}{1 + \frac{1}{4} + \gamma_n^2}.
\]

This together with (2.3) gives the following representation.
Lemma 1. The functions $\zeta(s)$ and $\zeta^*(s)$ can be expressed as

$$
\zeta(s) = \frac{\pi^\frac{s}{2} \xi(s)}{(s-1)\Gamma\left(\frac{s}{2} + 1\right)}, \quad \text{where} \quad \xi(s) = \frac{1}{2} \prod_{n=1}^\infty \left(1 - \frac{s}{\rho_n} \right) \left(1 - \frac{s}{1 - \rho_n} \right),
$$

$$
\zeta^*(s) = \frac{\pi^\frac{s}{2} \xi^*(s)}{(s-1)\Gamma\left(\frac{s}{2} + 1\right)}, \quad \xi^*(s) = \frac{1}{2} \prod_{n=1}^\infty \left(1 + \frac{s(s-1)}{\frac{1}{4} + \gamma^2_n} \right).
$$

3. The gamma function

From Lemma 1, we see that the proof of the assertion (1.2) relies on the estimation of the growth of the following gamma function (see [7, p.150]):

$$
\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + z} dx.
$$

This gives

$$
\log \Gamma\left(\frac{s}{2} + 1\right) = \left(\frac{s}{2} + \frac{1}{2}\right) \log\left(\frac{s}{2} + 1\right) - \left(\frac{s}{2} + 1\right) + \frac{1}{2} \log 2\pi + \Gamma^*(s)
$$

$$
= \left(\frac{s}{2} + \frac{1}{2}\right) \left\{ \log\left(\frac{s}{2} + 1\right) - \log\left(1 + \frac{2}{s}\right) \right\} - \frac{s}{2} - 1 + \frac{1}{2} \log 2\pi + \Gamma^*(s).
$$

Lemma 2. The function $\log \Gamma(s/2 + 1)$ can be expressed as

$$
\log \Gamma\left(\frac{s}{2} + 1\right) = \frac{s}{2} \log \frac{s}{2} - \frac{s}{2} + \frac{1}{2} \log s + \frac{1}{2} \log \pi + \frac{s + 1}{2} \log(1 + \frac{2}{s}) - 1 + \Gamma^*(s),
$$

where $\Gamma^*(s) = \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + s/2 + 1} dx.$

4. The proof of Theorem 4

To prove the assertion, we let $s > 1$ and $\gamma > 1$ be fixed and let $\rho = \sigma + i\gamma$, where $\sigma$ varies on $(0, 1)$. Consider the factor

$$
f(\sigma) = \left(1 - \frac{s}{\rho}\right) \left(1 - \frac{s}{1 - \rho}\right) \left(1 - \frac{s}{1 - \rho}\right),
$$

$$
= \left[1 + \frac{s(s - 2\sigma)}{\sigma^2 + \gamma^2} \right] \left[1 + \frac{s(s - 2(1 - \sigma))}{(1 - \sigma)^2 + \gamma^2} \right]
$$

$$
= 1 + \frac{s(s - 1)[2\sigma(1 - \sigma) + 2\gamma^2 + s(s - 1)]}{(\sigma^2 + \gamma^2)((1 - \sigma)^2 + \gamma^2)} = 1 + \frac{N(\sigma)}{D(\sigma)}, \text{say}.
$$
Clearly, we have that $N(\sigma) > 0, D(\sigma) > 0, N'(\sigma) = 2s(s-1)(1-2\sigma)$ and $D'(\sigma) = 2(2\sigma-1)(\gamma^2 - \sigma(1-\sigma))$. This gives

$$f'(\sigma) = \frac{2(1-2\sigma)(s(s-1)D(\sigma) + (\gamma^2 - \sigma(1-\sigma))N(\sigma))}{D(\sigma)^2}$$

and hence

$$\max_{0<\sigma<1} f(\sigma) = f\left(\frac{1}{2}\right).$$

We now replace $\rho$ by $\rho_n = \sigma + i\gamma_n$ and denote the associated factor by $f_n(s, \sigma)$. Then the product

$$\xi(s, \sigma) = \Pi_{n=1}^\infty f_n(s, \sigma), \text{ for } 0 < \sigma < 1 \text{ and } s > 1,$$

reaches its maximum at $\sigma = 1/2$ due to (4.1). This shows that the values $\xi(s)$ as well as $\zeta(s)$ increase and become maximum whenever a pair $\rho$ and $1-\rho$ move towards the critical line, and the theorem is proved.

5. The number of nontrivial zeros

Let $N(t)$ denote the number of zeros of $\zeta(s)$ in the rectangle

$$\{\sigma + iu: 0 \leq \sigma \leq 1 \text{ and } 0 < u \leq t\}.$$  

According to a theorem of Backlund (see [2, p.109, 119, 128]), the function $N(t)$ can be represented as follows:

$$N(t) = \frac{1}{\pi} \{\text{Im}[\log \Pi\left(\frac{it}{2} + \frac{1}{4}\right) - \log\left(\frac{it}{2} + \frac{1}{4}\right)] - \frac{t}{2\pi} \log \pi\} + 1 + S(t)$$

$$= \frac{1}{\pi} \text{Im}\left[\left(\frac{it}{2} - \frac{1}{4}\right) \log\left(\frac{it}{2} + \frac{1}{4}\right) - \left(\frac{it}{2} + \frac{1}{4}\right) + \frac{1}{2}\log 2\pi - I(t)\right] - \frac{t}{2\pi} \log \pi + 1 + S(t)$$

$$= \frac{t}{2\pi} \text{Re}\log\left(\frac{it}{2} + \frac{1}{4}\right) - \frac{1}{4\pi} \text{Im}\log\left(\frac{it}{2} + \frac{1}{4}\right) - \frac{t}{2\pi} - \frac{t}{2\pi} \log \pi + 1 + J(t) + S(t)$$

$$= \frac{t}{2\pi} \log \frac{\pi}{2} + \frac{1}{4\pi} \log(1 + \frac{1}{4t^2}) - \frac{1}{4\pi} \tan^{-1}2t - \frac{t}{2\pi} - \frac{t}{2\pi} \log \pi + 1 + J(t) + S(t),$$

where $\Pi(s) = \Gamma(s+1)$, $S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right)$, $J(t) = -\frac{1}{\pi} \text{Im} I(t)$ and

$$I(t) = \int_0^\infty \frac{x - [x] - \frac{1}{2}}{it + \frac{1}{4} + x} dx = \int_0^\infty \frac{(x - [x] - \frac{1}{2})(x + \frac{1}{4} - \frac{it}{2})}{(x + \frac{1}{4})^2 + \frac{t^2}{4}} dx.$$

Combining (5.1) with (5.2), we have the representation
Lemma 3. The function $N(t)$ can be expressed as
\[ N(t) = \frac{t}{2\pi} \left( \log \frac{t}{2\pi} - 1 \right) + 1 + \frac{t}{4\pi} \log(1 + \frac{1}{4t^2}) - \frac{1}{4\pi} \tan^{-1} 2t + J(t) + S(t), \]
where $J(t) = \frac{t}{2\pi} \int_0^\infty \frac{x-[x]-\frac{1}{2}}{(x+\frac{1}{2})^2 + \frac{1}{4}} dx$.

6. The expression of $\log \xi^*(s)$ and $\log \zeta^*(s)$

With the help of Lemmas 1 and 3, we are now able to state and prove the following main result:

Theorem 6. The functions $\log \xi^*(s)$ and $\log \zeta^*(s)$ can be expressed as
\[ \log \xi^*(s) = \frac{s}{2} \log \frac{s}{2\pi} - \frac{s}{2} - \frac{1}{2} \log s + \frac{1}{2}(1 + \log \pi - \log 2) \]
\[ + J^*(s) + S^*(s), \quad \text{for } \text{Re}s > \frac{1}{2}, \]
and
\[ \log \zeta^*(s) = S^*(s) - \log s(s - 1), \quad \text{for } \text{Re}s > \frac{1}{2}, \ s \neq 1, \]
where
\[ J^*(s) + \frac{1}{2}(1 - \log 2) = -\int_0^\infty \frac{2tJ(t)}{(s - \frac{1}{2})^2 + t^2} dt = \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + \frac{s}{2}} dx = \Gamma^*(s) + \frac{s + 1}{2} \log(1 + \frac{2}{s}) - 1 \]
and
\[ S^*(s) = \int_0^\infty \log(1 + \frac{s(s - 1)}{4 + t^2}) dS(t), \quad S(t) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + it). \]

