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Frederick Ira Moxley III[1]

[1]*Hearne Institute for Theoretical Physics, Department of Physics & Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA*

The Hamiltonian of a quantum mechanical system has an affiliated spectrum. If this spectrum is the sequence of prime numbers, a connection between quantum mechanics and the nontrivial zeros of the Riemann zeta function can be made. In this case, the Riemann zeta function is analogous to chaotic quantum systems, as the harmonic oscillator is for integrable quantum systems. Such quantum Riemann zeta function analogies have led to the Bender-Brody-Müller (BBM) conjecture, which involves a non-Hermitian Hamiltonian that maps to the zeros of the Riemann zeta function. If the BBM Hamiltonian can be shown to be Hermitian, then the Riemann Hypothesis follows. As such, herein we perform a symmetrization procedure of the BBM Hamiltonian to obtain a Hermitian Hamiltonian that maps to the nontrivial zeros of the analytic continuation of the Riemann zeta function, and provide an analytical expression for the eigenvalues of the results using Green's functions. A Gelfand triplet is then used to ensure that the eigenvalues are well defined. The holomorphicity of the resulting eigenvalues is demonstrated, and it is shown that that the expectation value of the Hamiltonian operator is also zero such that the nontrivial zeros of the Riemann zeta function are not observable. Moreover, a second quantization of the resulting Schrödinger equation is performed, and a convergent solution for the nontrivial zeros of the analytic continuation of the Riemann zeta function is obtained. Finally, from the holomorphicity of the eigenvalues it is shown that the real part of every nontrivial zero of the Riemann zeta function exists at $\sigma = 1/2$, and a general solution is obtained by performing an invariant similarity transformation.

I. INTRODUCTION

The unification of number theory with quantum mechanics has been the subject of many research investigations [1–5]. It has been proven that an infinitude of prime numbers exist [6]. In Ref. [7], it was shown that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator correspond to the nontrivial zeros of the Riemann zeta function. If the Riemann Hypothesis is correct [8], the zeros of the Riemann zeta function can be considered as the spectrum of an operator $\hat{R} = \hat{I}/2 + i\hat{H}$, where $\hat{H}$ is a self-adjoint Hamiltonian operator [5, 9], and $\hat{I}$ is identity. Hilbert proposed the Riemann Hypothesis as the eighth problem on a list of significant mathematics problems [10]. Although the BBM Hamiltonian is pseudo-Hermitian [11], it is consistent with the Berry-Keating conjecture [12–14], which states that when $\hat{x}$ and $\hat{p}$ commute, the Hamiltonian reduces to the classical $H = 2xp$. Berry, Keating and Connes proposed the classical Hamiltonian in an effort to map the Riemann zeros to the Hamiltonian spectrum. More recently, the classical Berry-Keating Hamiltonians were quantized, and were found to contain a smooth approximation of the Riemann zeros [15, 16]. This reformulation was found to be physically equivalent to the Dirac equation in Rindler spacetime [17]. Herein, the eigenvalues of the BBM Hamiltonian are taken to be the imaginary parts of the nontrivial zeroes of the analytical continuation of the Riemann zeta function

$$\zeta(s) = \frac{1}{1-2^{1-s}} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{1}$$

where the complex number $s = \sigma + it$, and $\Re(s) > 0$. The idea that the imaginary parts of the zeros of Eq. (1) are given by a self-adjoint operator was conjectured by Hilbert and Pólya [18]. Hilbert and Pólya asserted that the nontrivial zeros of Eq. (1) can be considered as the spectrum of a self-adjoint operator in a suitable Hilbert space. The Hilbert-Pólya conjecture has also found applications in quantum field theories [19]. The Riemann Hypothesis states that the zeros of Eq. (1) on $0 \leq \sigma < 1$ have real part equal to 1/2 [8, 20]. In Ref. [21], Hardy proved that infinitely many zeros are located at $\sigma = 1/2$. According to the Prime Number Theorem [22, 23], no zeros of Eq. (1) can exist at $\sigma = 1$. In this paper we present a Schrödinger equation that maps to the nontrivial zeros of the Riemann zeta function in Sec. II, and evaluate the convergence of the expression by imposing a normalization constraint on the density. A self-adjoint Hamiltonian is derived from the BBM Hamiltonian using a similarity transformation [24, 25], and a second quantization of the resulting Schrödinger equation is then performed to obtain the equations of motion. In Sec. II F, we study the holomorphic eigenvalues of the Riemann zeta function by taking the expectation values of the resulting Schrödinger equation to show that the real part of every nontrivial zero of the analytic continuation of the Riemann zeta function exists at $\sigma = 1/2$. Finally we obtain a general solution to the Riemann zeta Schrödinger equation by performing a similarity transformation in Sec. III, and make concluding remarks in Sec. IV.

