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A NEW APPROACH TO THE RENORMALIZATION OF UV DIVERGENCES USING ZETA REGULARIZATION TECHNIQUES

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ABSTRACT**:** In this paper we present a method to deal with divergences in perturbation theory using the method of the Zeta regularization, first of all we use the Euler-MacLaurin Sum formula to associate the divergent integral to a divergent sum in the form $1+2^{m}+3^{m}+4^{m}+\ldots\ldots$ After that we find a recurrence formula for the integrals and apply Zeta regularization techniques to obtain finite results for the divergent series. (Through all the paper we use the notation "m" for the power of the modulus of p, so we must not confuse it with the value of the mass of the quantum particle)

- *Keywords*: -Zeta regularization, Euler sum formula, regulator, Abel-Plana formula , renormalization

1. Zeta regularization and recurrence formula:

One of the most important problems in perturbative QFT is to give a meaning to clearly divergent integrals , in the form $\int_0^{\infty} dkk^m$, with $\lambda = \frac{2\pi}{|k|}$ and 'm' an integer, depending on if 'm' is either positive or negative , we will have UV (small wavelength and high momentum) or IR (small momentum) divergences , the case m=-1 is just known as a 'logarithmic divergence' , we will show how to get finite regularized values to these divergences using the mathematical tools of 'Zeta function regularization' .

- Small introduction to Zeta regularization

The Zeta regularization technique is useful when dealing with divergent series, we can use it to associate a finite value to a clearly divergent series in the form:

$1+2^a+3^a+\ldots \quad a>0 \ (1) \ \ a\in \mathrm{R}$ (1.1)

Now we could rewrite the series as:

$$1+2^{a-s}+3^{a-s}+\ldots=1+2^{-(s-a)}+3^{-(s-a)}+\ldots=\zeta(s-a) \qquad (1.2)$$

So we can apply the technique of analytic continuation for the Riemann Zeta function to extend it to negative values of the argument s, now if we set s=0 in (2) we could obtain a finite value for our first series as the value , ζ(-a) a>0 , this value can be calculated by means of the contour integral for a given path σ, on the complex plane:

$$\zeta(-a)=\frac{\Gamma(a+1)}{2\pi i}\int_{\sigma}dq\frac{q^{-(a+1)}}{e^{-q}-1} \qquad (1.3)$$

This form of re-calculate a series has been widely used in Physics, one of the most famous examples being the calculation of the force due to "Casimir Effect" using the Zeta regularized value ζ(-3)=-1/120, or as another example, the value ζ(-1)=-1/12 for the divergent sum 1+2+3+4+5+6+7+…. That appears in theoretical physics.

Euler himself, using the operator D=d/dx and Taylor series expansion got the identity for an infinite series with general term f(x+n):

$$f(x)+f(x+1)+f(x+2)+\ldots\ldots\ldots=\left(\sum_{n=0}^{\infty}e^{nD}\right)f=\left(\frac{1}{1-e^{D}}\right)f=-\frac{1}{D}\left(\sum_{n=0}^{\infty}B_n\frac{D^n}{n!}\right)f \qquad (1.4)$$

The 'B_n', numbers are known as 'Bernoulli Numbers' and are widely studied in math, they can be considered as the value at x=0 of the Bernoulli Polynomials, which have the next generating function:

$$\frac{te^{xt}}{e^{t}-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \qquad B_0(x)=1 \qquad B_2(x)=x^2-x+1/6 \qquad (1.5)$$

If we put f(x)=x or f(x)=x^3 inside (4) and take the limit x→0, we get the finite values –1/12 and –1/120 , which are in perfect agreement with modern version of Zeta regularization. If the exponent 'a' in (1.1) is an integer then we can relate the value of Bernoulli Polynomials at x=0 and the negative values of Riemann zeta in the form , $B_a(0)=-n\zeta(1-a)$, which can be used to simplify the Calculations

- *The Euler-Mc Laurin sum formula and UV divergences:*

Now if we introduce the Euler-MacLaurin sum formula in the form:

$$\int_0^b f(x)dx=\frac{f(b)}{2}+\sum_{n=1}^{b-1}f(n)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(b) \qquad (1.6)$$

Where for simplicity we have supposed that $f(0)=f^{(k)}(0)=0$, the B_{2n} that appear in the formula are the (even) "Bernoulli numbers" ,introduced previosly in formula (1.5)