We note that the left side of (6.1) is obviously analytic in the half-plane $\text{Re}s > \frac{1}{2}$, so is the right side (see Section 9). Therefore, to prove (6.1), it suffices to show the real case that $s > \frac{1}{2}$, since the result follows from the analytic continuation. To do this, we first apply Lemmas 1 and 3 to give the representation
\[ \log \xi^*(s) = \sum_{n=1}^\infty \log(1 + \frac{s(s - 1)}{\frac{1}{4} + \gamma_n^2}) - \log 2 = \int_0^\infty \log(1 + \frac{s(s - 1)}{\frac{1}{4} + t^2}) dN(t) - \log 2 \]
\[ = \int_0^\infty \log(1 + \frac{s(s - 1)}{\frac{1}{4} + t^2}) \left[ d\left[ \frac{t}{2\pi} \log \frac{t}{2\pi} - 1 \right] + d\left[ \frac{t}{4\pi} \log(1 + \frac{1}{4t^2}) \right] \right] \]
\[ - d\left( \frac{1}{4\pi} \tan^{-1} 2t \right) + dJ(t) + dS(t) \right] - \log 2 \]
\[ = I_1(s) + I_2(s) + I_3(s) + J^*(s) + S^*(s) - \log 2, \quad \text{say.} \]
We now calculate each of the above five integrals, and we begin with the first one. By observing the equality
\[ d\left[ \frac{t}{2\pi} (\log \frac{t}{2\pi} - 1) \right] = \frac{1}{2\pi} \log \frac{t}{2\pi} \, dt \]
and integrating by parts, we get
\[
I_1(s) = \frac{1}{2\pi} \int_0^\infty \log(1 + \frac{y}{4 + t^2})\log \frac{t}{2\pi} \, dt \\
= \frac{1}{2\pi} \int_0^\infty \log(1 + \frac{y}{4 + t^2}) \, dt + \frac{y}{\pi} \int_0^\infty \frac{t^2 \log \frac{t}{2\pi}}{(\frac{1}{4} + t^2)(\frac{1}{4} + y + t^2)} \, dt \\
= I_{11}(y) + I_{12}(y), \text{ say, where } y = s(s - 1). 
\]

To calculate the integrals, we use the residue theorem to evaluate the integrals at the two poles \( \frac{i}{2} \) and \( \sqrt{\frac{1}{4} + y} \) \( i = (s - \frac{1}{2})i \) of the integrand \( f(t) \) of \( I_{12}(y) \) in the upper half-disk \( D_R(|z| < R, \text{Im} z > 0) \) deleting \( |z| < \varepsilon \). Letting \( R \to \infty \) and \( \varepsilon \to 0 \), we find that
\[
\int_0^\infty [f(-t) + f(t)] \, dt = 2\pi i [\text{Res}(f, \frac{i}{2}) + \text{Res}(f, (s - \frac{1}{2})i)],
\]
where \( \log(-t) = i\pi + \log t \) in \( f(-t) \).

This yields that (by taking the real part)
\[
I_{12}(y) = \frac{y}{2\pi} \Re\left\{ 2\pi i \left[ \frac{(-\frac{1}{4}) \log \frac{i}{4\pi}}{i(\frac{1}{4} + y - \frac{1}{4})} - \frac{1}{4} \log((s - \frac{1}{2})\frac{i}{2\pi}) \right] \right\} \\
= \frac{1}{4} \log 4\pi + \frac{1}{2}(s - \frac{1}{2}) \log \frac{s - \frac{1}{2}}{2\pi}
\]
and (by taking the imaginary part)
\[
I_{11}(y) = -\frac{y}{\pi} \int_0^\pi \frac{t^2}{(\frac{1}{4} + t^2)(\frac{1}{4} + y + t^2)} \, dt = \frac{1}{4} - \frac{1}{2}(s - \frac{1}{2}) - \frac{1}{2}(s - \frac{1}{2}) + \frac{1}{4} \log 4\pi e.
\]

Combining the above three equations, we obtain
(6.4) \[ I_1(s) = \frac{1}{2}(s - \frac{1}{2}) \log \frac{s - \frac{1}{2}}{2\pi} - \frac{1}{2}(s - \frac{1}{2}) + \frac{1}{4} \log 4\pi e. \]

Next, we calculate the second integral. Integrating by parts, we get
\[
I_2(s) = \int_0^\infty \log(1 + \frac{y}{4 + t^2}) d\left[ \frac{t}{4\pi} \log(1 + \frac{1}{4t^2}) \right] \\
= \frac{1}{2\pi} \int_0^\infty \frac{yt^2 \log(1 + \frac{1}{4t^2})}{(\frac{1}{4} + t^2)(\frac{1}{4} + y + t^2)} \, dt.
\]
To calculate the integral, we consider the function

\[ f(z) = \frac{yt^2 \log(i - \frac{1}{2z})}{(\frac{1}{4} + z^2)(\frac{1}{4} + y + z^2)}. \]

By the same argument as before, we find that

\[
\int_0^\infty [f(-z) + f(z)]dz = 2\pi i \{Res(f, \frac{i}{2}) + Res(f, (s - \frac{1}{2})i)\}
\]
\[
= -\frac{\pi}{2} (\log 2 + \frac{i\pi}{2}) + \pi(s - \frac{1}{2})[\log(1 + \frac{1}{2(s - \frac{1}{2})}) + \frac{i\pi}{2}].
\]

Since

\[ f(-z) + f(z) = \frac{yz^2 \log(1 + \frac{i}{2z})(i - \frac{1}{2z})}{(\frac{1}{4} + z^2)(\frac{1}{4} + y + z^2)} \]
\[ = \frac{yz^2[\log(1 + \frac{1}{4z^2}) + i\pi]}{(\frac{1}{4} + z^2)(\frac{1}{4} + y + z^2)}, \]

it follows from the real part that

(6.5) \[ I_2(s) = -\frac{1}{4} \log 2 + \frac{1}{2} (s - \frac{1}{2}) \log(1 + \frac{1}{2(s - \frac{1}{2})}). \]

Turning to \( I_3(s) \), we find from the same argument that

\[ I_3(s) = -\int_0^\infty \log(1 + \frac{y}{4 + t^2})d(\frac{1}{4\pi} \tan^{-1} 2t) = -\frac{1}{8\pi} \int_0^\infty \frac{\log(\frac{1}{4} + y + t^2)}{t^2 + \frac{1}{4}} dt, \]

where \( \frac{1}{8\pi} \int_0^\infty \log(\frac{1}{4} + t^2) \frac{1}{t^2 + \frac{1}{4}} dt = 0 \), as calculated in \( D_R \) by setting

\[ f(z) = \frac{\log(z + i/2)}{z^2 + \frac{1}{4}}. \]

By considering the function

\[ f_s(z) = \frac{\log(z + (s - \frac{1}{2})i)}{z^2 + \frac{1}{4}}, \]

and using the residue theorem, we see that

\[ I_3(s) - \frac{i\pi}{8} = -\frac{1}{4} \log(\frac{i}{2} + (s - \frac{1}{2})) \]

and hence

(6.6) \[ I_3(s) = -\frac{1}{4} \log(\frac{1}{2} + s - \frac{1}{2}) = -\frac{1}{4} \log s. \]

7. The relation between \( J^*(s) \) and \( \Gamma^*(s) \)

In this section, we shall express the relation between \( J^*(s) \) and \( \Gamma^*(s) \) defined in (6.3) and Lemma 2 respectively. For this, we first state and prove the following
Lemma 4. The functions $J^*(s)$ and $J(t)$ satisfy

(7.1) \[ J^*(s) = \int_0^\infty \log(1 + \frac{y}{\frac{1}{4} + t^2}) dJ(t) = -\frac{1}{2}(1 - \log 2) - \int_0^\infty \frac{2tJ(t)}{\frac{1}{4} + y + t^2} dt \]

and

(7.2) \[ J(t) = -\frac{t}{6\pi(\frac{1}{4} + t^2)} + O\left(\frac{1}{t}\right), \text{ as } t \to \infty, \]

where $y = s(s - 1)$ and $J(t) = \frac{t}{2\pi} \int_0^\infty \frac{x-[x]-\frac{1}{2}}{(x+\frac{1}{4})^2+\frac{1}{4}} dx$.

Proof. Integrating by parts, we see that

\[ J^*(s) = \int_0^\infty \frac{2ytJ(t)}{(\frac{1}{4} + t^2)(\frac{1}{4} + y + t^2)} dt = \int_0^\infty \frac{2tJ(t)}{\frac{1}{4} + t^2} dt - \int_0^\infty \frac{2tJ(t)}{\frac{1}{4} + y + t^2} dt. \]

To prove (7.1), it suffices to show that

(7.3) \[ \int_0^\infty \frac{2tJ(t)}{\frac{1}{4} + t^2} dt = -\frac{1}{2}(1 - \log 2). \]

By Fubini’s theorem, we find that

\[ \int_0^\infty \frac{2tJ(t)}{\frac{1}{4} + t^2} dt = \frac{1}{\pi} \int_0^{\infty} \frac{x-[x]-\frac{1}{2}}{x+\frac{1}{2}} \int_0^{\infty} \frac{t^2}{(\frac{1}{4} + t^2)[(x+\frac{1}{4})^2+\frac{1}{4}]} dt dx \]

\[ = \int_0^{\infty} \frac{x-[x]-\frac{1}{2}}{x+\frac{1}{2}} dx = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{x-n-\frac{1}{2}}{x+\frac{1}{2}} dx \]

\[ = \sum_{n=0}^{\infty} \left\{1 - (n+1)\log\left(\frac{2n+3}{2n+1}\right) \right\} = -\frac{1}{2}(1 - \log 2), \]

where the last equality is known, see Bromwich [1, p.526] (follows easily from Stirling’s formula). This proves (7.3).

