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SHEFFER'S STROKE FOR PRIME NUMBERS

In 1922 J. Łukasiewicz introduced $n+1$-valued $(n \geq 2, n \in N)$ logical matrix (see [5]):

$\mathcal{M}_{n+1} = < M_{n+1}, \sim, \rightarrow, \{n\} >$

where $M_{n+1} = \{0, 1, 2, \ldots, n\}$, the functions $\sim x$ and $x \rightarrow y$ are defined by the following way:

$\sim x = n - x$,

$x \rightarrow y = min(n, n - x + y)$,

and $\{n\}$ is the set of designated values.

In what follows the set of all functions of $\mathcal{M}_{n+1}$ that is generated by superpositions of $\sim x$ and $x \rightarrow y$ will be denoted by $\mathbf{Ł}_{n+1}$.

J. C. C. McKinsey [6] replaced functions of $\sim x$ and $x \rightarrow y$ by the single function $x \rightarrow^E y$, which is called Sheffer's stroke for $\mathbf{Ł}_{n+1}$. The set of all superpositions of $x \rightarrow^E y$ will be denoted by $\mathbf{E}_{n+1}$. Thus, $\mathbf{E}_{n+1} = \mathbf{Ł}_{n+1}$.

Let $\mathbf{P}_{n+1}$ be the set of all $n+1$-valued functions defined on the set M_{n+1}. A set of functions $\mathbf{R}_{n+1}$ is called *functionally precomplete* (in $\mathbf{P}_{n+1}$) if every enlargement $\mathbf{R}_{n+1} \cup \{f\}$ by a function f such that $f \notin \mathbf{R}_{n+1}$ and $f \in \mathbf{P}_{n+1}$ is functionally complete.

For example, let $\mathbf{T}_{n+1}$ denote the set of all functions from $\mathbf{P}_{n+1}$, which preserve 0 and n, i.e. $f(x_1, \ldots, x_k) \in \mathbf{T}_{n+1}$ iff $f(x_1, \ldots, x_k) \in \{0, n\}$, where $x_i \in \{0, n\}, 1 \leq i \leq k$. S. V. Jabłoński proved in [3] a theorem concerning functionally precomplete sets in $n+1$-valued logic from which it follows, that the given set $\mathbf{T}_{n+1}$ is precomplete in $\mathbf{P}_{n+1}$ for each $n \geq 2$.

THEOREM 1. (D. A. Bochvar, V. K. Finn [1]). *For any $n \geq 2, n$ is prime number iff $\mathbf{Ł}_{n+1} = \mathbf{T}_{n+1}$.*

A similar result was rediscovered later in [2] and [7].

Let $x \rightarrow^K y$ be $n+1$-valued function defined in the following way:

$$x \to^K y = \begin{cases} \text{(i) } x, \text{ if } 0 < x < y < n, (x,y) \neq 1 \text{ and } (x+y) \leq n \\ \text{(ii) } y, \text{ if } 0 < x < y < n, (x,y) \neq 1 \text{ and } (x+y) > n \\ \text{(iii) } y, \text{ if } 0 < x = y < n \\ \text{(iv) } x \to y \text{ otherwise,} \end{cases}$$

where $(x, y) \neq 1$ denote that x and y are not relatively prime numbers.

The set of all superpositions of $x \to^K y$ will be denoted by $\mathbf{K}_{n+1}$.

LEMMA 1. *For any $n > 2, n$ is a prime number iff $n \in \mathbf{K}_{n+1}$.*

Thus, a formula α (i.e. a superposition of function $x \to^K y$) takes the designated value under the all valuations iff n is prime number.

LEMMA 2. *For any $n > 2$ such that n is a prime number, $\mathbf{K}_{n+1} = \mathbf{Ł}_{n+1}$.*

PROOF.

I. $\mathbf{Ł}_{n+1} \subseteq \mathbf{K}_{n+1}$.

(1) $x \to^1 y = \sim y \to^K \sim x$

(2) $x \to^S y = x \to^1 ((y \to^1 y) \to^1 \sim y$

(3) $x \to^2 y = \sim y \to^S \sim x$

(4) $x \to^3 y = \sim ((y \to^K x) \to^K \sim (y \to^K x)) \to^K (x \to^K y)$.

