

**APPROXIMATE SOLUTIONS OF THE URYSOHN INTEGRAL EQUATION
WITH A NONLINEARITY OF THE FORM $K(u, x, \varphi(x)) = R(u, x)e^{iu\varphi(x)}$ AND
ITS CONNECTION WITH RIEMANN HYPOTHESIS**

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ABSTRACT: In this paper we study the approximate solutions to solve the Urysohn integral equation of first and second kind with an exponential nonlinearity . This study is motivated due to a reinterpretation of the Chebyshev function in Number theory as the Trace of certain Hamiltonian operator , which in the end yields to the fact that the potential $V(x)$ of a Hamiltonian whose Energies are just the Non-trivial zeroes of the Riemann Zeta function is the solution of a certain Urysohn equation of First kind . Also we discuss the relationship between these type of integrals and linear PDE and ODE with certain initial value or boundary conditions and propose a new Hilbert-Polya operator (Hamiltonian) as a possible candidate to satisfy Riemann Hypothesis.

- *Keywords:* Urysohn equation , exponential nonlinearity , existence theorem, Riemann Hypothesis Hilbert-Polya operator.

1.Urysohn integral equation of First and Second kind:

The general definition of a Nonlinear Urysohn equation is given by:

$$\lambda\varphi(u) + g(u) = \int_a^b dt K(u, x, \varphi(x)) \quad (1.1)$$

With the value of lambda $\lambda = 0,1$ depending on if it is an integral equation of First or Second kind, $g(s)$ is the Non-homogeneous part of the equation and $K(s,t,\xi)$ is a real or complex function . The special case whenever the Kernel 'K' is separable:

$$K(u, x, \xi) = l(u, x)R(x, \xi) \quad l(u, x) = l(x, u) \quad (1.2)$$

Is known as a ‘Hammerstein integral equation’, the integral equation (1.1) includes the cases of ‘Hammerstein’ and Fredholm equation of First and Second kind when the Kernel $K(u,x,\xi)$ is of the form (1.2) or when the Kernel is just a linear function of ‘u’.

The main purpose of this paper is to obtain the solution to Urysohn integral equations with a complex exponential nonlinearity in the form $e^{iu\varphi(x)}$, this kind of Kernel appears for example when ones tries to find a Hilbert-Polya operator that satisfies

$\zeta\left(\frac{1}{2} + iE_n\right) = 0$, so all the Non-trivial zeros of the Riemann Zeta function (these different from $-2, -4, -6, \dots$) have real part equal to $1/2$. One of the first step in proving RH using the Hilbert-Polya conjecture was due to Berry and Keating, [2], they proposed and studied the operator $H_b = xp + \alpha W(x)$ where ‘alpha’ is a coupling constant then, its Quantum Mechanical version satisfied the linear differential equation:

$$-ix \frac{d\Phi}{dx} - i \frac{\Phi}{2} + \alpha W(x)\Phi = E_n \Phi = H_b \Phi \quad \zeta\left(\frac{1}{2} + iE_n\right) = 0 \quad (1.3)$$

An even more surprising relationship between Quantum mechanics and Number theory comes from the fact that if Riemann Hypothesis is, so all the Non-trivial zeros are of the form $\rho_n = \frac{1}{2} + it_n$, $t_n \in R$, then the Von Mangoldt explicit formula for the Chebyshev function involving a sum over these Non-trivial zeros :

$$\Psi_0(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\zeta'(s) x^s}{\zeta(s) s} = \begin{cases} x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1-x^{-2}) & u > 0 \\ 0 & u < 0 \end{cases} \quad (1.4)$$

$$\Psi_0(x) = \begin{cases} \Psi(x) - \frac{1}{2} \Lambda(x) & \text{when } x = p^m \quad m \in Z^+ \\ \Psi(x) & \text{otherwise} \end{cases} \quad \lim_{x \rightarrow \infty} \frac{\Psi_0(x)}{x} = 1 \quad \text{so} \quad (1.5)$$