It remains to show (7.2). For this, we first observe from the Fourier series (see [8, p.15]) that

\[ x-[x]-\frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}. \]
It follows from the term-by-term integration and the integration by parts that

\[ J_R(t) = \frac{t}{2\pi} \int_0^R \frac{x - [x] - \frac{1}{2}}{(x + \frac{1}{4})^2 + \frac{t^2}{4}} \, dx = -\frac{t}{2\pi} \sum_{n=1}^\infty \frac{1}{n\pi} \int_0^R \frac{\sin 2n\pi x}{(x + \frac{1}{4})^2 + \frac{t^2}{4}} \, dx \]

\[ = \frac{t}{4\pi^3} \sum_{n=1}^\infty \frac{1}{n^2} \int_0^R \frac{1}{(x + \frac{1}{4})^2 + \frac{t^2}{4}} \, d\cos 2n\pi x \]

\[ = \frac{t}{4\pi^3} \sum_{n=1}^\infty \frac{1}{n^2} \left( \frac{\cos 2n\pi R}{(R + \frac{1}{4})^2 + \frac{t^2}{4}} - \frac{1}{\frac{16}{15} + \frac{t^2}{4}} + \int_0^R \frac{2(x + \frac{1}{4})\cos 2n\pi x}{[(x + \frac{1}{4})^2 + \frac{t^2}{4}]^2} \, dx \right) \]

\[ = \frac{t}{4\pi^3} \sum_{n=1}^\infty \frac{1}{n^2} \left( \frac{\cos 2n\pi R}{(R + \frac{1}{4})^2 + \frac{t^2}{4}} - \frac{4}{4 + t^2} + \frac{(R + \frac{1}{4})\sin 2n\pi R}{n\pi[(R + \frac{1}{4})^2 + \frac{t^2}{4}]^2} \right. 

\[ - \frac{1}{n\pi} \left. \int_0^R \frac{\sin 2n\pi x}{[(x + \frac{1}{4})^2 + \frac{t^2}{4}]^2} - \frac{2(x + \frac{1}{4})^2\sin 2n\pi x}{((x + \frac{1}{4})^2 + \frac{t^2}{4})^3} \, dx \right) \}

Letting \( R \to \infty \) and using \( \zeta(2) = \pi^2/6 \), the equation (7.4) yields

\[ J(t) = -\frac{t\zeta(2)}{\pi^3(\frac{1}{4} + t^2)} - \frac{t}{4\pi^4} \sum_{n=1}^\infty \frac{1}{n^3} \int_0^\infty \frac{\sin 2n\pi x}{[(x + \frac{1}{4})^2 + \frac{t^2}{4}]^2} - \frac{2(x + \frac{1}{4})^2\sin 2n\pi x}{((x + \frac{1}{4})^2 + \frac{t^2}{4})^3} \, dx \]

\[ = -\frac{t}{6\pi(\frac{1}{4} + t^2)} + O\left(\frac{1}{t}\right), \text{ as } t \to \infty. \]

This proves (7.2).

With the help of Lemma 4, we can now express the relation between \( J^*(s) \) and \( \Gamma^*(s) \) as follows:

Lemma 5. The two functions \( J^*(s) \) and \( \Gamma^*(s) \) satisfy

\[ J^*(s) + \frac{1}{2}(1 - \log 2) = -\int_0^\infty \frac{2tJ(t)}{\frac{1}{4} + y + t^2} \, dt \]

\[ = \int_0^\infty \frac{x - x + \frac{1}{2}}{x + \frac{y}{2}} \, dx = \Gamma^*(s - 2), \quad y = s(s - 1), \]

and the difference

\[ J^*(s) - \Gamma^*(s) = \int_0^\infty \frac{x - x + \frac{1}{2}}{x + \frac{y}{2} + \frac{1}{2}} \, dx \]

\[ = \frac{s + 1}{2} \log(1 + \frac{2}{s}) - \frac{1}{2}(3 - \log 2), \]
which is analytic everywhere except at the two points \( s = 0 \) and \(-2\).

**Proof.** We first prove (7.5). Recalling the definition of \( J(t) \) in (7.2) and using Fubini’s theorem, we see that

\[
\int_0^\infty \frac{2tJ(t)}{\frac{1}{4} + y + t^2} dt = \frac{1}{\pi} \int_0^\infty \frac{t^2}{(\frac{1}{4} + y + t^2)[(x + \frac{1}{4})^2 + t^2]} dt dx
\]

This proves (7.5). The particular case \( s = 1 \) has been used in the proof of (7.3).

It remains to prove (7.6). As in (7.3), we find that

\[
\int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + \frac{s}{2}} dx = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{n - x + \frac{1}{2}}{x + \frac{s}{2}} dx
\]

(7.7)

\[
= \sum_{n=0}^{\infty} \left\{ \left( n + \frac{s + 1}{2} \right) \log \frac{2(n + 1) + s}{2n + s} - 1 \right\}
\]

and

(7.8)

\[
\int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + \frac{s}{2} + 1} dx = \sum_{n=0}^{\infty} \left\{ \left( n + \frac{s + 3}{2} \right) \log \frac{2(n + 2) + s}{2(n + 1) + s} - 1 \right\}.
\]

It then follows from (7.7) and (7.8) that

\[
J^*(s) - \Gamma^*(s) = \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + \frac{s}{2}} dx - \frac{1}{2}(1 - \log 2) - \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x + \frac{s}{2} + 1} dt
\]

\[
= \frac{s + 1}{2} \log \frac{2 + s}{s} - \frac{1}{2}(3 - \log 2).
\]

This proves (7.6). Clearly, this function \( J^*(s) - \Gamma^*(s) \) is analytic everywhere except at \( s = 0 \) and \(-2\). The lemma is proved.
We remark that the cancellation of the same logarithmic poles between the two functions \( J^*(s) \) and \( \Gamma^*(s) \) is the key to cancel out the trivial zeros of the zeta-like function \( \zeta^*(s) \).

8. The expression of \( S^*(s) \)

In this section, we shall reduce the expression of \( S^*(s) \). Recalling from the definition in (6.3), we have that

\[
S^*(s) = \int_0^\infty \log\left(1 + \frac{y}{1 + t^2}\right) dS(t), \quad y = s(s - 1), \quad S(t) = \frac{1}{\pi} \arg\zeta\left(\frac{1}{2} + it\right).
\]

To calculate this integral, we first observe that \( S(0) = -1 \). In fact, we have \( N(t) = 0 \), for \( 0 < s < 14 \) (see [8, 15.2]), so by Lemma 3 we get \( 0 = 1 + S(0) \) or \( S(0) = -1 \). This together with a well known theorem of Backlund (see [8, Theorems 9.4 and 9.9]) gives the following

**Lemma 6.** The functions \( S(t) \) and \( S_1(t) \) satisfy

\[
S(0) = -1, \quad S_1(0) = 0 \quad \text{and} \quad S(t) = O(\log t) = S_1(t), \quad \text{where} \quad S_1(t) = \int_0^t S(x) dx.
\]

Applying Lemma 6 and integrating by parts, we find that

\[
S^*(s) = \int_0^\infty \log\left(1 + \frac{y}{1 + t^2}\right) dS(t)
\]

\[
= -\log(1 + 4y)S(0) + 2y \int_0^\infty \frac{tS(t)}{(1 + t^2)(\frac{1}{4} + y + t^2)} dt
\]

\[
= \log(1 + 4y) + 2 \int_0^\infty \frac{S_1(t)y}{(1 + t^2)(\frac{1}{4} + y + t^2)} \left\{ \frac{2t^2}{\frac{1}{4} + t^2} + \frac{2t^2}{\frac{1}{4} + y + t^2} - 1 \right\} dt
\]

\[
= \log(1 + 4y) + S_1^*(y), \quad \text{say.}
\]

Applying Lemma 6 again, and using the dominated convergence theorem, we get

\[
\lim_{y \to \infty} S_1^*(y) = 2 \int_0^\infty \frac{S_1(t)(t^2 - \frac{1}{4})}{(\frac{1}{4} + t^2)^2} dt = H < \infty.
\]

Substituting (8.2) into (8.1) and writing

\[
\frac{y}{\frac{1}{4} + y + t^2} = 1 - \frac{1}{\frac{1}{4} + y + t^2},
\]

we obtain the following
Lemma 7. The function $S^*(s)$ satisfies

$$S^*(s) = \log(1 + 4y) + H + S^*_{11}(y) + S^*_{12}(y), \quad y = s(s - 1),$$

where $H = 2 \int_0^\infty \frac{S_1(t)(t^2 - \frac{1}{4})}{(\frac{1}{4} + t^2)^2} \, dt$, $S^*_{11}(y) = \int_0^\infty \frac{2S_1(t)}{4+y+t^2} \, dt$ and $S^*_{12}(y) = - \int_0^\infty \frac{4t^2S_1(t)}{(\frac{1}{4} + y + t^2)^2} \, dt$.