The divergent integrals that appear when doing perturbation theory in Quantum Field theory take the form: (except the case of a Logarithmic-divergent integral m=-1)

$$\int_0^{\Lambda} p^m dp \quad m \in \mathbf{Z} \qquad m > 0 \qquad (1.7)$$

with "p" the modulus of the momentum vector of the particle, now we introduce the regulator 'Λ' acting as a fictional cut-off inside our physical theory and ater performing calculations we will make it tend to infinite. Using this and the Euler sum formula we can write the divergent integral in the next form with $f(p)=p^m$:

$$\int_0^{\Lambda} p^m dp = \frac{\Lambda^m}{2} + \sum_{n=1}^{\Lambda} n^m - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} a_{mr} \Lambda^{m-2r+1} \qquad a_{mr} = \frac{\Gamma(m+1)}{\Gamma(m-2r+2)} \qquad (1.8)$$

Now we can rewrite the powers of the regulator in the form of divergent integrals:

$$\Lambda^b = b \int_0^{\Lambda} dp p^{b-1} \qquad (1.9)$$

With the regulator/cut-off tending to infinite $\Lambda \to \infty$, the first sum on the left of (1.8) can be regularized to a finite value in the form:

$$\sum_{n=1}^{\infty} n^m = 1 + 2^m + 3^m + \ldots + \Lambda^m \to \zeta(-m) \qquad (1.10)$$

From the definition of the a_{mr} we can see that for m and r positive integers satisfying m-2r+2<0 they are are 0, so the last series in (1.8) will be finite, now if we call:

$$I(m,\Lambda) = \int_0^{\Lambda} p^m dp \qquad (1.11)$$

and use the result obtained in (7) we can write the Euler sum formula (5) for our case as:

$$I(m,\Lambda) = (m/2) I(m-1,\Lambda) + \zeta(-m) - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} a_{mr} (m-2r+1) I(m-2r,\Lambda) \qquad (1.12)$$

m=1,2,3,……………

So to re-calculate our infinite integral I(m,Λ) we should know the values of I(m-1,Λ), I(m-2,Λ) , I(m-3,Λ) , … … … I(0,Λ), this last UV divergence with m=0 is :

$$I(0,\Lambda) = \int_0^{\Lambda} dp = 1+1+1+1+\ldots\ldots = \Lambda \to \zeta(0) = -1/2 \qquad (1.13)$$

To illustrate some examples of our final version given at (1.12) of the 'Euler-Mc Laurin recurrence for UV divergences involving m= 1,2,3 and 4 we have (note that the Riemann Zeta function has trivial zeros when s=-2n n=1,2,3,4,…. So we have omitted these values to simplify calculations) :

$$I(0,\Lambda)=I(0,\Lambda)$$

$$I(1,\Lambda)=\frac{I(0,\Lambda)}{2}+\zeta(-1)$$

(1.14)

$$I(2,\Lambda)=\left(I(0,\Lambda)\frac{1}{2}+\zeta(-1)\right)-\frac{B_2}{2}a_{21}I(0,\Lambda)$$

$$I(3,\Lambda)=\frac{3}{2}\left(\frac{1}{2}\big(I(0,\Lambda)+\zeta(-1)\big)-\frac{B_2}{2}a_{21}I(0,\Lambda)\right)+\zeta(-3)-B_2a_{31}I(0,\Lambda)$$

$$I(4,\Lambda)=3\left(\frac{1}{2}\big(I(0,\Lambda)+\zeta(-1)\big)-\frac{B_2}{2}a_{21}I(0,\Lambda)\right)+2\zeta(-3)-2B_2a_{31}I(0,\Lambda)-$$
$$\frac{B_2}{2}\left(\frac{1}{2}\big(I(0,\Lambda)+\zeta(-1)\big)-\frac{B_2}{2}a_{21}I(0,\Lambda)\right)3a_{41}-\frac{B_4}{24}a_{42}I(0,\Lambda)$$