A. Preliminaries

Definition 1. *The complex valued function (eigenstate)* $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x): \ \mathbb{X} \to \mathbb{C}$ *is measurable if* $\mathbb{E}$ *is a measurable subset of the measure space* $\mathbb{X}$ *and for each real number* r*, the sets* $\{x \in \mathbb{E}: \ \phi_\sigma(x) > r\}$ *and* $\{x \in \mathbb{E}: \ \phi_t(x) > r\}$ *are measurable for* $\sigma, t \in \mathbb{R}$*.*

Definition 2. *Let* ϕ_s *be a complex-valued eigenstate on a measure space* $\mathbb{X}$*, and* $\phi_s = \phi_\sigma + i\phi_t$*, with* ϕ_σ *and* ϕ_t *real. Therefore,* ϕ_s *is measurable iff* ϕ_σ *and* ϕ_t *are measurable.*

Suppose μ is a measure on the measure space $\mathbb{X}$, and $\mathbb{E}$ is a measurable subset of the measure space $\mathbb{X}$, and ϕ_s is a complex-valued eigenstate on $\mathbb{X}$. It follows that $\phi_s \in (\mathscr{H} = \mathscr{L}(\mu))$ on $\mathbb{E}$, and ϕ_s is complex square-integrable, if ϕ_s is measurable and

$$\int_{\mathbb{E}} \mid \phi_s \mid d\mu < +\infty. \tag{2}$$

Definition 3. *The complex valued function (eigenstate)* $\phi_s = \phi_\sigma + i\phi_t$ *defined on the measurable subset* $\mathbb{E}$ *is said to be integrable if* ϕ_σ *and* ϕ_t *are integrable for* $\sigma, t \in \mathbb{R}$*, where* μ *is a measure on the measure space* $\mathbb{X}$*. The Lebesgue integral of* ϕ_s *is defined by*

$$\int_{\mathbb{E}} \phi_s d\mu = \int_{\mathbb{E}} \phi_\sigma d\mu + i \int_{\mathbb{E}} \phi_t d\mu. \tag{3}$$

Definition 4. *Let* $\mathbb{X}$ *be a measure space, and* $\mathbb{E}$ *be a measurable subset of* $\mathbb{X}$*. Given the complex eigenstate* ϕ_s*, then* $\phi_s \in (\mathscr{H} = \mathscr{L}^2(\mu))$ *on* $\mathbb{E}$ *if* ϕ_s *is Lebesgue measurable and if*

$$\int_{\mathbb{E}} \mid \phi_s \mid^2 d\mu < +\infty, \tag{4}$$

such that ϕ_s *is square-integrable. For* $\phi_s \in (\mathscr{H} = \mathscr{L}^2(\mu))$ *we define the* $\mathscr{L}^2$*-norm of* ϕ_s *as*

$$\parallel \phi_s \parallel_2 = \left(\int_{\mathbb{E}} \mid \phi_s \mid^2 d\mu \right)^{1/2}, \tag{5}$$

where μ *is the measure on the measure space* $\mathbb{X}$*.*

Definition 5. *Let* $\mathbb{X}$ *be a measure space, and* $\mathbb{E}$ *be a measurable subset of* $\mathbb{X}$*. Given the complex eigenstate* ϕ_s*, then* $\phi_s \in (\mathscr{H} = \mathscr{L}^p(\mu))$ *on* $\mathbb{E}$ *if* ϕ_s *is Lebesgue measurable and if*

$$\int_{\mathbb{E}} \mid \phi_s \mid^p d\mu < +\infty, \tag{6}$$

such that ϕ_s *is p-integrable. For* $\phi_s \in (\mathscr{H} = \mathscr{L}^p(\mu))$ *we define the* $\mathscr{L}^p$*-norm of* ϕ_s *as*

$$\parallel \phi_s \parallel_p = \left(\int_{\mathbb{E}} \mid \phi_s \mid^p d\mu \right)^{1/p}, \tag{7}$$

where μ *is the measure on the measure space* $\mathbb{X}$*.*

Definition 6. *A rigged Hilbert space (i.e., a Gelfand triplet [33]) is a triplet* $(\Phi, \mathscr{H}, \Phi^*)$*, where* Φ *is a dense subspace of* $\mathscr{H}$ *and* Φ^* *is its continuous dual space.*