We note that the above lemma will be used much later in Section 12.

9. The proof of Theorem 6

With the help of the above lemmas, we are now able to complete the proof of Theorem 6. Recalling from (6.3)-(6.6), we have that

$$\log \xi^*(s) = \frac{1}{2}(s - \frac{1}{2}) \log((s - \frac{1}{2})/2\pi) - \frac{1}{2}(s - \frac{1}{2}) + \frac{1}{4} \log 4\pi e - \frac{1}{4} \log 2$$

$$+ \frac{1}{2}(s - \frac{1}{2}) \log(1 + \frac{1}{2s - 1}) - \frac{1}{4} \log s + J^*(s) + S^*(s) - \log 2$$

$$= \frac{1}{2}(s - \frac{1}{2})[\log(\frac{s}{2\pi}) + \log(1 - \frac{1}{2s}) + \log(1 + \frac{1}{2s - 1})] - \frac{s}{2} - \frac{1}{4} \log s + \frac{1}{2}$$

$$- \frac{3}{4} \log 2 + \frac{1}{4} \log \pi + J^*(s) + S^*(s)$$

$$= \frac{s}{2} \log \frac{s}{2\pi} - \frac{s}{2} - \frac{1}{2} \log s + \frac{1}{2}(1 + \log \pi - \log 2) + J^*(s) + S^*(s).$$

Clearly, we have

$$(s - \frac{1}{2})^2 + x^2 = (\sigma + it - 1/2)^2 + x^2 = (\sigma - 1/2)^2 - t^2 + x^2 + 2it(\sigma - 1/2) \neq 0$$

for all $\text{Re}s = \sigma > 1/2$ and hence both functions $J^*(s)$ and $S^*(s)$ are analytic on the half-plane $\sigma > 1/2$. Thus (9.1) holds for all $\sigma > 1/2$. This proves (6.1).
It remains to prove (6.2). In view of (9.1), Lemmas 1, 2 and 5, we conclude that

\[
\log \zeta^*(s) = \frac{s}{2} \log \pi + \log \zeta^*(s) - \log(s - 1) - \log \Gamma\left(\frac{s}{2} + 1\right)
\]

\[
= \frac{s}{2} \log \pi + \frac{s}{2} \log \frac{s}{2\pi} - \frac{1}{2} \log s + \frac{1}{2} (1 + \log \pi - \log 2) + J^*(s) + S^*(s)
\]

\[- \log(s - 1) - \left[ \frac{s}{2} \log \frac{s}{2} - \frac{s}{2} + \frac{1}{2} \log s + \frac{1}{2} \log \pi \right]
\]

\[+ \frac{s + 1}{2} \log\left(1 + \frac{2}{s}\right) - 1 + \Gamma^*(s)\]

\[= [J^*(s) - \Gamma^*(s) - \frac{s + 1}{2} \log(1 + \frac{2}{s}) + \frac{1}{2} (3 - \log 2)] + S^*(s) - \log s(s - 1)
\]

\[= S^*(s) - \log s(s - 1), \text{ where } S^*(s) = \int_0^\infty \log(1 + \frac{s(s - 1)}{\frac{1}{4} + t^2})dS(t).
\]

This proves (6.2). The relation between $J^*(s)$ and $\Gamma^*(s)$ was proved in Lemma 5, and the proof is complete.

10. The proof of Theorem 3

With the help of Theorem 6, we are now able to prove Theorem 3. According to (9.2), we have the first equality in (1.2). This gives

\[
(10.1) \quad \zeta^*(s) = \frac{e^{S^*(s)}}{s(s - 1)}
\]

and

\[
(10.2) \quad S^*(1 - s) = \int_0^\infty \log(1 + \frac{(1 - s)(1 - s - 1)}{\frac{1}{4} + t^2})dS(t) = S^*(s).
\]

Combining (10.1) with (10.2), we obtain

\[
(10.3) \quad \zeta^*(1 - s) = \frac{e^{S^*(1-s)}}{(1 - s)(1 - s)} = \zeta^*(s), \text{ for each } s > 1.
\]

This proves the second equality in (1.2).

It remains to prove the two inequalities in (1.2). According to Theorem 4, it suffices to show that

\[
(10.4) \quad \zeta^*(s) \neq \zeta(s), \text{ for some } s.
\]
Now, by (10.3) we see that
\[
\zeta^*(-2n) = \zeta^*(1 - (1 + 2n)) = \zeta^*(1 + 2n) = \frac{e^{S^*(1+2n)}}{2n(1+2n)} > 0, \quad \text{for } n > 0,
\]
and hence the function \( \zeta^*(s) \) has no trivial zeros. On the other hand, it is well known that
\[
\zeta(-2n) = 0 \quad \text{(see [8, p.19])}
\]
so that
\[
\zeta(-2n) = 0 < \zeta^*(-2n), \quad \text{for } n = 1, 2, \ldots.
\]
This proves (10.4). It then follows from Theorem 4 that
\[(10.5) \quad \zeta(-2n) = 0 < \zeta(1 + 2n) < \zeta^*(1 + 2n) = \zeta^*(-2n), \quad \text{for } n = 1, 2, \ldots,
\]
and
\[
\zeta^*(s) > \zeta(s), \quad \text{for each } s > 1.
\]
This proves all the assertions in (1.2).

Finally, we show that the function \( S^*(s) \) is analytic everywhere except at the nontrivial zeros \( \rho_n^* = \frac{1}{2} + i\gamma_n, \quad n = 1, 2, \ldots \). To do this, we let \( I_n \) be the open interval between \( \rho_n^* \) and \( \rho_{n+1}^* \). Then it suffices to prove that the function \( S^*(s) \) can be analytically acrossed \( I_n \).

For this, we first observe that if \( s = \frac{1}{2} + iy \), then the value \( s(s - 1) = -(\frac{1}{4} + y^2) \) is real, so is the value
\[
S^*(s) = \int_0^\infty \log(1 + \frac{s(s - 1)}{\frac{1}{4} + t^2})dS(t) = \frac{1}{2} \int_0^\infty \log(\frac{t^2 - y^2}{\frac{1}{4} + t^2})^2dS(t).
\]
Therefore, it suffices to show that \( S^*(s) \) is continuous on \( I_n \). To do this, we decompose the function \( S(t) \) into the sum of the singular and absolute parts, namely \( S(t) = N(t) + A(t) \), see (5.1), where \( A'(t) \) is continuous. This together with (6.3) gives
\[(10.6) \quad S^*(s) = \sum_{n=1}^\infty \log(1 + \frac{s(s - 1)}{\frac{1}{4} + \gamma_n^2}) + A^*(s)
\]
\[= \log \xi^*(s) + \log 2 + A^*(s), \quad A^*(s) = \int_0^\infty \log(1 + \frac{s(s - 1)}{\frac{1}{4} + t^2})A'(t)dt.
\]
Recalling from (2.4), we see immediately that the function $\log \xi^*(s)$ is continuous on $I_n$. Hence it suffices to show the continuity of $A^*(s)$. For this, we write

$$A^*(\frac{1}{2} + iy) = \frac{1}{2} \left\{ \int_0^{\gamma_n} + \int_{\gamma_n}^{\gamma_{n+1}} + \int_{\gamma_{n+1}}^{\infty} \right\} \log \left( \frac{t^2 - y^2}{4 + t^2} \right)^2 A'(t) dt$$

$$= A_1(y) + A_2(y) + A_3(y), \text{ say, where } 0 < \gamma_n < y < \gamma_{n+1}.$$}

Clearly, both of $A_1(y)$ and $A_3(y)$ are continuous. To prove $A_2(y)$, we write

$$A_2(y) = \int_{\gamma_n}^{y} \log(y - t) A'(t) dt + \int_{y}^{\gamma_{n+1}} \log(t - y) A'(t) dt$$

$$+ \int_{\gamma_n}^{\gamma_{n+1}} \log \left( \frac{t + y}{4 + t^2} \right) A'(t) dt = A_{21}(y) + A_{22}(y) + A_{23}(y), \text{ say.}$$

The last one $A_{23}(y)$ is obviously continuous. As for the first two, we need only verify one of them, say $A_{21}(y)$. By the mean-value theorem, we see that

$$A_{21}(y) = A'(\bar{y}) \int_{\gamma_n}^{y} \log(y - t) dt$$

$$= A'(\bar{y}) \{(y - \gamma_n) \log(y - \gamma_n) - (y - \gamma_n)\}, \text{ where } \gamma_n < \bar{y} < y.$$}

This shows that $A_{21}(y)$ is continuous, so is $A_{22}(y)$. Combining (10.6), (10.7) and (10.8), we conclude that the function $S^*(s)$ is continuous on $I_n$. It then follows from the reflection principle that $S^*(s)$ can be analytically acrossed $I_n$. This completes the proof of Theorem 3.