This formula in (1.11) is a recurrence relation to re-calculate the divergent integrals by using Zeta regularization, we still have to study the case where m is any real number or when m=-1 in that cases by making a change of variable $p=e^{\omega u}$ with w a real number:

$$I(m,\Lambda)=\omega\int_{-\Delta}^{\Delta}due^{u\omega(m+1)}=\sum_{n=0}^{\infty}\frac{c_n}{(2n)!}\int_0^{\Delta}duu^{2n} \qquad c(n)=2\omega[(m+1)\omega]^{2n} \quad (1.15)$$

$\Delta=\frac{\ln\Lambda}{\omega}\to\infty$ for m=-1 we have $I(-1,\Lambda)=2\omega\int_0^{\Delta}du$ (ultraviolet divergence for the new regulator Δ) $\qquad 1/\Lambda=\frac{\omega}{\ln\Delta}\to 0 \qquad$ (1.16)

The "Delta" is the Logarithm of the regulator tending also to infinite, m and w can be arbitrary real numbers, due to the fact that u^{2n+1} are Odd functions then:

$$\int_{-\Delta}^{\Delta}duu^{2n+1}=0 \quad \text{, n= positive integer} \qquad (1.17)$$

Although (1.12) is correct , we may have some problems if m is big , for example if m=100 then our recurrence formula could take too much time to express $I(100,\Lambda)$ as a linear combination of the negative values involving Riemann Zeta function.

If we introduce the function $\zeta_H(s,\beta)=\sum_{n=0}^{\infty}(n+\beta)^{-s}$ $\Re e(s)>1$, with 'beta' different from a negative integer, the sum above can be analytically continued to negative values of argument s via the functional equation:

$$\zeta_H\left(1-s,\frac{\tau}{\mu}\right)=\frac{2\Gamma(s)}{(2\pi\mu)^s}\sum_{k=1}^{\mu}\cos\left(\frac{\pi s}{2}-\frac{2\pi k\tau}{\mu}\right)\zeta_H\left(s,\frac{k}{\mu}\right) \qquad (1.18)$$

In case 1-s is an integer then we have , $\zeta_H(-m,\beta) = -\frac{B_m(\beta)}{m+1}$ then using the 'rectangle method' to approximate a divergent integral by a divergent sum we find:

$$\int_0^\infty dx(x+\beta)^m \approx \sum_{n=0}^\infty (\beta+\varepsilon n)^m \varepsilon \to \zeta\left(\frac{\beta}{\varepsilon}, -m\right)\varepsilon^{m+1} \quad m>0 \qquad (1.19)$$

Formula (1.19) is valid for every positive 'm' no matter if it is real, rational or integer, however it is just an approximation to the true regularized value of divergent integral and since it involves a term ε^{m+1} for small values of epsilon it can be Numerically unstable for big 'm' .

If we suppose that can differentiate respect the parameter 'm' , we can define another formula to obtain regularized finite values to logarithmic integral as:

$$\int_0^\infty dx \log^r(x) \approx \frac{\partial^r}{\partial m^r}\zeta_H(-m,\beta\varepsilon^{-1})\varepsilon^{m+1}\Big|_{m=0} \qquad (1.20)$$

Another application of Zeta regularization to Dirichlet series in the form $\zeta_Q(s) = \sum_{n=1}^\infty Q(n)n^{-s}$, where Q(x) is a Rational function given by the quotient of two Polynomials A(x) and B(x) , if used the expansion for B(x) in the form:

$$\frac{1}{B(x)} = \sum_{n=0}^\infty u_n x^n \qquad \frac{1}{2\pi i}\int_C dz \frac{z^{-(n+1)}}{B(z)} = u_n \qquad (1.21)$$

The last is just a line integral over a closed circuit 'C' on the complex plane. Using (1.121) we could write:

$$\sum_{n=1}^\infty Q(n)n^{-s} = \sum_{k=0}^\infty \sum_{m=0}^d a_m \sum_{n=1}^\infty u(k)n^{-(s-k-m)} = \zeta_Q(s) \qquad \frac{A(x)}{B(x)} = Q(x) \qquad (1.22)$$

Then performing the summation over index 'n' we could obtain the identity :

$$\zeta_Q(s) = \sum_{k=0}^\infty \sum_{m=0}^d a_m u_k \zeta(s-(m+k)) \qquad (1.23)$$

Here 'd' is the degree of Polynomial A(x) , note that since $\lim_{n\to\infty}\zeta(-n) = \infty$ the series (1.23) is divergent for almost any value of s and it will need to be regularized by Borel or other summation technique. In many cases the most important value of s inside (1.23) is obtained setting s=0