II. RIEMANN ZETA SCHRÖDINGER EQUATION

We consider the eigenvalues of the Hamiltonian

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}), \tag{8}$$

where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$. In Ref. [7], it is conjectured that if the Riemann Hypothesis is correct, the eigenvalues of Eq. (8) are non-degenerate. Next, we let $\Psi_s(x)$ be an eigenfunction of Eq. (8) with an eigenvalue $t = i(2s - 1)$, such that

$$\hat{H} \left| \Psi_s(x) \right\rangle = t \left| \Psi_s(x) \right\rangle, \tag{9}$$

and $x \in \mathbb{R}^+$, $s \in \mathbb{C}$. Solutions to Eq. (9) are given by the analytic continuation of the Hurwitz zeta function

$$\begin{aligned} |\Psi_s(x)\rangle &= -\zeta(s, x+1) \\ &= -\Gamma(1-s)\frac{1}{2\pi i}\oint_C \frac{z^{s-1}e^{(x+1)z}}{1-e^z}dz, \end{aligned} \tag{10}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2s-1)$, $s \in \mathbb{C}$, $\Re(s) \leq 1$, the contour C is a loop around the negative real axis, and Γ is the Euler gamma function for $\Re(s) > 0$

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx. \tag{11}$$

As $-|\Psi_s(x=1)\rangle$ is $1-\zeta(s^*)$, this implies that s belongs to the discrete set of nontrivial zeros of the Riemann zeta function when $s^* = \sigma - it$ and $\sigma = 1/2$, and as $-|\Psi_s(x=-1)\rangle$ is $\zeta(s)$, this implies that s belongs to the discrete set of nontrivial zeros of the Riemann zeta function when $s = \sigma + it$ and $\sigma = 1/2$.

Remark 1. *Solutions to Eq. (9) are symmetric about the origin, i.e.,* $x \in (-\infty, -1] \cup [1, \infty)$, *and subject to the singularity at* $\phi_s(x=0) = 0$ *[27].*

From inserting Eq. (9) into Eq. (8), we have the relation

$$\frac{1}{1-e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1-e^{-i\hat{p}})\,|\Psi_s(x)\rangle = t\,|\Psi_s(x)\rangle\,. \tag{12}$$

Given that Eq. (8) is not Hermitian, it is useful to symmetrize the system. This can be accomplished by letting

$$\begin{aligned} |\phi_s(x)\rangle &= [1-\exp(-\partial_x)]\,|\Psi_s(x)\rangle\,, \\ &= \hat{\Delta}\,|\Psi_s(x)\rangle \\ &= |\Psi_s(x)\rangle - |\Psi_s(x-1)\rangle\,, \end{aligned} \tag{13}$$

and defining a shift operator

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x). \tag{14}$$

For $s > 0$ the only singularity of $\zeta(s, x)$ in the range of $0 \leq x \leq 1$ is located at $x = 0$, behaving as x^{-s}. More specifically,

$$\zeta(s, x+1) = \zeta(s, x) - \frac{1}{x^s}, \tag{15}$$

with $\zeta(s, x)$ finite for $x \geq 1$ [27]. As such, it can be seen from Eq. (13) that the Berry-Keating eigenfunction [12, 13]

$$\begin{aligned} |\phi_s(x)\rangle &= \frac{1}{x^s} \\ &= \exp\Big(\ln(x)(-\sigma - it)\Big) \\ &= \exp\Big(-\sigma\ln(x) - it\ln(x))\Big) \\ &= \exp\Big(-\sigma\ln(x)\Big)\Big(\cos(t\ln(x)) - i\sin(t\ln(x))\Big) \\ &= x^{-\sigma}\Big(\cos(t\ln(x)) - i\sin(t\ln(x))\Big). \end{aligned} \tag{16}$$

Furthermore, the distributional orthonormality relation at $x = 1$ is satisfied such that [30]

$$\langle\phi_s|\phi_{s'}\rangle = \delta_{ss'}. \tag{17}$$

Upon inserting Eq. (13) into Eq. (12) we obtain

$$-i[x\partial_x + \partial_x x]\,|\phi_s(x)\rangle = t\,|\phi_s(x)\rangle\,. \tag{18}$$