We remark from (10.5) that the function $\zeta^*(s)$ has no trivial zeros. This is the key difference between $\zeta^*(s)$ and $\zeta(s)$. The reason why the trivial zeros disappear is because they have been cancelled out by the logarithmic poles of the function $J^*(s)$, see Lemma 5.

11. The proof of Theorem 5

We now prove Theorem 5 via the following well known theorem of Hamburger and Siegel (see [8, p.31] or the one-page proof of ours [3]):
Theorem 7. Let \( f(s) \) be an entire meromorphic function of finite order having at most finitely many poles, and let \( f(s) \) be expanded as a Dirichlet series

\[
(11.1) \quad f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},
\]

which converges absolutely for \( \Re s \geq 1 + \varepsilon \), for each \( \varepsilon > 0 \). If \( f(s) \) satisfies the functional equation

\[
(11.2) \quad \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) f(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) f(1-s),
\]

then \( f(s) = a_1 \zeta(s) \).

With the help of Theorem 7, we can now easily prove Theorem 5. In view of the function \( \xi^*(s) \) in Lemma 1, we see that

\[
(11.3) \quad \xi^*(s) = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 + \frac{s(s-1)}{1 + \gamma_n^2}\right) = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 + \frac{(1-s)(1-s-1)}{1 + \gamma_n^2}\right) = \xi^*(1-s).
\]

Since

\[
\Gamma\left(\frac{s}{2} + 1\right) = \frac{s}{2} \Gamma\left(\frac{s}{2}\right),
\]

it follows from the expression of \( \zeta^*(s) \) in Lemma 1 that

\[
(11.4) \quad \xi^*(s) = s(s-1) \Gamma\left(\frac{s}{2}\right) \xi^*(s) \pi^{-\frac{s}{2}}.
\]

Replacing \( s \) by \( 1-s \) in (11.4), we get

\[
(11.5) \quad \xi^*(1-s) = (1-s)(-s) \Gamma\left(\frac{1-s}{2}\right) \xi^*(1-s) \pi^{-(1-s)/2}.
\]

Combining (11.3), (11.4) and (11.5), we obtain

\[
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \xi^*(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \xi^*(1-s).
\]

This shows that the function \( \zeta^*(s) \) satisfies the functional equation (11.2). Therefore, if \( \zeta^*(s) \) could be expanded as a Dirichlet series in (11.1), then \( \zeta^*(s) = a_1 \zeta(s) \) due to Theorem
7. This however is impossible since the values $\zeta^*(-2n) > 0 = \zeta(-2n)$, see (10.5). The theorem is proved.

12. The proof of Theorem 2

To prove the assertion, we suppose on the contrary that $N^*(t) = o(t)$. We shall prove that the function $\log \zeta^*(s) \equiv \log \zeta(s)$ or $\zeta^*(s) \equiv \zeta(s)$ which contradicts Theorem 1. For this, we write

\begin{equation}
\log \zeta^*(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{for } \sigma \geq \sigma_0 > 1, \text{ where } \sigma = \text{Res.}
\end{equation}

Then we have for each $T > 0$ and $\sigma \geq \sigma_0$,

\[
\frac{1}{2T} \int_{-T}^{T} \log \zeta^*(s)m^s dt = \frac{m^\sigma}{2T} \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \int_{-T}^{T} \left( \frac{m}{n} \right)^{it} dt
\]

\[
= a_m + m^\sigma \sum_{n \neq m} \frac{a_n \sin(T \log m/n)}{n^\sigma T \log m/n}
\]

and hence

\begin{equation}
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log \zeta^*(s)m^s dt = a_m, \quad \text{for } m = 1, 2, \ldots.
\end{equation}

We shall show that $a_m = \Lambda_1(m) = 1$, for all $m = p^q$, and $a_m = 0$, otherwise. This yields $\log \zeta^*(s) \equiv \log \zeta(s)$ (see [8, p.64]).

In view of (9.2) and Lemma 7, we see that

\begin{equation}
\frac{1}{2T} \int_{-T}^{T} \log \zeta^*(s)m^s dt
\end{equation}

\[
= \frac{1}{2T} \int_{-T}^{T} \left\{ (2 \log 2 + H) + \log(1 + \frac{1}{4s(s-1)}) + S_{11}(y) + S_{12}(y) \right\} m^s dt,
\]

where $H = \int_0^\infty \frac{S_1(x)(x^2-\frac{1}{4})}{(1+x^2)^2} dx$, $S_{11}(y) = \int_0^\infty \frac{2S_1(x)}{y+x^2} dx$, $S_{12}(y) = -\int_0^\infty \frac{4x^2S_1(x)}{(1+y+x^2)^2} dx$.

We now calculate each of the above four integrals on the right side of (12.3), and we begin with the first one. Clearly, we have

\begin{equation}
\frac{1}{2T} \int_{-T}^{T} (2 \log 2 + H)m^s dt = 2 \log 2 + H, \quad \text{for } m = 1,
\end{equation}

\[
= \frac{(2 \log 2 + H)m^\sigma \sin(T \log m)}{T \log m}, \quad \text{for } m > 1.
\]
We note that the number $H_R = -2 \log 2$ which is not needed in our proof.

Next, we estimate the second integral, namely

$$\frac{1}{2T} \int_{-T}^{T} \log(1 + \frac{1}{4s(s-1)})m^s dt$$

(12.5)

$$= \frac{m^\sigma}{2T} \int_{-T}^{T} \{ \log(1 - \frac{1}{4[t^{2} - \sigma(\sigma - 1) - it(2\sigma - 1)]})m^{it} \} dt$$

$$= \frac{m^\sigma}{2T} \int_{-T}^{T} f(t) dt = \frac{m^\sigma}{2T} \{ \int_{-T}^{-5\sigma} + \int_{-5\sigma}^{5\sigma} + \int_{5\sigma}^{T} \} f(t) dt,$$

where $s(s-1) = (\sigma + it)(\sigma - 1 + it) = \sigma(\sigma - 1) - t^2 + it(2\sigma - 1)$, and the function $f(t)$ is analytic everywhere except at $i(2\sigma - 1)/2$, $i\sigma$ and $i(\sigma - 1)$. Observing that

$$|m^{it}| = 1, \text{ for } -T < t < T \text{ and } |\log(1 - \varepsilon)| < 2|\varepsilon|, \text{ for } |\varepsilon| < \frac{1}{2},$$

we get from (12.5) for $\sigma > 1$, the upper bound

$$\left| \frac{1}{2T} \int_{-T}^{T} \log(1 + \frac{1}{4s(s-1)})m^s dt \right|$$

(12.6)

$$< \frac{m^\sigma}{2T} \{ C_\sigma + 2 \int_{-5\sigma}^{5\sigma} \frac{1}{t^2 - t(2\sigma - 1) - \sigma(\sigma - 1)} | dt \}$$

$$< \frac{m^\sigma(C_\sigma + 2)}{2T}, \text{ for } m = 1, 2, ..., \text{ where } \int_{-5\sigma}^{5\sigma} | f(t) | dt < C_\sigma.$$

Thirdly, by Fubini's theorem we have that

$$\frac{1}{2T} \int_{-T}^{T} S_{11}^*(y)m^s dt = \frac{1}{2T} \int_{-T}^{T} S_{1}(x)m^s dt$$

(12.7)

$$= \frac{m^\sigma}{T} \int_{-T}^{T} S_{1}(x) \left\{ \frac{m^{it}}{(t + x - ia)(t - x - ia)} \right\} dt dx$$

$$= \frac{m^\sigma}{T} \left( \int_{-T}^{T} S_{1}(x) \int_{-T}^{T} g(t) dt dx \right. = G_1(T) + G_2(T), \text{ say},$$

where

$$\frac{1}{4} + y + x^2 = (s - \frac{1}{2})^2 + x^2 = (\sigma - \frac{1}{2} + it)^2 - i^2 x^2$$

$$= [t + x - i(\sigma - \frac{1}{2})][t - x - i(\sigma - \frac{1}{2})]$$

$$= -(t + x - ia)(t - x - ia), \sigma = \frac{1}{2} > \frac{1}{2}.$$
We now estimate each of the above two integrals and we begin with the first one. By (12.8) and Lemma 6, we see that

\begin{equation}
| G_1(T) | < \frac{C_1 m^\sigma (\log 2T)^2}{T} \int_0^T \frac{1}{(x^2 + a^2)^{\frac{1}{2}}} dx < \frac{2C_1 m^\sigma (\log 2T)^3}{T}.
\end{equation}

Turning to the second, using the first inequality in (12.8), we find that

\begin{equation}
| G_2(T) | < \frac{C_2 m^\sigma}{T} \int_2^\infty \frac{\log x}{x} \int_0^T \frac{1}{x - T} dt dx < 2C_2 m^\sigma \int_2^\infty \frac{\log x}{x^2} dx = \frac{C_2 m^\sigma (\log 2T + 1)}{T}.
\end{equation}