Another interesting case involving Zeta regularization is this, let be an operator $\hat{A}$ with a discrete set of Eigenvalues $\{\lambda_n\}_{n=0,1,2,3,4,....}$, then its Determinant can be defined:

$$\prod_{i=1}^\infty \lambda_i = e^{-\zeta_A'(0)} \qquad \sum_{n=1}^\infty \frac{1}{\lambda_1^s} = \zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\sum_n e^{-t\lambda_n} \qquad (1.24)$$

ABEL-PLANA FORMULA , RAMANUJAN AND BOREL RESUMMATION AS MATHEMATICAL TOOLS FOR THE RENORMALIZATION:

- *Abel-Plana formula and renormalization:*

In mathematics a useful formula involved in calculations for Casimir effect is given by:

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} dx f(x) = \frac{f(0)}{2} + i\int_0^{\infty} \frac{f(it)-f(-it)}{e^{2\pi t}-1} dt \quad \frac{f(it)-f(-it)}{i} = 2\Im m[f(it)] \quad (2.1)$$

The derivation of the formula comes from the argument principle for complex (Meromorphic) functions

$$\frac{1}{2\pi i}\int_{\gamma} dz f(z)\frac{g'(z)}{g(z)} = \sum_{\mu} f(x_{\mu}) - \sum_{\tau} f(x_{\tau}) \quad \text{With} \quad g(z) = -if(z)Cot(\pi z) \quad (2.2)$$

Where the series is made over all the zeros and poles included in closed curve γ, an appropriate election for the contour gives formula (9) and $\lim_{t\to 0}\frac{f(it)-f(-it)}{t} = a$ $|a|<\infty$

The conditions that f(z) must satisfy ($z = x+iy$) are (exponential growth) :

$$\int_{a+i\infty}^{b+i\infty} dz\,|f(z) \pm g(z)| = 0 \quad , \quad \lim_{y\to\infty} e^{2\pi|y|}|f(x+iy)| = 0 \text{ for any 'x'} \quad (2.3)$$

Abel-Plana formula allows you to get finite results for divergent integrals in the form :

$$\zeta(-m,\beta) - i\int_0^{\infty}\frac{(it+\beta)^m - (-it+\beta)^m}{e^{2\pi t}-1}dt - \frac{\beta^2}{2} = \int_o^{\infty} dp(p+\beta)^m \qquad m \in \square \qquad (2.4)$$

For some cases with 'm' integer ,the identity below could be useful:

$$\int_0^{\infty} dt \frac{t^m}{e^{2\pi t}-1} = \frac{\Gamma(m+2)\zeta(m+1)}{(2\pi)^{m+1}} \quad \text{m>0} \qquad (2.5)$$

For a logarithmic-type divergence with b>0 integer or real we find:

$$\sum_{n\geq 1}^{[R]}(n+b)^{-1} - 2\int_0^{\infty} dt \frac{t}{(t^2+b^2)(e^{2\pi t}-1)} + \frac{1}{2b} = \int_0^{\infty}\frac{dp}{p+b} \quad (2.6)$$

Where 'R' here means that the series is regularized by the process known as 'Ramanujan resummation' , that can be defined by the formula:

$$\sum_{n\geq 1}^{[R]} a(n) = a(1)+a(2)+a(3)+.......+a(N-1)+\sum_{n\geq 0}^{[R]} a(n+N) - \int_1^N dt a(t) \quad (2.7).$$

putting a(x)=1/x we have $\sum_{n\ge 1}^{[R]} \frac{1}{n} \to \gamma$ as N→∞ (Euler-Mascheroni constant)

So we could get a finite value for the integrals $\int_0^\infty dp(p+\beta)^r$ using Abel-Plana formula and the values $\zeta(-r,\beta)$ for r∈ R>0, for other cases we should make a change of variables to convert the integral with divergence at p=0 into another integral which diverges for p→∞ with the zeta-regularized value

- *Borel generalized resummation of divergent series:*

Although we have studied only the cases with $f(x)=x^m$ (the case for m negative can be handled if we make the change of variable x=1/q) for every analytic function that admits a Laurent series expansion:

$$f(z+\beta)=\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} + C \qquad \text{(C=constant)} \qquad (2.8)$$