Can be considered as the Trace of certain Unitary operator, to see this we put $x = \exp(u)$ and differentiate respect to ‘u’ inside the definition (1.4), multiplying by a factor $e^{u/2}$ at both sides we get the expression:

$$e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} = \sum_{n=-\infty}^{\infty} e^{iu(t_n = E_n)} = Tr\{\hat{U} = e^{iu\hat{H}}\} \quad (1.6)$$

So, we would get that the Trace of the Unitary operator given at (1.6) involving the Hamiltonian \hat{H} , Which maps the imaginary parts of the Non-trivial zeros, in the Semiclassical WKB expansion should be equal to :

$$\left(e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \right) \sqrt{\frac{u}{\pi}} \approx \int_{-\infty}^{\infty} dx \exp\left(iuV(x) + i\frac{\pi}{4} \right) \quad u > 0 \quad (1.7)$$

To get (1.7) we have approximated a sum over the Energies of the Hamiltonian, by an integral over the whole phase space (x,p) of the system y the form:

$$\sum_{m=-\infty}^{\infty} e^{iuE_n} \approx \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp e^{iu(p^2+V(x))} = \sqrt{\frac{\pi}{u}} \int_{-\infty}^{\infty} dx e^{iuV(x) + i\frac{\pi}{4}} \quad (1.8)$$

Once we have found a solution to (1.7) we can get another integral constraint to the energies as $n \rightarrow \infty$ in the form :

$$\int_C dx \sqrt{E_n - V(x)} = \left(n + \frac{1}{2}\right)\pi \quad \zeta\left(\frac{1}{2} + iE_n\right) = 0 \quad (C = \text{closed path}) \quad (1.9)$$

The ‘Trace condition’ (1.7) is a necessary condition that must be satisfied by every proposed Hilbert-Polya operator, it is not an hypothesis but it is imposed by the explicit formula for the Chebyshev function, ANY operator whose energies are precisely the imaginary part of the Non-trivial zeros , must have a Trace $Tr\{e^{iu\hat{H}}\}$ equal to the expression (1.7) solving this integral constraint we will give some possible potential functions V(x) for the Hilbert-Polya Hamiltonian operator.

A direct method to evaluate the integral equation given in (1.7) when $u \rightarrow \infty$, is expanding the potential inside the exponential integral near its extremum $V'(\sigma) = 0$

$$g(u) \approx \frac{2\pi}{\sqrt{u|V''(\sigma)|}} e^{i\frac{\pi}{4} + iV(\sigma)} \left(1 - \frac{d\Psi_0(e^u)}{du} \right) e^{u/2} \approx \frac{1}{u} \frac{\pi\sqrt{2}}{|V''(\sigma)|^{1/2}} e^{i\frac{\pi}{4}} e^{iuV(\sigma)} \quad u \rightarrow \infty \quad (1.10)$$

Here (1.10) tells us that since $u = \log(x)$ the difference between the derivative of Chebyshev function and the function $f(x) = 1$ tends to 0 as $1/\log(x)$.

Since the expressions $\sum_{m=-\infty}^{\infty} e^{iuE_n} = Tr\{U_+ = e^{iu\hat{H}}\} = \sum_{m=-\infty}^{\infty} e^{-iuE_n} = Tr\{U_- = e^{-iu\hat{H}}\}$ are equal (and real) for every real ‘u’ we can propose a set of real-valued Urysohn integral equations similar to (1.7) but involving cos(x) and sin(x) functions:

$$\left(e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \right) \sqrt{\frac{u}{\pi}} \approx \int_{-\infty}^{\infty} \cos\left(iuV(x) + i\frac{\pi}{4} \right) dx \quad u > 0 \quad (1.11)$$

$$\left(1 - \frac{d\Psi_0(e^u)}{du} \right) e^{u/2} \approx \frac{\pi\sqrt{2}}{|V''(\sigma)|^{1/2}} \text{Cos}\left(uV(\sigma) + \frac{\pi}{4} \right) \quad u \rightarrow \infty \quad (1.12)$$

$$\int_{-\infty}^{\infty} \sin\left(iuV(x) + i\frac{\pi}{4}\right) dx \approx 0 \quad (1.13)$$