Let $\mathscr{H}$ be a Hilbert space, and from Eq. (18) we have the Hamiltonian operator

$$\begin{aligned} \hat{H} &= -i\hbar\Big[x\partial_x + \partial_x x\Big] \\ &= -i\hbar\Big[2x\partial_x + 1\Big], \end{aligned} \tag{19}$$

for $x \in \mathbb{R}$ acting in $\mathscr{H}$, such that

$$\langle \hat{H}f, g\rangle = \langle f, \hat{H}g\rangle \ \forall \ f, g \in \mathscr{D}(\hat{H}). \tag{20}$$

Restricting $x \in \mathbb{R}^+$, Eq. (19) is then written

$$\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}, \tag{21}$$

where $s \in \mathbb{C}$, and $x \in \mathbb{R}^+$. For the Hamiltonian operator as given by Eq. (21), the Hilbert space is $\mathscr{H} = \mathscr{L}^{p=2}(1, \infty)$ [28–30]. We then impose on Eq. (21) the following minimal requirements, such that its domain is not too artificially restricted.

i $\hat{H}$ is a symmetric (Hermitian) linear operator;

ii $\hat{H}$ can be applied on all functions of the form

$$g(x, s) = P(x, s)\exp\Big(-\frac{x^2}{2}\Big), \tag{22}$$

where P is a polynomial of x and s. Here, it should be pointed out that $\hat{H} = \hat{T} + \hat{V}$, and from Eq. (19), it can be seen that $\hat{T} = -2i\hbar x\partial_x$, $\hat{V} = -i\hbar$. From (ii), $\hat{V}g(x, s)$ must belong to the Hilbert space $\mathscr{H} = \mathscr{L}^2$ defined over the space x. This is guaranteed as

$$|-i\hbar| \leq \hbar, \tag{23}$$

where $\hbar$ is the reduced Planck constant or Dirac constant. The domain $\mathscr{D}_{\hat{V}}$ of the potential energy $\hat{V}$ consists of all $\phi \in \mathscr{H}$ for which $\hat{V}\phi \in \mathscr{H}$. As such, $\hat{V}$ is self-adjoint. It is not necessary to specify the domain of Eq. (21), as it is only necessary to admit that Eq. (21) is defined on a certain $\mathscr{D}_{\hat{H}}$ such that (i) and (ii) are satisfied. If we denote by $\mathscr{D}_1$ the set of all functions in Eq. (22), then (ii) implies that $\mathscr{D}_{\hat{H}} \supseteq \mathscr{D}_1$. By letting $\hat{H}_1$ be the contraction of $\hat{H}$ with domain $\mathscr{D}_1$, i.e., $\hat{H}$ is an extension of $\hat{H}_1$, and letting $\tilde{H}_1$ be the closure of $\hat{H}$, it can be seen that $\tilde{H}_1$ is self-adjoint. Since $\hat{H}$ is symmetric and $\hat{H} \supseteq \hat{H}_1$, i.e., $\hat{H}$ is an extension of $\hat{H}_1$, it follows that $\tilde{H} = \tilde{H}_1$ and $\hat{H}$ is essentially self-adjoint, where $\tilde{H}$ is the unique self-adjoint extension [31]. Other than eigenfunctions $\phi_s(x)$ in configuration space as seen in Eq. (16), it is useful to represent eigenfunctions in momentum space $\Phi_s(p)$. The transformation between configuration space eigenfunctions and momentum space eigenfunctions can be obtained via Plancherel transforms [32], where the one-to-one correspondence $\phi_s(x) \rightleftharpoons \Phi_s(p)$ is linear and isometric.

A. Green's function

In order to obtain eigenstates that are orthonormal when $x \neq 1$, as seen in Eq. (17), we begin by writing Eq. (21) as the eigenvalue equation

$$-2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x) = t\phi_s(x). \tag{24}$$

Dividing by $-2i\hbar$ on both sides and rearranging the terms, we obtain

$$\phi_s' + \frac{1}{x}\frac{t}{2i\hbar}\phi_s = -\frac{1}{2x}\phi_s. \tag{25}$$

This can be written as

$$\phi_s' + k^2 = Q, \tag{26}$$

where

$$k \equiv \sqrt{\frac{t}{2i\hbar x}}, \tag{27}$$

and

$$Q \equiv -\frac{1}{2x}\phi_s. \tag{28}$$

Therefore, we can express Eq. (24) as

$$(\partial_x + k^2)\phi_s = Q. \tag{29}$$

In order to solve an inhomogeneous differential equation such as Eq. (29), we can find a Green's function that uses a delta function source, viz.,