Substituting (12.9) and (12.10) into (12.7), we obtain

\begin{equation}
\frac{1}{2T} \int_{-T}^T S^*_1(y)m^\sigma dt < \frac{C_{11} m^\sigma (\log T)^3}{T}.
\end{equation}

Finally, we estimate the last integral in (12.3). As in (12.7) and (12.8), we have that

\begin{equation}
\frac{1}{2T} \int_{-T}^T S^*_2(y)m^\sigma dt = -\frac{1}{2T} \int_0^\infty \int_0^T \frac{4x^2 S_1(x)}{\left( \frac{1}{4} + y + x^2 \right)^2} m^\sigma dt dx
\end{equation}

\begin{align*}
&= -\frac{2m^\sigma}{T} \int_0^\infty x^2 S_1(x) \int_T^T \frac{m^\sigma}{(t + x - i\alpha)^2(t - x - i\alpha)^2} dt dx \\
&= -\frac{2m^\sigma}{T} \left( \int_0^{T-U} + \int_{T-U}^{T+U} + \int_{T+U}^{\infty} \right) x^2 S_1(x) \int_T^h(t) dt dx \\
&= H_1(T) + H_2(T) + H_3(T),
\end{align*}

where \( U = T^\frac{3}{4} \), and

\begin{equation}
\int_{-T}^T | h(t) | dt = 2 \int_0^T \frac{1}{\left( (t + x)^2 + a^2 \right)((t - x)^2 + a^2)} dt
\end{equation}

\begin{align*}
&< \frac{2}{x^2 + a^2} \int_0^T \frac{1}{(t - x)^2 + a^2} dt \\
&= \frac{2}{x^2 + a^2} \left( \tan^{-1} \frac{T-x}{a} + \tan^{-1} \frac{x}{a} \right) < \frac{2\pi}{a(x^2 + a^2)}.\end{align*}
Instead of the first one, we now estimate the second which is much easier. By (12.13) and Lemma 6, we get

\begin{equation}
| H_2(T) | < \frac{C_3 m^\sigma \log T}{T} \int_{T-U}^{T+U} dx = \frac{2C_3 m^\sigma \log T}{T^{1/4}}.
\end{equation}

Next, we estimate \( H_3(T) \). Using the first inequality in (12.13), we find that

\begin{equation}
| H_3(T) | < \frac{C_3 m^\sigma}{T} \int_{T+U}^{\infty} \log x \int_{0}^{T} \frac{1}{(x-T)^2} dt \, dx
\end{equation}

\[ = C_3 m^\sigma \left( \frac{\log(T + U)}{U} + \frac{1}{T} \log \frac{T + U}{U} \right) < \frac{2C_3 m^\sigma \log T}{T^{3/4}}.
\]

Thirdly, we estimate \( H_1(T) \). To do this, we first note that the function \( h(t) \) is analytic everywhere except the two poles \( p_+ = x + ia \) and \( p_- = -x + ia \). Since \( |x| = T - U \), these two poles lie in the upper half-disk \( D_T \). Using the residue theorem, we find that

\begin{equation}
\int_{-T}^{T} h(t) \, dt = -\int_{0}^{\pi} h(T e^{i\theta}) e^{i\theta} \, d\theta + 2\pi i \{ \text{Res}(h, p_+) + \text{Res}(h, p_-) \}
\end{equation}

\[ = -iT \int_{0}^{\pi} h(T e^{i\theta}) e^{i\theta} \, d\theta + \frac{\pi im^{-a}}{2x^2} \left\{ i \log m (m^{ix} + m^{-ix}) - \frac{m^{ix} - m^{-ix}}{x} \right\}.
\]

It follows from (12.12), (12.16) and Lemma 6 that

\begin{equation}
H_1(T) = \frac{2m^\sigma}{T} \int_{0}^{T-U} \left\{ x^2 O(\log x) \left[ iT \int_{0}^{\pi} h(T e^{i\theta}) e^{i\theta} \, d\theta + \frac{\pi im^{-a}(m^{ix} - m^{-ix})}{2x^2} \right] + \frac{\pi m^{-a}}{2} \log m S_1(x)(m^{ix} + m^{-ix}) \right\} dx
\end{equation}

\[ = O\{ m^\sigma T^3 \log T \mid h(T e^{i\theta}) \mid \} + O\left\{ \frac{m^{1/2} \log T}{T} \int_{0}^{T} \left| \frac{\sin(x \log m)}{x} \right| \, dx \right\}
\]

\[ + \frac{\pi m^{1/2}}{T} \log m \int_{0}^{T-U} S_1(x)(m^{ix} + m^{-ix}) \, dx.
\]

Since \( x \leq T - U \) and \( a < U/10 \), say, we get

\[ | Te^{i\theta} - x - ia |^2 > |T - x - a |^2 \geq (U - a)^2 > \frac{T^{3/2}}{2}.
\]

It is geometrically clear that for all \( 0 \leq \theta \leq \pi/2 \),

\[ | Te^{i\theta} + x - ia |^2 > | Te^{i\pi/2} + x - ia |^2 > (T - a)^2 > \frac{T^2}{2}.
\]
The above two inequalities give

\begin{equation}
(12.18) \quad |Te^{i\theta} + x - ia|^2 |Te^{i\theta} - x - ia|^2 > \frac{T^{7/2}}{4}.
\end{equation}

By the symmetrization, we see that (12.18) holds for all $\pi/2 \leq \theta \leq \pi$ and hence it holds through the range $0 \leq \theta \leq \pi$. This in turn implies that

\begin{equation}
(12.19) \quad |h(Te^{i\theta})| < \frac{4}{T^{7/2}} \frac{|m^{iTe^{i\theta}}|}{T^{7/2}}, \quad \text{for all } 0 \leq \theta \leq \pi.
\end{equation}

Substituting (12.19) into (12.17), we obtain

\begin{equation}
(12.20) \quad H_1(T) = O\left(\frac{m^\sigma \log T}{\sqrt{T}}\right) + O\left(\frac{m^\sigma (\log T)^2}{T}\right)
+ \frac{\pi m^{\frac{1}{2}} \log m}{T} \int_0^{T-U} S_1(x)(m^{ix} + m^{-ix})dx.
\end{equation}

It remains to estimate the last integral. For this, we need the following theorem of Littlewood, see [8, (9.9.4)], cf. [6, (1.9)]:

**Lemma 8.** The function $S_1(x)$ can be expressed as

\[
S_1(x) = \frac{1}{\pi} \int_{\frac{1}{2}}^\infty \log |\zeta(\sigma + ix)| \, d\sigma - \frac{1}{\pi} \int_{\frac{1}{2}}^\infty \log |\zeta(\sigma)| \, d\sigma.
\]

Since

\[
\log |\zeta(\sigma + ix)| = \log \left| 1 + \frac{1}{2\sigma + ix} + \frac{1}{3\sigma + ix} + \cdots \right| < \frac{1}{2\sigma} + \frac{1}{3\sigma} + \cdots
\]

\[
< \frac{1}{2\sigma} + \int_2^\infty \frac{1}{t^\sigma} dt = \frac{\sigma + 1}{(\sigma - 1)2^\sigma}, \quad \text{for all } \sigma \geq 2 \text{ and } x \geq 0,
\]

it follows that

\begin{equation}
(12.21) \quad S_1(x) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\log_2 T} \log |\zeta(\sigma + ix)| \, d\sigma - \frac{1}{\pi} \int_{\frac{1}{2}}^\infty \log |\zeta(\sigma)| \, d\sigma + O\left(\frac{1}{T}\right).
\end{equation}
By (12.21) and Fubini’s theorem, we find that
\[
\int_0^{T-U} S_1(x)(m^{ix} + m^{-ix})dx = \frac{1}{\pi} \int_{\frac{1}{2}}^{\log_2 T} \int_0^T \log |\zeta(\sigma + ix)| \cdot (m^{ix} + m^{-ix})dx d\sigma + O\left(\frac{1}{\log m}\right), \quad m > 1,
\]
where
\[
\int_0^{T-U} \cos(x \log m)dx = \frac{\sin((T-U) \log m)}{\log m} < \frac{1}{\log m}, \quad \text{for } m > 1.
\]

We now consider the function
\[
f(z) = \log \zeta(\sigma - iz)m^{iz}, \quad \text{where } \frac{1}{2} \leq \sigma \leq \log_2 T,
\]
and separate the range of \(\sigma\) into two cases: either \(\frac{1}{2} \leq \sigma \leq 1\) or \(1 < \sigma \leq \log_2 T\). We begin with the second case which is much easier. In this case, by observing that
\[
\sigma - i(x + iy) = \sigma + y - ix \quad \text{and} \quad \sigma - iz = 1 \quad \text{if} \quad z = -i(\sigma - 1), \quad \text{where} \quad \sigma > 1,
\]
we see that the function \(\zeta(\sigma - iz)\) has no zeros and poles on the upper half-disk \(D_R\), where \(R = T - U\). Then by Cauchy’s theorem, we get
\[
(12.23) \quad \int_{-R}^{R} f(z)dz = -\int_{0}^{\pi} f(Re^{i\theta})Re^{i\theta}i\theta d\theta.
\]