Unfortunately (1.7) and (1.11-12) are nonlinear integral equations (and since we have used the WKB approximation even we knew an exact analytic solution to them we could only obtain an approximation to potential $V(x)$) of Urysohn type with singular values at the points $u = k \log(p)$ where these singular values could be understood to be some kind of ‘Hagedorn Temperatures’ of a certain Quantum Statistical system.

Only a few analytic results are known for these kind of Urysohn integral equations , in the next section we will discuss some existence theorem and (approximate) solutions to (1.7) in order to get the potential $V(x)$ of the Hamiltonian $H = p^2 + V(x)$. Many of the results presented in this paper had been previously introduced in reference [6] the main task of this paper is to redefine them in terms of the language of nonlinear integral operators so we can use the tools of Functional analysis to solve RH .

2.Existence theorems and approximate solutions:

Georgescu [7] has developed several methods to determine if a certain Urysohn equation of the form:

$$\varphi(u) + g(u) = K[\varphi] = \int_0^1 dx K(u, x, \varphi(x)) \quad (2.1)$$

has any (or several) solutions, using the Leray-Schauder theory she reached to the following conclusions:

- If the operator K defining the integral equation (2.1) is bounded , so its norm $\|K\| < \infty$, then exist a solution for this Urysohn integral defined on the interval $x \in (0,1)$
- Let be a Borel function $B(x)$, so the Kernel is related to $B(x)$
 $|K(u, x, \xi)| \leq B(\xi) \quad B : [0, \infty) \rightarrow [0, \infty)$ and the limit $\overline{\lim}_{x \rightarrow \infty} \frac{B(x)}{x} < 1$, then the Urysohn integral equation (2.1) has a solution
- From previous theorem , if the Kernel of the integral equation behaves like $\alpha x^\beta \quad 0 < \alpha < 1 \quad 0 < \beta < 1$, or the Kernel is oscillating then the Urysohn equation (.21) has a solution.

We can write (1.11) and (1.12) into a similar form to equation (2.1) , introducing a cut-off , so instead of integrating over the whole real line we perform integration over the interval $(-N, N)$, then making a change of variable we have:

$$Tr \left\{ e^{iu\hat{H}} \right\} \sqrt{\frac{u}{\pi}} e^{-i\pi/4} \approx \int_0^1 dt f(u\varphi(t)) \quad \varphi(x) = V(2Nx - N) \quad N \gg 1 \quad (2.2)$$

In the following paragraphs we will give some descriptions of how an Urysohn equation with nonlinearity of the form $R(u, x)e^{iu\varphi(x)}$ could be solved, since the potential function $V(x)$ of a Hamiltonian which solves the Riemann Hypothesis is related to an Urysohn integral equation of first kind we will try to obtain a form to get $V(x)$, finding a solution to the integral equations described before in (1.7), (1.11) and (1.12)

○ *The linearized integral equation:*

Given the Urysohn integral of the form, with ‘lambda’ being 1 or 0 depending on if it is an integral equation of first or second kind:

$$\lambda\varphi(u) + g(u) = \int_{-\infty}^{\infty} dx R(u, x) e^{iu\varphi(x)} \quad \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} du |R(u, x)|^2 < \infty \quad (2.3)$$

We could try the ansatz $\varphi(x) = \varphi_0(x) + \alpha Q(x)$, with ‘alpha’ being an small coupling constant and $Q(x)$ it is an unknown function of x , expanding the complex exponential keeping only the linear terms in the coupling constant our initial integral equation becomes a linear integral equation of the form:

$$\lambda\varphi_0(u) + \lambda\alpha Q(u) + g(u) - F(u) = i\alpha u \int_{-\infty}^{\infty} dx R(u, x) Q(x) \quad \int_{-\infty}^{\infty} dx R(u, x) \varphi_0(x) = F(u) \quad (2.4)$$