$$(\partial_x + k^2)G(x) = \delta(x), \tag{30}$$

where the delta potential is given by

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{31}$$

It then follows from Eq. (30) that we can express ϕ_s as an integral to obtain $Q(x)$, i.e.,

$$\phi_s(x) = \int G(x - x_0)Q(x_0)dx_0, \tag{32}$$

and it must satisfy

$$\begin{aligned} (\partial_x + k^2)\phi_s(x) &= \int \Big[(\partial_x + k^2)G(x - x_0)\Big]Q(x_0)dx_0 \\ &= \int \delta(x - x_0)Q(x_0)dx_0 = Q(x). \end{aligned} \tag{33}$$

In order to obtain the Green's function $G(x)$ such that a solution to Eq. (30) can be obtained, we take the Fourier transform which turns the differential equation into an algebraic one, like

$$G(x) = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x)g(\omega)d\omega, \tag{34}$$

where $g(\omega)$ is the projection, and $\exp(i\omega x)$ is the complete basis set. Upon inserting Eq. (34) into Eq. (30), we obtain

$$(\partial_x + k^2)G(x) = \frac{1}{\sqrt{2\pi}} \int g(\omega)(\partial_x + k^2)\exp(i\omega x)d\omega = \delta(x). \tag{35}$$

However, since

$$\partial_x \exp(i\omega x) = i\omega \exp(i\omega x), \tag{36}$$

and

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x)d\omega, \tag{37}$$

Eq. (30) can be expressed as

$$\frac{1}{\sqrt{2\pi}} \int (i\omega + k^2)\exp(i\omega x)g(\omega)d\omega = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x)d\omega, \tag{38}$$

where

$$g(\omega) = \frac{1}{\sqrt{2\pi}(i\omega + k^2)}. \tag{39}$$

Hence we have poles at

$$k = \pm\sqrt{i\omega}. \tag{40}$$

Now consider the contour integral

$$\frac{1}{\sqrt{2\pi}} \int_C f(z)dz = \frac{1}{\sqrt{2\pi}} \int_C \frac{\exp(izx)}{(iz + k^2)} dz. \tag{41}$$

Since $\exp(izx)$ is an entire function, Eq. (41) has singularities only at the poles, as given in Eq. (40), i.e., $z = ik^2$. As $f(z)$ is

$$\frac{\exp(izx)}{(iz + k^2)} = \frac{\exp(izx)}{i} \frac{1}{(z - ik^2)}, \tag{42}$$

the residue of $f(z)$ at $z = ik^2$ is

$$\mathrm{Res}_{z=ik^2} f(z) = \frac{\exp(-k^2 x)}{i}. \tag{43}$$

According to the residue theorem, we then obtain

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_C f(z)dz &= \frac{2\pi i}{\sqrt{2\pi}} \mathrm{Res}_{z=ik^2} f(z) \\ &= \sqrt{2\pi} \exp(-k^2 x) = G(x). \end{aligned} \tag{44}$$

Hence, the most general solution to Eq. (30) is

$$\phi_s(x) = \sqrt{2\pi} \int \exp(-k^2 x_0) \Big(-\frac{1}{2x_0} \phi_s(x_0) \Big) dx_0. \tag{45}$$

From Eq. (16) it can be seen that $\phi_s(x_0) = x_0^{-s}$. As such,

$$\begin{aligned} \phi_s(x) &= -\sqrt{2\pi} \int \exp(-k^2 x_0) \Big(\frac{x_0^{-s-1}}{2} \Big) dx_0 \\ &= -\sqrt{\frac{\pi}{2}} \int \frac{\exp(-k^2 x_0)}{x_0^{s+1}} dx_0 \\ &= -\sqrt{\frac{\pi}{2}} \int \frac{\exp(-\frac{t x_0}{2i\hbar x})}{x_0^{s+1}} dx_0 \\ &= -\sqrt{\frac{\pi}{2}} \int \cos\Big(\frac{t x_0}{2\hbar x}\Big) \frac{1}{x_0^{s+1}} dx_0 - i\sqrt{\frac{\pi}{2}} \int \sin\Big(\frac{t x_0}{2\hbar x}\Big) \frac{1}{x_0^{s+1}} dx_0, \end{aligned} \tag{46}$$