Where the coefficients can be calculated, either applying the inverse Z-transform to get every ‘b_n’ , or simply using the Cauchy’s Residue theorem. Now to obtain the regularized value for $\int_0^\infty dx f(x+\beta)$, we can use the term-by-term integration inside (2.8) together with the change of variable x=1/q , Ramanujan resummation and formula (1.12) to obtain a numerical series:

$$\sum_{n=0}^{\infty} d_n \zeta(1-n) \qquad (2.9)$$

Where the d(n) are a known sequence of numbers, the problem with (2.9) comes from the fact that , the negative values of Riemann Zeta function are related to the Bernoulli Numbers, which diverge as the factorial (2n)! , hence (2.9) is clearly divergent, to illustrate the problem, let be the series expansion :

$$g(x)=1-1!x+2!x^2-3!x^3+\dots\dots\dots\dots \qquad g(0)=1 \qquad (2.10)$$

Borel and Euler, managed to get a finite result for (2.10) valid on the positive x-axis in the form of the integral $\int_0^\infty \frac{dt}{1+xt} e^{-t} = g(x)$, a similar result to (2.9) can be hold , so:

$$\int_0^\infty dt e^{-\sqrt{t}} F(t) = B(S) \qquad \sum_{n=0}^{\infty} \frac{d_n \zeta(1-2n)}{2(2n+1)!} t^n = F(t) \qquad (2.11)$$

However, we can only give approximations to accelerate the convergence of the power series (2.11) , using the ‘Euler-Abel’ method to our power series:

$$F(t) \approx \sum_{k=0}^{r-1} (-1)^k \Delta^k \left[(-1)^n \frac{d_n \zeta(1-2n)}{2(2n+1)!} \right]_{n=0} \frac{t^k}{(t+1)^{k+1}} + \dots \qquad (2.12)$$

If we had that $\Delta^{k}\left[(-1)^{n}\dfrac{d_{n}\zeta(1-2n)}{2(2n+1)!}\right]_{n=0}$ tends to 0 as , $k\to\infty$ using only a finite number of terms 'r' inside (2.12) and integrating term by terms we could get the "Borel sum" of our divergent series.

EXAMPLE: Let us calculate the integrals of the Rational functions $\dfrac{A(x)}{B(x)}\approx\dfrac{1}{x^{r}}$ as x tends to '0' and $\dfrac{C(x)}{D(x)}\approx x^{u}$ as x tends to infinity with r and u positive and $r\neq 1$ on the interval $(0,\infty)$ we can separate the divergent and the finite part of the integral in both cases

$$\int_{0}^{\infty}dx\frac{A(x)}{B(x)}\approx\int_{0}^{\varepsilon_0}dx\frac{1}{x^{r}}+\int_{\varepsilon_0}^{\infty}dx\frac{A(x)}{B(x)}=\int_{\varepsilon_0}^{\infty}dx\frac{A(x)}{B(x)}+I(r-2,\Lambda)-\frac{\varepsilon_0^{1-k}}{r-1} \tag{2.13}$$

$$\int_{0}^{\infty}dx\frac{C(x)}{D(x)}\approx\int_{\Delta_0}^{\infty}dxx^{u}+\int_{0}^{\Delta_0}dx\frac{C(x)}{D(x)}=\int_{0}^{\Delta_0}dx\frac{C(x)}{D(x)}+I(u,\Lambda)-\frac{\Delta_0^{u+1}}{u+1} \tag{2.14}$$

Where (Δ_0,ε_0) are finite constant, $\varepsilon_0<<1$ (small) and $\Delta_0>>1$ (big) ,the equations inside (2.13) are just approximations , however they allow us to separate the divergent integral in two parts a finite one that can be computed via Numerical methods and a divergent integral in the form $I(m,\Lambda)$ $\Lambda\to\infty$ that can be computed using the recurrence relation (1.12) . For a more general case with $k=|k|=\dfrac{2\pi}{\lambda}=\sqrt{k_{\mu}k^{\mu}}$ the 'modulus' of momentum vector , if C is a small Volume inside R^{n} including the origin of coordinates and n<u then after integration over the angular variables $d\Omega$ we can split the integral into

$$\int d^{n}kF(\vec{k})\approx\int_{C}\frac{dkd\Omega}{|k|^{u}}+\int_{R^{n}-C}d^{n}kF(\vec{k})\qquad F(\vec{k})\approx\frac{1}{|k|^{u}} \tag{2.15}$$