If we use a Quadrature method, we can obtain a relation between an integral equation and a system of linear equation in the form:

$$\lambda\varphi_0(u_j) + \lambda\alpha Q(u_j) + g(u_j) - F(u_j) = i\alpha u_j \sum_{i,j} C_{i,j} W(x_i) R(u_j, x_i) \varphi(x_j) \quad (2.5)$$

This system of linear equation will only have a solution (different from $\varphi(x) = 0$) for the set $\{\varphi(x_i)\}_{i=1,2,3,\dots}$ iff the Determinant of the Matrix

$A_{i,j} = C_{i,j} u_j R(u_j, x_i) W(x_i) - \lambda \delta_{i,j}$ is different from 0 (a similar formulation of Fredholm alternative theorem), so our linear integral operator $R[f] = \int_{-\infty}^{\infty} dx R(u, x) f(x)$ will have an unique solution iff:

$$R[f] = \int_{-\infty}^{\infty} dx R(u, x) f(x) = \beta_n f(x) \quad \text{and} \quad \beta_n \neq 0, 1 \quad (2.6)$$

If we consider the Berry Hamiltonian $H_b = xp + \alpha W(x)$, and the Trace condition given previously in the equation (1.7), we would obtain:

$$e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \approx \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp e^{iuxp} (1 + iu\alpha W(x) + \dots) \quad (2.7)$$

Then , using the Fourier transform representation for the Dirac delta we would get the linearized integral equation after integration over the variable ‘x’ :

$$e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} - \frac{2\pi}{|u|} \approx iu\alpha \int_{-\infty}^{\infty} dp \hat{F}\{W(x), u\} \quad u > 0 \quad (2.8)$$

Here $\hat{F}\{W(x), u\} = \int_{-\infty}^{\infty} dx e^{iux} W(x)$ is the Fourier transform of the perturbative term $W(x) = -W(-x) \in L^2(R)$ (from the definition of the linearized integral, the term $W(x)$ must be an odd function of argument since the non-homogeneous part of the integral equation is a real number).

○ *The inverse Fourier and Hartley transform method:*

In case $R(u, x)$ is equal to 1 then our Urysohn integral equation can be turned into a Fourier transform by making the change of variable $x = \varphi^{-1}(\tau)$:

$$g(u) = \int_{-\infty}^{\infty} dx e^{i\varphi(x)} \rightarrow g(u) = \int_{-\infty}^{\infty} d\tau W_b^a(\tau) \frac{d\varphi^{-1}(\tau)}{d\tau} e^{iu\tau} = -iu \int_{-\infty}^{\infty} d\tau W_b^a(\tau) \varphi^{-1}(\tau) e^{iu\tau} \quad (2.9)$$

$$W_b^a(\tau) = H(\tau - b) - H(\tau - a) \quad \varphi(-\infty) = b \quad \varphi(\infty) = a \quad (2.10)$$

$$H(x) = \text{Heaviside function} \quad \varphi^{-1}(x) W_b^a(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \frac{g(u)}{u} e^{-iux + i\pi/2} \quad (2.11)$$

The solution to the Urysohn equation is just the inverse Fourier transform of $g(u)/u$ multiplied by a constant . For the special case of our Hilbert-Polya operator:

$$\left(e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \right) \sqrt{\frac{u}{\pi}} e^{-i\frac{\pi}{4}} = g(u) \quad \frac{d\Psi_0(x)}{dx} = \sum_{v=1}^{\infty} \sum_p \delta(x - p^v) \quad (2.12)$$

$$\begin{aligned} \sqrt{4\pi^3} W_b^a(x) V^{-1}(x) &\approx \int_0^{\infty} dx \left(e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \right) \sqrt{\frac{1}{u}} e^{-iux + i\frac{\pi}{4}} = \\ &\sum_{\gamma > 0} \left(\sqrt{\frac{\pi}{|x - \gamma|}} + \sqrt{\frac{\pi}{|x + \gamma|}} \right) \approx \left(\sqrt{\frac{8\pi}{4x^2 + 1}} + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dr}{|x + r|^{1/2}} \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{ir}{2} \right) - D e^{ix} + c.c \right) \end{aligned} \quad (2.13)$$