Moreover, by using Eq. (16) it can be seen that

$$\begin{aligned} \int \cos\Big(\frac{t x_0}{2\hbar x}\Big) \frac{1}{x_0^{s+1}} dx_0 = &\int \cos\Big(\frac{t x_0}{2\hbar x}\Big) \frac{x_0^{-\sigma}}{x_0} \cos\Big(t \ln(x_0)\Big) dx_0 \\ &- i \int \cos\Big(\frac{t x_0}{2\hbar x}\Big) \frac{x_0^{-\sigma}}{x_0} \sin\Big(t \ln(x_0)\Big) dx_0, \end{aligned} \tag{47}$$

and

$$\int \sin\left(\frac{tx_0}{2\hbar x}\right)\frac{1}{x_0^{s+1}}dx_0 = \int \sin\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\sin\Big(t\ln(x_0)\Big)dx_0 + i\int \sin\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\cos\Big(t\ln(x_0)\Big)dx_0. \tag{48}$$

Since $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$, it can be seen that

$$\begin{aligned}\phi_\sigma(x) &= -\sqrt{\frac{\pi}{2}}\int \cos\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\cos\Big(t\ln(x_0)\Big)dx_0 - \sqrt{\frac{\pi}{2}}\int \sin\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\sin\Big(t\ln(x_0)\Big)dx_0 \\ &= -\sqrt{\frac{\pi}{2}}\int x_0^{-\sigma-1}\cos\Big(ix_0k^2 - t\log(x_0)\Big)\,dx_0 \\ &= -\sqrt{\frac{\pi}{2}}\int x_0^{-\sigma-1}\Big(\cosh\Big(k^2x_0\Big)\cos(t\log(x_0)) + i\sinh\Big(k^2x_0\Big)\sin(t\log(x_0))\Big)\,dx_0,\end{aligned} \tag{49}$$

and

$$\begin{aligned}\phi_t(x) &= \sqrt{\frac{\pi}{2}}\int \cos\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\sin\Big(t\ln(x_0)\Big)dx_0 - \sqrt{\frac{\pi}{2}}\int \sin\left(\frac{tx_0}{2\hbar x}\right)\frac{x_0^{-\sigma}}{x_0}\cos\Big(t\ln(x_0)\Big)dx_0 \\ &= -\sqrt{\frac{\pi}{2}}\int x_0^{-\sigma-1}\sin\Big(ix_0k^2 - t\log(x_0)\Big)\,dx_0 \\ &= -\sqrt{\frac{\pi}{2}}\int x_0^{-\sigma-1}\Big(-\cosh\Big(k^2x_0\Big)\sin(t\log(x_0)) + i\sinh\Big(k^2x_0\Big)\cos(t\log(x_0))\Big)\,dx_0.\end{aligned} \tag{50}$$

Here, we can use the identities

$$\cos\Big(t\log(x_0)\Big) = \frac{1}{2}x_0^{-it} + \frac{1}{2}x_0^{it}, \tag{51}$$

and

$$\sin\Big(t\log(x_0)\Big) = \frac{i}{2}x_0^{-it} - \frac{i}{2}x_0^{it}, \tag{52}$$

to rewrite Eqs. (49)-(50) as

$$\begin{aligned}\phi_\sigma(x) &= -\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} x_0^{-\sigma-1}\cos\Big(t\log(x_0)\Big)\exp(-k^2x_0)dx_0 \\ &= -\frac{1}{2}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} e^{-k^2x_0}\left(1 + x_0^{2it}\right)x_0^{-\sigma-it-1}dx_0,\end{aligned} \tag{53}$$

and

$$\begin{aligned}\phi_t(x) &= \sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} x_0^{-\sigma-1}\sin\Big(t\log(x_0)\Big)\exp(-k^2x_0)dx_0 \\ &= -\frac{1}{2}i\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} e^{-k^2x_0}\left(-1 + x_0^{2it}\right)x_0^{-\sigma-it-1}dx_0.\end{aligned} \tag{54}$$

Taking $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$, we arrive at the expression using Eq. (27)

$$\begin{aligned}\phi_s(x) &= \sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty}\left(-e^{-k^2x_0}\right)x_0^{-\sigma-it-1}dx_0 \\ &= \frac{\sqrt{\frac{\pi}{2}}\left(k^* - ik\right)e^{-\frac{1}{2}\pi(t+3i\sigma)}\left(e^{2\pi t} - e^{2i\pi\sigma}\right)\left(-k^4\right)^{\frac{1}{2}(\sigma+it)}\Gamma(-it-\sigma)}{2k^*} \\ &= \frac{\sqrt{\pi}2^{-\sigma-it-\frac{3}{2}}e^{-\frac{1}{2}\pi(t+3i\sigma)}\left(e^{2\pi t} - e^{2i\pi\sigma}\right)\left(t - x\sqrt{\frac{t^2}{x^2}}\right)\Gamma(-it-\sigma)\left(\frac{t^2}{x^2}\right)^{\frac{1}{2}(\sigma+it)}}{t} = 0\;\forall\; x \in \mathbb{R}^+_{\geq 1}.\end{aligned} \tag{55}$$