Equation (2.15) tells that if the function F has a 'pole/divergence' for high wavelength (IR divergence) it can be split into a finite part and a singular integral , making a change of coordinates to these of a n-dimensional sphere we can integrate over the angles and apply the recurrence (1.12) as we did in (2.13-14)

- *Regularization of Dirac delta type distributions:*

From the definition of the derivatives of Dirac delta function in terms of the Fourier transform, we could find the Zeta regularized value:

$$\begin{aligned}&(2\pi)D^{m}\delta(x)=i^{m}\int_{-\infty}^{\infty}dxx^{m}e^{ikx}\\&(2\pi)D^{m}\delta(0)^{[R]}\ \to\ i^{m}(1+(-1)^{m})\left(\int_{-\beta}^{0}dt(t+\beta)^{m}+\varepsilon^{m+1}\zeta_{H}(-m,\beta\varepsilon^{-1}\right)\end{aligned} \tag{2.16}$$

However the case of the regularization involving the product of two distributions is not in general well-defined, if the Convolution theorem always holds no matter what Fourier transform we use ,with 'A' being a renormalization constant then:

$$(2\pi)i^{m+n}D^{m}\delta(\omega)D^{n}\delta(\omega)=AF_{\omega}\left\{\int_{-\infty}^{\infty}dtt^{m}(x-t)^{n}\right\} \qquad (2.17)$$

Using the Binomial theorem (for 'n' positive integer) and the definition of the Fourier transform (in variable 't') for integer powers of 'x' we find:

$$i^{m+n}D^{m}\delta(\omega)D^{n}\delta(\omega)^{[R]} \rightarrow \sum_{k=0}^{n}\binom{n}{k}i^{m+k}AD^{n-k}\delta(\omega)(-1)^{k}i^{n-k}D^{m+k}\delta(0)^{[R]} \qquad (2.18)$$

Where [R] means we must take the 'regularized value of the derivatives involving Dirac delta function given at (2.16) . We could use the recurrence relation (1.8) since

$$(2\pi)D^{m}\delta(0)^{[R]}=i^{m}\int_{-\infty}^{\infty}x^{m}dx=i^{m}\lim_{\Lambda\to\infty}\left(\int_{-\Lambda}^{0}x^{m}dx+\int_{0}^{\Lambda}x^{m}dx\right)=i^{m}\lim_{\Lambda\to\infty}\left(1+(-1)^{m}\right)\int_{0}^{\Lambda}dxx^{m} \qquad (2.19)$$

Note that the constant 'A' depends on the definition of the Fourier transform taken when defining the Harmonic representation of the Dirac delta and its derivatives, in any case the constant 'A' is just a normalization constant and does not provide any relevant physical information .

- *Propagators and Fourier transforms:*

For the case of the Fourier transforms/propagators in the form:

$$F(\omega)=\int_{-\infty}^{\infty}dx\frac{A(x)}{B(x)}e^{i\omega x} \qquad F^{(k)}(0)=\int_{-\infty}^{\infty}dx\frac{A(x)}{B(x)}(ix)^{k} \qquad (2.20)$$

With k=0,1,2,3,4,……….,|B|-1 , here $|B|=d_2$ is the degree of Polynomial B(x) , so |A| > |B| , and the second part inside (2.17) it is an UV divergence.

To obtain the regularized values of the derivative for F(w) at the origin , we can take:

$$F^{(k)\,[R]}(0) \rightarrow \sum_{k=0}^{\infty}\sum_{m=0}^{d}a_{m}u_{k}\varepsilon^{m+k+1}\zeta(-m-k,\beta\varepsilon^{-1})i^{k} \qquad (2.21)$$

If we define $P(D)=a_{0}+a_{1}D+a_{2}D^{2}+.........+a_{n}D^{n}$, then applying the operator $B(-i\partial_{x})$ to F(w) and taking the definition of the derivatives for the Dirac delta :

$$B(-i\partial_{x})F(\omega)=2\pi A(-i\partial_{x})\delta(\omega) \qquad B(-i\partial_{x})K(\omega-s)=\delta(\omega-s) \qquad (2.22)$$

The equation (2.22) is just a linear ODE , 'K' here is just the Kernel of the Cauchy method to solve linear equation, the solution to (2.21) can be expressed as the integral $\int_{0}^{\omega}dsK(w-s)A(-i\partial_{s})\delta(s)$, note that to obtain this K we have used the regularized values of the initial conditions given at (2.21) .