Last expression is just the Fourier integral representation for the inverse of the potential $V^{-1}(x)$ here the quantities ‘ γ ’ are just the imaginary part of the non-trivial Riemann zeta zeros , (c.c) stands for the complex conjugate of the last parenthesis in (2.13)

, $D = g(0) \log \pi + 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\log n)$ and the function $g(u)$ is obtained by means of the

Fourier integral $4\pi g(u) = \int_{-\infty}^{\infty} d\xi |\xi|^{-\frac{1}{2}} e^{-iu\xi}$, where $|x|$ is interpreted as the absolute value

in case 'x' is real and the modulus of $|z|$ in case z is complex, $\frac{\Gamma'(x)}{\Gamma(x)}$ is Digamma

function, the last equality comes from the Riemann-Weyl formula that allows to handle with sums over the non-trivial zeros of the zeta function. Except for a cosine term the main contribution to the inverse of the potential is due to the last improper integral involving a Digamma function note that the integral in (2.13) must be understood in Cauchy's principal value to avoid the singularity whenever $x+r=0$, a necessary condition for the expression (2.13) to converge is that the integral

$$\int_1^{\infty} \frac{dx}{\log(x)} \left| 1 - \frac{d\Psi_0(x)}{dx} \right|^2 \text{ has a finite value} \quad (2.14)$$

In many cases such as (1.11) when we are looking for real-valued solutions only, is more interesting to find some kind of Fourier transform mapping real functions into real functions, if we introduce the 'Hartley transform' in terms of the Fourier exponential transform:

$$H(u) = \frac{F(-u) + F(u)}{2} + i \frac{F(-u) - F(u)}{2} \quad 2\pi H \{Hf(x)\} = f(x) \quad (2.15)$$

The Hartley transform has a Kernel defined only as a linear combination of cosine and sine functions, maps a real function into another real function and is its own inverse and direct transform making the calculations look simpler, if we introduce:

$$cas(ux) = \cos(ux) - \sin(ux) = \sqrt{2} \cos\left(ux + \frac{\pi}{4}\right) \quad (2.16)$$

$$\frac{d}{dx} cas(ux) = -u(\cos(ux) + \sin(ux)) = -u\sqrt{2} \cos\left(ux - \frac{\pi}{4}\right) \quad (2.17)$$

The inverse of the solution for integral equation (1.11) can be written in terms of Hartley transform and the cas(ux) function:

$$2W_b^a(x) \sqrt{2\pi^3} V^{-1}(x) \approx \int_0^{\infty} dx \left(e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \right) \sqrt{\frac{1}{u}} cas(ux) \quad (2.18)$$

A necessary condition for (2.18) to converge is that for any fixed value of 'x' there will be a constant defined by the equality:

$$A_x = 2 \int_0^{\sqrt{T}} du e^{u^2/2} cas(xu^2) - \sum_{p, p^v \leq e^T} \frac{cas(xv \log p)}{\sqrt{v} p^v \log p} \quad T \rightarrow \infty \quad (2.19)$$

Where the last sum is over all the primes and prime powers not exceeding e^T . The validity of (2.19) for certain values of x would imply our Hilbert-Polya Hamiltonian operator is correct and RH is true.