(5) $x \vee^1 y = (x \to^3 y) \to^3 y$

(6) $x \to y = ((x \to^K y) \to^2 (\sim y \to^K \sim x)) \vee^1 ((\sim y \to^K \sim x) \to^2 (x \to^K y)) = min(n, n - x + y)$.

II. $\mathbf{K}_{n+1} \subseteq \mathbf{Ł}_{n+1}$.

From the definition of $x \to^K y$ it follows that the set $\mathbf{K}_{n+1}$ is not functionally complete for $n \geq 2$. But, as we have showed above $\mathbf{Ł}_{n+1}$ is included in $\mathbf{K}_{n+1}$. Since the set $\mathbf{Ł}_{n+1}$ is a functionally precomplete when n is a prime number [1], then for this case $\mathbf{K}_{n+1} \subseteq \mathbf{Ł}_{n+1}$. Thus, $\mathbf{K}_{n+1} = \mathbf{Ł}_{n+1}$.

From Lemma 1, Lemma 2 and properties of $\mathbf{Ł}_{n+1}$ it follows

THEOREM 2. *For any $n > 2, n$ is prime number iff $\mathbf{K}_{n+1} = \mathbf{Ł}_{n+1}$.*

Let $\mathbf{S}_{n+1}$ denote the set of all superpositions of $x \to^S y$ (see formula (2)).

LEMMA 3. *For any $n > 2$ such that n is a prime number, $\mathbf{S}_{n+1} = \mathbf{K}_{n+1}$.*

PROOF.

I. $\mathbf{S}_{n+1} \subseteq \mathbf{K}_{n+1}$.

Formulae (1) and (2).

II. $\mathbf{K}_{n+1} \subseteq \mathbf{S}_{n+1}$.

(7) $\sim x = x \to^S x$
(8) $n = \sim (x \to^S (x \to^S x)) \to^S \sim ((x \to^S x) \to^S x)$
(9) $x \to^1 y = x \to^S (n \to^S y)$.
(10) $x \to^K y = \sim y \to^1 \sim x$.

Thus, for any $n > 2$ such that n is a prime number, the function $x \to^S y$ is Sheffer's stroke for $\mathbf{K}_{n+1}$.

From Lemma 3 and Lemma 2 it follows

LEMMA 4. *For any $n > 2$ such that n is a prime number,* $\mathbf{S}_{n+1} = \mathbf{Ł}_{n+1}$.

Now, from Lemma 4, Lemma 3 and Theorem 2 we obtain

THEOREM 3. *For any $n > 2$, the function $x \to^S y$ is Sheffer's stroke for* $\mathbf{Ł}_{n+1}$ *iff $n > 2$ is a prime number.*

From that and from McKinsey's result it follows

THEOREM 4. *For any $n > 2, n$ is a prime number iff* $\mathbf{S}_{n+1} = \mathbf{E}_{n+1}$.

Note that the following formulae: (7), (8), (9), (10), (3), (4), (5), (6) and

(11) $x \to^E y = x \to (n \to^S y)$

is a direct proof of $\mathbf{E}_{n+1} \subseteq \mathbf{S}_{n+1}$.

In conclusion we should stress that equality $\mathbf{S}_{n+1} = \mathbf{E}_{n+1}$ holds not for the whole row of natural numbers but only for *the sequence of prime numbers*. The equivalence of the two classes of functions is essentially expressed by the equation (11), right part of which is a superposition of the function $x \to^S y$. The number of occurrences of $x \to^S y$ in this superposition essentially depends on the definition of the function $x \to^K y$ and, consequently, on the definition of the notion of a prime number (Lemma 1). For example, omitting (i) and the condition $(x + y) > n$ in (ii) we obtain a function $x \to^{K'} y$. When $\to^{K'}$ is taken instead of $\to^K$ then formula corresponding to (11) becomes extremely complex. In [4] the

Łukasiewicz's $\to$ is defined using $\sim x$ and $x \to^{K'} y$ for n prime, $n \geq 2$. The corresponding formula contains 21 345 281 occurrences of the function $x \to^{K'} y$.

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