CONCLUSIONS AND FINAL REMARKS:

We have developed a method to deal with UV divergences (for IR divergences a change of variable xq=1 would fit) using the tools of 'Zeta regularization' , we believe this can be important since:

- Zeta regularization is a consistent procedure , and have been used previously with success to regularize infinite sums , in the calculations of 'Casimir effect' and other problems of theoretical physics.
- Zeta regularization of UV divergences is compatible with other renormalization methods commonly used in Physics
- Zeta regularization is a rigorous mathematical procedure (Elizalde et. al)
- Zeta regularization can be applied no matter how many divergent integrals of the form $I(m,\infty)=\int_0^\infty dx x^m$ exist.

The Zeta function regularization of divergent integrals can be made in two ways.

- We use the Euler-McLaurin formula to relate a divergent integral to a divergent series involving the negative values of Riemann Zeta function:

$$I(m,\Lambda)=(m/2)I(m-1,\Lambda)+\zeta(-m)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}a_{mr}(m-2r+1)I(m-2r,\Lambda) \quad (3.1)$$

This recurrence relation is exact for Polynomials , however if m is big the recurrence relation is hard to solve , also we have introduced an 'artificial' cut-off so the divergence only becomes apparent when $\Lambda\to\infty$
Another method is to replace (using the rectangle method) the divergent integral by a divergent series that can be regularized in terms of the negative values of the Hurwitz Zeta function $\zeta_H(-m,\beta\varepsilon^{-1})\varepsilon^{m+1}$, this avoids a recurrence similar to (3.1) , however if the step epsilon $'\varepsilon'$ it is very small , the finite regularized values of the divergences can be numerically unstable due to an ill-possed problem , we think this can be fixed up using a set of finite small steps and extrapolating to the value $\varepsilon\to 0$

Another regularization method to relate divergent series and integrals in case f(x) can not grow faster that exp(2πt) is given by means of 'Abel-Plana' formula:

$$\sum_{n=0}^{\infty}f(n)-\int_0^\infty dx f(x)=\frac{f(0)}{2}+i\int_0^\infty\frac{f(it)-f(-it)}{e^{2\pi t}-1}dt \quad (3.2)$$

Then the first sum can be regularized taking the value s=0 of the Dirichlet series $\sum_{n=0}^{\infty}f(n)n^{-s}=\zeta_f(s)$. For the case of logarithmic divergences the 'ramanujan resummation' technique can provide the finite result $\sum_{n\ge 0}^{[R]}\frac{1}{an+b}\to-\frac{1}{a}\frac{\Gamma'(b/a)}{\Gamma(b/a)}$

For the case of the integral $\int_0^b dxD^n\delta(x\pm\tau)f(x)=(-1)^n D^n f(0)/2=I(\pm\tau)$, for every small and positive 'tau' to avoid problems with the distributions being not defined at x=0 , the result above comes from taking the mean value $\frac{I(\tau)+I(-\tau)}{2}$.

The study of the theory (and possible definition of product) of Distributions is not casual, since many Fourier integrals although divergent can be expressed as distributions for example

$$\int_{-\infty}^{\infty} dxf(x)e^{ikx} = 2\pi\hat{f}(-i\partial_k)\delta(k) \qquad \hat{f}(-i\partial_k)=\sum_{n=0}^{\infty} c_n \frac{(-i\partial_k)^n}{n!} \qquad (3.3)$$

the constants c_n come from the Taylor expansion of f(x) although (3.3) is an infinite series , in many cases can be described as a finite linear combination,

$$\int_{-\infty}^{\infty} dp\frac{1}{p^2-m^2}e^{ipx}=\frac{\pi}{mi}\left(e^{-imx}\int_{-\infty}^{x} d\xi e^{im\xi}\delta(\xi)-e^{imx}\int_{-\infty}^{x} d\xi e^{-im\xi}\delta(\xi)\right) \qquad (3.4)$$

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