In view of (1.1) and the functional equation (2.1), we have (by the reflection principle)
\[
\zeta(\sigma + ix) = \overline{\zeta(\sigma - ix)}, \quad \text{for all} \quad 0 \leq x < \infty.
\]

It follows from (12.23) that
\[
(12.24) \quad \int_{0}^{R} \{ \log |\zeta(\sigma + ix)| \cdot (m^{ix} + m^{-ix}) - i \arg \zeta(\sigma + ix)(m^{ix} - m^{-ix}) \}dx
= -iR \int_{0}^{\pi} \log \zeta(\sigma - iRe^{i\theta})m^{iRe^{i\theta}}e^{i\theta}d\theta, \quad \text{where} \quad \sigma > 1, \quad R = T - U.
\]

Turning to the first case: \(\frac{1}{2} \leq \sigma \leq 1\), the pole and zeros of the function \(\zeta(\sigma - iz)\) occur at \(z_p = i(1 - \sigma)\) and \(z_n = -\gamma_n + i(\sigma_n - \sigma), \quad \sigma_n > \sigma, \quad \text{which lie in} \quad D_R\). We then cut \(D_R\)
from \( z_p \) to 0 and from \( z_n \) to \(-\gamma_n\) along the imaginary axis and line segments respectively. It then follows from the residue theorem that

\[
(12.25) \quad \int_{-R}^{R} f(z)dz - 2\pi i \int_{0}^{\infty} m^{iz}dz + 2\pi i \sum_{|\gamma_n|<R} \int_{-\gamma_n}^{\gamma_n} m^{iz}dz = -\int_{0}^{\pi} f(Re^{i\theta})Re^{i\theta}i\theta d\theta,
\]

where the integrations are on the positive direction so that the values of \( \log \zeta(\sigma - iz) \) on the left and the right side of the cut differ by \(-2\pi i\) for the cut from \( i(1-\sigma) \) to 0, and by \(2\pi i\) for the cut from \( z_n \) to \(-\gamma_n\), cf. [6, Lemma 16]. The second and third integrals are obviously equal to

\[
(12.26) \quad \int_{0}^{i(1-\sigma)} m^{iz}dz = i(1-\sigma), \text{ for } m = 1,
\]

\[
= \frac{m^{-(1-\sigma)} - 1}{i\log m}, \text{ for } m = 2, 3, ..., \]

and

\[
(12.27) \quad \int_{-\gamma_n}^{z_n} m^{iz}dz = i(\sigma_n - \sigma), \text{ for } m = 1,
\]

\[
= \frac{m^{-i\gamma_n}(m^{-(\sigma_n-\sigma)} - 1)}{i\log m} \text{ for } m = 2, 3, ....
\]

Substituting (12.26) and (12.27) into (12.25), we obtain

\[
(12.28) \quad \int_{0}^{1} \log |\zeta(\sigma + ix)| \left| (m^{ix} + m^{-ix}) - i \arg \zeta(\sigma + ix)(m^{ix} - m^{-ix}) \right| dx
\]

\[
= \frac{2\pi (m^{-(1-\sigma)} - 1)}{\log m} - \frac{2\pi m^{-i\gamma_n}(m^{-(\sigma_n-\sigma)} - 1)}{\log m}
\]

\[
- iR \int_{0}^{\pi} \log \zeta(\sigma - iRe^{i\theta})Re^{i\theta}e^{i\theta}d\theta,
\]

where \(1/2 \leq \sigma \leq 1\), in the first term and \(1/2 \leq \sigma \leq \sigma_n\) in the second term, \(m > 1\), and the case \(m = 1\) is not needed.

Integrating both sides of (12.28) and (12.24) over the two ranges \(1/2 \leq \sigma \leq 1\) and
1 \leq \sigma \leq \log_2 T \text{ respectively, we find that}

\[ \int_{\frac{1}{2}}^{\log_2 T} \int_0^R \{ \log | \zeta(\sigma + ix) | (m^{ix} + m^{-ix}) \\
- i \arg \zeta(\sigma + ix)(m^{ix} - m^{-ix}) \} \, dx \, d\sigma \]

\begin{equation}
(12.29) \quad = \frac{2\pi(1 - m^{-\frac{1}{2}})}{(\log m)^2} - \frac{2\pi}{\log m} \sum_{m^{-i\gamma_n} \{ 1 - m^{-\left(\sigma_n - \frac{1}{2}\right)} \} / \log m} - \left( \sigma_n - \frac{1}{2} \right) \}
- i R \int_0^\pi \ell(T)m^{iR\theta} e^{i\theta} \, d\theta,
\end{equation}

where \( m > 1, \sigma_n > \frac{1}{2} \), for all \( n \) and

\[ \ell(T) = \int_{\frac{1}{2}}^{\log_2 T} \log \zeta(\sigma + iR^{i\theta}) d\sigma = O(\log T)^2, \quad R = T - U < T. \]

Since we supposed \( N^*(t) = o(t) \), it follows that

\begin{equation}
(12.30) \quad \sum_{\gamma_n < R} m^{-i\gamma_n} \left\{ \frac{1 - m^{-\left(\sigma_n - \frac{1}{2}\right)}}{\log m} - \left( \sigma_n - \frac{1}{2} \right) \right\} = o(R) = o(T), \quad \frac{1}{2} < \sigma_n < 1.
\end{equation}

Now, replacing \( f(z) \) and \( D_R \) by \( f^-(z) = \log \zeta(\sigma - iz)m^{-iz} \) and \( D_R \), the conjugate domain respectively, then we have the same two cases as before. The pole \( z_p = -i(\sigma - 1) \) now occurs in the second case while the nontrivial zeros \( z_n \) appear in both cases, where

\[ \sigma - iz_n = \sigma_n + i\gamma_n \text{ or } z_n = -\gamma_n + i(\sigma_n - \sigma). \]

We then cut \( D_R \) from \( z_n \) to \( p_n = -\gamma_n - i(R^2 - \gamma_n^2)^{\frac{1}{2}} \), and we have in the first case that cf. (12.25),

\begin{equation}
(12.31) \quad - \int_{-R}^{R} f^-((z) \, dz + 2\pi i \sum_{\gamma_n < R} \int_{p_n}^{z_n} m^{-iz} \, dz = \int_{0}^{\pi} f^-((Re^{i\theta})Re^{i\theta} \, i d\theta,
\end{equation}

where the values of \( \log \zeta(\sigma - iz) \) on the left and the right side of the cut differ by \( 2\pi i \), and the integrations are on the positive direction. By adding the term

\begin{equation}
(12.32) \quad -2\pi i \int_{0}^{z_p} m^{-iz} \, dz = -\frac{2\pi(1 - m^{-(\sigma - 1)})}{\log m}, \quad m > 1, \sigma > 1,
\end{equation}

to the left side of (12.31), we get the second case cf. (12.23). The sum in (12.31) is obviously equal to
where \( \rho_n = \sigma_n + i\gamma_n \), \( \sigma_n \geq \frac{1}{2} \) and \( \frac{1}{2} \leq \sigma \leq \log_2 T \). Since \( \rho_n \) and \( \bar{\rho}_n \) are both zeros of \( \zeta(s) \), it follows from Landau's theorem [4, Theorem 1] that

\[
\sum_{|\gamma_n| < R} m^{\rho_n} = -\frac{R}{\pi} \Lambda(m) + O_m(\log R).
\]

By (12.31)-(12.34) and the same argument as before, we obtain

\[
\int_{\frac{1}{2}}^{\log_2 T} \int_0^R \left\{ \log |\zeta(\sigma + ix)| (m^{ix} + m^{-ix}) + i \arg \zeta(\sigma + ix)(m^{ix} - m^{-ix}) \right\} dx d\sigma
\]

\[
= \frac{2\pi (\log_2 T - \frac{1}{2})}{\log m} \sum_{|\gamma_n| < R} m^{-ip_n} + \frac{2\pi}{\log m} L(T, m) \left[ -\frac{R}{\pi} \Lambda(m) + O_m(\log R) \right]
\]

\[
- \frac{2\pi}{\log m} \left( \log_2 T - 1 - \frac{1 - m^{-\log_2 T + 1}}{\log m} \right) - iR \int_{\frac{1}{2}}^{\pi} \ell(T)m^{-iRe^{i\theta}} e^{i\theta} d\theta,
\]

where \( L(T, m) = \int_{\frac{1}{2}}^{\log_2 T} (-m^{-\sigma}) d\sigma = \frac{m^{-\log_2 T} - m^{-\frac{1}{2}}}{\log m} \).