Hence, the nontrivial zeros of the Riemann zeta function can be considered as the spectrum of an operator $\hat{R} = \hat{I}/2 + i\hat{H}$, where $\hat{H}$ is a self-adjoint Hamiltonian operator [5, 9], and $\hat{I}$ is identity, such that

$$\begin{aligned}\langle \hat{R} \rangle &= \hat{I}/2 + i \langle \hat{H} \rangle \\ &= \hat{I}/2\end{aligned} \tag{56}$$

and the eigenvalues $\langle \hat{H} \rangle = t$ are not observable, as seen from Eq. (55).

B. Measure

Theorem 1. *The eigenstate* $\phi_s(x) = x^{-s} : \mathbb{X} \to \mathbb{C}$ *is measurable. That is,* $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$ *where* $\phi_\sigma, \phi_t : \mathbb{E} \to (-\infty, -1] \cup [1, \infty)$ *are measurable for* $s = \sigma + it$ *and* $\sigma, t \in \mathbb{R}$.

Proof. Owing to the one-to-one correspondence obtained from Plancherel transforms between configuration space and momentum space eigenstates, it can be seen that

$$\begin{aligned}\Phi_s(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(x) \exp(-ipx) dx \\ &= \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2} i\pi s\Big)(\text{sgn}(p) + 1) \sin(\pi s)\Gamma(1-s) \left|p\right|^{s-1},\ 0 < \Re(s) < 1.\end{aligned} \tag{57}$$

and

$$\phi_s(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_s(p) \exp(ipx) dp. \tag{58}$$

Since

$$\| \phi_s \|_1 \equiv \int_{-\infty}^{-1} | \phi_s(x) | \, dx + \int_{1}^{\infty} | \phi_s(x) | \, dx = \int_{-\infty}^{-1} | \Phi_s(p) | \, dp + \int_{1}^{\infty} | \Phi_s(p) | \, dp \equiv \| \Phi_s \|_1, \tag{59}$$

from which

$$\| \Phi_s \|_1 = \| \phi_s \|_1 = -\frac{1}{s\pi^{1/2}} \exp\Big(\frac{1}{2}\pi \Im(s)\Big) \sqrt{\sin^2(\pi s)} \sqrt{\Gamma(1-s)^2}. \tag{60}$$

It then follows that ϕ_s is complex square-integrable, i.e.,

$$\phi_s(x) \in \mathscr{H} \iff \int_{\mathbb{E}} |\phi_s(x)| d\mu < +\infty. \tag{61}$$

□

Theorem 2. *Let the complex valued eigenstate* $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$ *where* $s = \sigma + it$*, and let the measurable subset* $\mathbb{E} \to [1, \infty)$*. The* $\mathscr{H} = \mathscr{L}^2$*-norm of the complex-valued eigenstate* $\phi_s = x^{-s}$ *is* ∞*, i.e.,* ϕ_s *is not* $p = 2$ *integrable at* $\sigma = 1/2$.

Proof. Owing to the one-to-one correspondence obtained from Plancherel transforms between configuration space and momentum space eigenstates, it can be seen that

$$\begin{aligned}\Phi_s(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(x) \exp(-ipx) dx \\ &= \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2} i\pi s\Big)(\text{sgn}(p) + 1) \sin(\pi s)\Gamma(1-s) \left|p\right|^{s-1},\ 0 < \Re(s) < 1.\end{aligned} \tag{62}$$

and

$$\phi_s(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_s(p) \exp(ipx) dp, \tag{63}$$

where

$$\phi_\sigma(x) = (x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \cos\Big(\sigma \cdot \arg(x) + \frac{t}{2}\log(x^2)\Big), \tag{64}$$

and

$$\phi_t(x) = -(x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \sin\Big(\sigma \cdot \arg(x) + \frac{t}{2}\log(x^2)\Big) \tag{65}$$

for $x \in \mathbb{R}^+_{\geq 1}$. Since

$$\| \phi_s \|_p = \Big(\int_1^\infty | \phi_s(x) |^p \, dx\Big)^{\frac{1}{p}}, \tag{66}$$

and[40]

$$\| \Phi_s \|_p = \Big(\int_1^\infty | \Phi_s(p) |^p \, dp\Big)^{\frac{1}{p}}, \tag{67}$$

from which

$$\| \Phi_s \|_p = \| \phi_s \|_p = \Big(\frac{1}{p\sigma - 1}\Big)^{\frac{1}{p}}. \tag{68}$$