The real solution (2.17-19) had previously been discussed by the author (reference [6]), note that we can reformulate our integral equations (1.11-12) to write them in terms of the Hartley Kernel function $\text{cas}(x)$ or its derivative:

$$\int_{-\infty}^{\infty} dx \frac{d}{d\xi} \text{cas}(\xi = uV(x)) \approx 0 \quad \text{Tr} \left\{ e^{iu\hat{H}} \right\} \sqrt{\frac{2u}{\pi}} \approx \int_{-\infty}^{\infty} dx \text{cas}(uV(x)) \quad (2.20)$$

Where the Trace of the operator $\text{Tr} \left\{ e^{iu\hat{H}} \right\} = \sum_{n=-\infty}^{\infty} e^{iuE_n}$ had previously been defined in step (1.6) when dealing with Chebyshev function.

Another example of how this method works come from the use of WKB approximation in Statistical Physics and the Laplace inversion theorem, we will consider the next example for the Harmonic oscillator.

Example: given the Energies for $n > 0$ $E_n = n + 1/2$, what is the Hamiltonian which generates them, so $\hat{H}\Phi_n = (n + 1/2)\Phi_n$?, with $V^{-1}(0)=0$ and $V(\infty)=\infty$

If we assume a Hamiltonian of the Form $H=T+V$ where 'T' is the Kinetic energy and 'V' is the potential, then using the WKB approximation for the Trace of $\exp(-u\hat{H})$ where $u > 0$ is a real number we would find:

$$\sum_{n=0}^{\infty} e^{-u(n+1/2)} = \frac{e^{-u/2}}{1 - e^{-u}} \approx \frac{1}{2} \sqrt{\frac{\pi}{u}} \int_0^{\infty} dx e^{-uV(x)} \quad \text{taking} \quad x = V^{-1}(\tau) \quad (2.21)$$

$$\frac{e^{-u/2}}{u} \approx \frac{1}{2} \sqrt{\frac{\pi}{u}} \int_0^{\infty} d\tau \frac{dV^{-1}(\tau)}{d\tau} e^{-u\tau} \rightarrow \sqrt{\frac{4}{\pi u}} e^{-u/2} \approx L \left\{ \frac{dV^{-1}(\tau)}{d\tau} \right\} \quad (2.22)$$

Taking the inverse Laplace transform inside (1.12) and integrating over parameter 'tau' we get that the inverse of the potential should be $V^{-1}(x) \approx \frac{4}{\pi} \sqrt{x - \frac{1}{2}}$, in other words the Hamiltonian whose energies are of the form $E_n = n + 1/2$ is a Harmonic potential, this means that at least for the Harmonic oscillator the WKB semiclassical approach gives almost exact results compared with exact Quantum Mechanical methods, we have also used a Taylor series expansion for the expression $(1 - e^{-u}) \approx u + \dots$ inside () for a better comprehension of the method

- *1-Dimensional Schwinger-Dyson equation and Urysohn integral:*

One of the most powerful Non-perturbative techniques to calculate Feynmann Path integrals was due to Schwinger and Dyson, they independently got a Functional differential equation for the Generating functional in the form:

$$Z(J) = \int D\phi e^{iS[\phi] + i \int d^4x J(x)\phi(x)} \quad \frac{\delta S}{\delta \phi} \left[-i \frac{\delta}{\delta J} \right] Z(J) + Z(J)J(x) = 0 \quad (2.23)$$

To illustrate how their method work , we will apply it to our theory of Urysohn integral equation , if we define the ‘Generating function’ Z in the form:

$$Z_u(\eta) = \int_{-\infty}^{\infty} dx e^{iu\varphi(x) + iu\eta x} = -iu \int_{-\infty}^{\infty} dx x e^{iu\varphi(x) + iu\eta x} \left(\frac{d\varphi(x)}{dx} + \eta \right) \quad (2.24)$$

$$\int_{-\infty}^{\infty} dx x^n e^{iu\varphi(x) + iu\eta x} = \left(-\frac{i}{u} \frac{\partial}{\partial \eta} \right)^n Z_u(\eta) \quad \lambda\varphi(u) + g(u) = \int_{-\infty}^{\infty} dx e^{iu\varphi(x)} = Z_u(0) \quad (2.25)$$