The sum of (12.29) and (12.35) together with (12.30) yields the desired integral

\[
\int_{\frac{1}{2}}^{\log_2 T} \int_0^R \log |\zeta(\sigma + ix)| (m^{ix} + m^{-ix}) dx d\sigma
\]

\[
= -\frac{R}{\log m} \Lambda(m)L(T, m) + o(T) + O_m(\frac{\log T}{\log m}) - \frac{iR}{2} \left\{ \int_{0}^{\pi} \ell(T)m^{iR\sin \theta} e^{i\theta} d\theta + \int_{0}^{\pi} \ell(T)m^{-iR\sin \theta} e^{i\theta} d\theta \right\}
\]

\[
+ \pi(\log_2 T - \frac{1}{2}) \sum_{|\gamma_n| < R} m^{-ip_n}.
\]

The last two integrals satisfy

\[
| \int_{0}^{\pi} \ell(T)m^{iR\sin \theta} e^{i\theta} d\theta | < C_1(\log T)^2 \int_{0}^{\pi} m^{-R\sin \theta} d\theta
\]

\[
= 2C_1(\log T)^2 \int_{0}^{\frac{\pi}{2}} m^{-R\sin \theta} d\theta < 2C_1(\log T)^2 \int_{0}^{\pi/2} m^{-R\theta/2} d\theta
\]

\[
= \frac{4C_1(\log T)^2}{R\log m} (1 - m^{-R\pi/4}) < \frac{4C_1(\log T)^2}{R\log m}, \text{ for } m > 1,
\]
and

\begin{equation}
(12.38) \quad \left| \int_{0}^{\pi} \ell(T) m^{-i Re^{i\theta}} e^{i\theta} d\theta \right| < 2 \left\{ \int_{-\pi/2}^{-10 \log R/R} + \int_{10 \log R/R}^{0} \right\} |\ell(T)| m^{R\theta/2} d\theta < \frac{C_2 (\log T)^2 m^{-5 \log R}}{\frac{1}{2} R \log m} + \frac{C_2 (\log T)^3}{R \log m},
\end{equation}

where \( R = T - T^{\frac{3}{4}} \) and \( \zeta(\sigma + it) = O(t^{\frac{1}{2} - \frac{1}{2}}), \) see \([8, (5.1.1)]\), so that

\[
\ell(T) = \int_{\frac{1}{2}}^{\log_2 T} \log \zeta(\sigma - iRe^{i\theta}) d\sigma = \int_{\frac{1}{2}}^{\log_2 T} O\left\{ (\frac{1}{2} - \sigma - R \sin \theta) \log R \right\} d\sigma = O\{T(\log T)^2\}, \quad \text{for } -\frac{\pi}{2} \leq \theta \leq -\frac{10 \log R}{R},
\]

\[ = O(\log T)^3, \quad \text{for } -\frac{10 \log R}{R} \leq \theta \leq 0.
\]

It remains to estimate the series in (12.36). For this, we separate it into two subseries and then apply Lemma 3, namely

\begin{equation}
(12.39) \quad \sum_{|\gamma_n| < R} m^{-i\gamma_n} = \sum_{|\gamma_n| < (R^2 - \log^2 T)^{\frac{1}{2}}} + \sum_{r \leq |\gamma_n| < R} m^{i\gamma_n - (R^2 - \gamma_n^2)^{\frac{1}{2}}} = O(R \log R m^{-\log_2 T}) + O(\log m) = O(\log T).
\end{equation}

On combining (12.20)-(12.22) with (12.36)-(12.39), we conclude that for \( m > 1, \)

\[
H_1(T) = O\left( \frac{m^\sigma \log T}{\sqrt{T}} \right) + O\left( \frac{m^\sigma (\log T)^2}{T} \right)
\]

\begin{equation}
(12.40) \quad + \frac{m^{\frac{1}{2}} \log m}{T} \left\{ - \frac{R}{\log m} \Lambda(m)L(T, m) + o(T) + O_m\left( \frac{R(\log T)^3}{R \log m} \right) \right\}
\end{equation}

\[ = - \frac{Rm^{\frac{1}{2}}}{T} \Lambda(m)L(T, m) + o_m(1) + O_m\left( \frac{m^\sigma \log T}{\sqrt{T}} \right).
\]

Substituting (12.14), (12.15) and (12.40) into (12.12), we obtain the following desired estimation

\begin{equation}
(12.41) \quad \frac{1}{2T} \int_{-T}^{T} S_{12}^*(y)m^s dt = - \frac{Rm^{\frac{1}{2}}}{T} \Lambda(m)L(T, m) + o_m(1) + O_m\left( \frac{m^\sigma \log T}{T^{\frac{1}{2}}} \right),
\end{equation}
where \( o_m(1) \to 0 \), as \( T \to \infty \) and \( m = 2, 3, \ldots \).

In view of (12.1)-(12.4), (12.6), (12.11) and (12.41), we finally conclude that

\[
a_m = \lim_{T \to \infty} \left\{ m^\sigma \left[ O\left( \frac{1}{T \log m} \right) + O\left( \frac{1}{T} \right) + O\left( \frac{\log T}{T} \right) \right] \right. \\
- \frac{Rm^{\frac{1}{2}}}{T} \Lambda(m) L(T, m) + o_m(1) + O\left( \frac{\log T}{T^\frac{1}{4}} \right) \right\} \\
= -m^{\frac{1}{2}} \Lambda(m) \lim_{T \to \infty} \frac{m^{-\log_2 T} - m^{-\frac{1}{2}}}{\log m} = \frac{\Lambda(m)}{\log m} = \Lambda_1(m) = 1,
\]

for all prime powers \( m = p^k \), and \( a_m = 0 \), otherwise. Hence the function \( \log \zeta^*(s) \equiv \log \zeta(s) \) or \( \zeta^*(s) \equiv \zeta(s) \) which contradicts Theorem 1. This completes the proof of Theorem 2.

13. Disproofs of the Mertens, Koch and Riesz conjectures

The Riemann hypothesis is the most famous of all unsolved problems and therefore, the assertion of Theorem 1 can have many applications. In closing this paper, we shall present three major applications, namely the disproofs of the Mertens, Koch and Riesz conjectures. More precisely, we shall briefly prove the following three theorems:

**Theorem 8.** The Möbius sum \( M(x) = \sum_{n \leq x} \mu(n) \neq O(x^{\frac{1}{2} + \epsilon}) \), for some \( \epsilon > 0 \).

**Theorem 9.** The prime number \( \pi(x) \neq Li(x) + O(x^{\frac{1}{2} + \epsilon}) \), \( \epsilon > 0 \).

**Theorem 10.** The sum \( R(x) = \sum_{n=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)! \zeta(2k)} \neq O(x^{\frac{1}{2} + \epsilon}) \), \( \epsilon > 0 \).

**13.1. Proof of Theorem 8**

To prove the assertion, we suppose on the contrary that

\[
M(x) = \sum_{n \leq x} \mu(n) = O(x^{\frac{1}{2} + \epsilon}), \text{ for each } \epsilon > 0.
\]

Then by partial summation, we see that the series (see [8, p.372])

\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx \quad (s = \sigma + it),
\]

where \( \sigma > 1 \).
converges for $\sigma > \frac{1}{2}$ so that the function $\zeta(s)$ has no zeros on $\sigma > \frac{1}{2}$. This contradicts Theorem 1 and hence the sum $M(x) \neq O(x^{\frac{1}{2}+\epsilon})$, for some $\epsilon > 0$.

As a consequence, we disprove immediately the following Mertens conjecture (see [8, p.374]):

$$|M(n)| < \sqrt{n}, \text{ for each } n = 2, 3, \ldots$$

In fact, the above conjecture was first disproved by Odlyzko and te Riele [5] via numerical computations.

13.2. Proof of Theorem 9

To prove the assertion, we suppose on the contrary that

$$\pi(x) = Li(x) + O(x^{\frac{1}{2}+\epsilon}), \text{ for each } \epsilon > 0,$$

where $\pi(x)$ denotes the number of all primes $p \leq x$ and

$$Li(x) = \int_0^x \frac{dt}{\log t}.$$ 

Then by a well known theorem of de la Vallée Poussin and Koch, we have that

$$\psi(x) - x = O(x^{\frac{1}{2}+\epsilon})$$

and

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} - 1 - x \int_1^\infty (\psi(x) - x)x^{-s}dx.$$ 

Hence the integral converges for $\sigma > \frac{1}{2}$ and the function $\zeta(s)$ has no zeros on $\sigma > \frac{1}{2}$. This again contradicts Theorem 1 and the theorem follows.

13.3. Proof of Theorem 10

Suppose to the contrary that

$$R(x) = O(x^{\frac{1}{4}+\epsilon}), \text{ for each } \epsilon > 0.$$
Then by Mellin's inversion formula (see [8, 14.32])

\[
\frac{\Gamma(1-s)}{\zeta(2s)} = - \int_0^\infty R(x)x^{-1-s}dx,
\]

we conclude that the integral converges for \( \sigma > \frac{1}{4} \) and the function \( \zeta(2s) \) has no zeros on \( \sigma > \frac{1}{4} \). This again contradicts Theorem 1, and the proof is complete.

REFERENCES