Then if we knew an exact analytic expression for $Z_u(\eta)$, we would have solved the initial Urysohn equation with nonlinearity of the form $K(u, x, \varphi(x)) = e^{iu\varphi(x)}$, in the special case of our Urysohn integral equation, the Generating functional ‘ Z ’ can be formulated as the solution of a certain linear ODE with initial conditions:

$$Z_u(\eta) + \eta \frac{\partial Z_u(\eta)}{\partial \eta} + iux\varphi(x) \left(\hat{x} = \frac{-i}{u} \frac{\partial}{\partial \eta} \right) Z_u(\eta) = 0 \quad Z_u(0) = g(u) + \lambda\varphi(u) \quad (2.26)$$

Differentiating respect to the parameter ‘ u ’ we would also get the PDE:

$$\frac{\partial}{\partial u} \left(iux \frac{d\hat{\varphi}}{dx} \left(-\frac{i}{u} \frac{\partial}{\partial \eta} \right) \right) f + \eta \frac{\partial^2}{\partial \eta \partial u} f + \frac{\partial}{\partial u} f = 0 \quad Z_u(\eta) = f(u, \eta) \quad (2.27)$$

$$\frac{\partial f(u, 0)}{\partial u} = g'(u) + \lambda\varphi'(u) \quad \lim_{(u, \eta) \rightarrow \infty} f(u, \eta) = 0$$

Where, in all cases using the Taylor series we can define :

$$x \frac{d\varphi}{dx} \left(x = \frac{-i}{u} \frac{\partial}{\partial \eta} \right) \approx a_1 \left(\frac{-i}{u} \frac{\partial}{\partial \eta} \right) + a_2 \left(\frac{-i}{u} \frac{\partial}{\partial \eta} \right)^2 + \dots \quad (2.28)$$

If we truncate the Taylor series above after the M -th term, then the order of the PDE and ODE given in (2.27) and (2.28) will be ‘ M ’, also note that the condition for the limit

$$\lim_{(\eta, x) \rightarrow \infty} f(\eta, x) = 0 \text{ must be imposed to get a real function } \varphi(x)$$

For our Hilbert-Polya Hamiltonian then $\lambda = 0$ and $g(u)$ has the value:

$$\left(e^{u/2} - e^{-u/2} \frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u} - e^u} \right) \sqrt{\frac{u}{\pi}} e^{-i\frac{\pi}{4}} = g(u) \quad (2.29)$$

In case we had $K(u, x, \varphi(x)) = R(u, x)e^{iu\varphi(x)}$ with $R(u, x)$ a Polynomial on the variable x , then an extra linear term of the form $R\left(u, \hat{x} = \frac{-i}{u} \frac{\partial}{\partial \eta}\right) Z_u(\eta)$ would appear inside (2.26)

These results can also be generalized to Generating function involving exponential sums of the form $\sum_{n=0}^N e^{iaf(n)+ia\eta n}$, where $f(x)$ is a Polynomial and ‘a’ is a real number, if we use summation by parts and the approximate identity for the difference operator:

$$\sum_{n=1}^N e_a(f(n) + \eta n) = N e_a(f(N) + \eta N) - e_a(f(1) + \eta) - \sum_{n=1}^N n \Delta e_a(f(n) + \eta n) \quad (2.30)$$

$$\Delta = e^D - 1 \approx D + \frac{D^2}{2} + \dots \quad e_a(f(x)) = e^{iaf(x)} \quad (2.31)$$

$$G(\eta) = \sum_{n=1}^N e^{iaf(n)+ia\eta n} \quad D^{(u)}G(\eta) = \sum_{n=1}^N (ian)^u e^{iaf(n)+ia\eta n} \quad (2.32)$$

The approximate Schwinger-Dyson one dimensional differential equation for ‘ $G=G_a(\eta)$ ’ is:

$$\begin{aligned} G_a(\eta) - \frac{1}{2} a^2 \left(x f'(x)^2 \left(\frac{-i}{a} \frac{\partial}{\partial \eta} \right) + \eta^2 x \left(\frac{-i}{a} \frac{\partial}{\partial \eta} \right) + 2\eta x f'(x) \left(\frac{-i}{a} \frac{\partial}{\partial \eta} \right) \right) G_a(\eta) \\ + \frac{1}{2} i a x f''(x) \left(\frac{-i}{a} \frac{\partial}{\partial \eta} \right) G_a(\eta) + \eta \frac{\partial}{\partial \eta} G_a(\eta) + i a x f'(x) \left(\frac{-i}{a} \frac{\partial}{\partial \eta} \right) G_a(\eta) = H(\eta) \end{aligned} \quad (2.33)$$

With $H(\eta) = H_N(\eta) = N e_a(f(N) + \eta N) - e_a(f(1) + \eta)$. The value $G_a(\eta = 0) \approx \sum_{n=1}^N e^{iaf(n)}$

Obtained after solving the differential equation (2.33) could be used to obtain estimates for any exponential series involving a Polynomial $f(x)$.

○ *Polyanin - Zhurov method for Urysohn integral equation:*

Polyanin and Zhurov have recently developed a method to solve the Urysohn integral:

$$\varphi(u) + \int_a^b dx \left(R(u, x) \varphi(x) + \sum_{j=0}^M \gamma_j(u) \Phi_j(x, \varphi(x)) \right) = g(u) \quad (2.34)$$

They tried the ansatz to solve (2.34) in the form $\varphi(x) = Y_g(x) + \sum_{j=1}^M A_j Y_{\gamma_j}(x)$, with this our initial nonlinear equation can be split into a system of linear ones :

$$\int_a^b dx R(u, x) Y_g(x) = g(u) \quad \int_a^b dx R(u, x) Y_{\gamma_j}(x) = \gamma_j(u) \quad (2.35)$$

and, a Transcendent equation to set the values of the constants A_j in the form:

$$A_m + \int_a^b dx \Phi_m(x, Y_g(x) + \sum_{j=1}^M A_j Y_{\gamma_j}(x)) = 0 \quad m=0,1,2,3,\dots,M \quad (2.36)$$

For our type of Urysohn equations, the Polyanin-Zhurov method can be applied if we make use of the approximate Taylor expansion (provided the Kernel $R(u,x)$ is analytic on the variable u near the point $u=c$):

$$\int_a^b dx R(u, x) e^{iu\varphi(x)} \approx \int_a^b dx \left(\sum_{j=0}^M \frac{D_u^j (R(c, x) e^{c i \varphi(x)})}{j!} (u-c)^j \right) \quad (2.37)$$

$$\text{then } \Phi_j(x, \varphi(x)) = D_u^j (R(c, x) e^{c i \varphi(x)}) \quad \gamma_j(u) = \frac{(u-c)^j}{j!} \quad R(u, x) = 1 \quad (2.38)$$

We can find a finite analogue to (1.8) using Landau's formula, if we consider the difference $\Psi_0(n) - \Psi_0(n-1) = \Lambda(n) \approx \frac{d\Psi_0}{dx}$ for Chebyshev function, studying the sum over all the energies less than a given limit 'E' we would have

$$\sum_{E_n \leq E} e^{iuE_n} = e^{-u/2} F(u) O_u(\log E) - \frac{E}{2\pi} F(u) e^{-u/2} \Lambda(e^u) \approx \int_I dx e^{iuW(x)} \quad (2.39)$$

Here $O(g(x))$ must be understood in the sense of Big-O notation and

$$f(x) = O_u(g(x)) \quad \rightarrow \quad f(x) \leq C(u)g(x) \quad \frac{1}{F(u)} = \int_0^{x_0} dp e^{iup^2} \quad (2.40)$$

The interval I inside equation (2.39) can be turned by a linear change of variable into the interval $[0,1]$, with $x_0 = x_0(E)$ and $I = I(E)$, in case $E \rightarrow \infty$ we recover the expression obtained before in (1.8)

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