It then follows that as $\sigma \to 1/2$,

$$\| \Phi_s \|_p = \| \phi_s \|_p = \Big(\frac{1}{\frac{p}{2} - 1}\Big)^{\frac{1}{p}}, \tag{69}$$

such that the $\mathscr{L}^{p=2}$-norm of ϕ_s is of indeterminant form. Furthermore, it can be seen from

$$\lim_{p\to 2} \Big(\frac{1}{\frac{p}{2} - 1}\Big)^{\frac{1}{p}}, \tag{70}$$

and letting

$$y = \Big(\frac{1}{\frac{p}{2} - 1}\Big)^{\frac{1}{p}}, \tag{71}$$

then

$$\begin{aligned} \ln(y) &= \frac{1}{p}\ln\Big(\frac{1}{\frac{p}{2} - 1}\Big) \\ &= \frac{1}{p}\Big(\ln(1) - \ln\Big(\frac{p}{2} - 1\Big)\Big) \\ &= -\frac{1}{p}\ln\Big(\frac{p}{2} - 1\Big), \end{aligned} \tag{72}$$

and

$$\begin{aligned} \lim_{p\to 2} \ln(y) &= \lim_{p\to 2}\Big(- \frac{1}{p}\ln\Big(\frac{p}{2} - 1\Big)\Big) \\ &= \infty. \end{aligned} \tag{73}$$

Exponentiating both sides, we obtain

$$\begin{aligned} \exp\Big[\lim_{p\to 2} \ln(y)\Big] &= \lim_{p\to 2}\Big[\exp\Big(\ln(y)\Big)\Big] \\ &= \lim_{p\to 2} y = \exp(\infty) = \infty, \end{aligned} \tag{74}$$

such that we obtain the infinite density [24]

$$\| \Phi_s \|_{p=2} = \| \phi_s \|_{p=2} = \infty. \tag{75}$$

□

Corollary 1. *Let $\mathscr{H} = \mathscr{L}^2[1,\infty)$ and consider the Hamiltonian observable given by*

$$\hat{H}\phi_s(x) = -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x). \tag{76}$$

Although the action of $\hat{H}$ is in principle well-defined for all $\phi_s(x) \in \mathscr{L}^2$, there are functions which are in $\mathscr{L}^2$, but for which $\hat{H}\phi_s(x)$ is no longer an element of $\mathscr{L}^2$, e.g., when $\sigma = 1/2$,

$$\phi_{\frac{1}{2}+it}(x) = \frac{e^{t\arg(x)}\cos\left(\frac{\arg(x)}{2} + \frac{1}{2}t\log\left(x^2\right)\right)}{\sqrt[4]{x^2}} - \frac{ie^{t\arg(x)}\sin\left(\frac{\arg(x)}{2} + \frac{1}{2}t\log\left(x^2\right)\right)}{\sqrt[4]{x^2}}. \tag{77}$$

Therefore the domain of $\hat{H}$ is given by

$$\mathscr{D}(\hat{H}) = \left\{\phi_s(x) \in \mathscr{L}^2 : \int_{-\infty}^{-1}\left| -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\right|^2 dx + \int_1^{\infty}\left| -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\right|^2 dx < \infty\right\} \subset \mathscr{L}^2. \tag{78}$$

Similarly, the domain of $\hat{H}^2$ is

$$\mathscr{D}(\hat{H}^2) = \left\{\phi_s(x) \in \mathscr{L}^2 : \int_{-\infty}^{-1}\left| -4\hbar^2 x\partial_x^2 x\phi_s(x)\right|^2 dx + \int_1^{\infty}\left| -4\hbar^2 x\partial_x^2 x\phi_s(x)\right|^2 dx < \infty\right\} \subset \mathscr{D}(\hat{H}), \tag{79}$$

etc. As such, we define

$$\Phi \equiv \bigcap_{n=0}^{\infty} \mathscr{D}(\hat{H}^n), \tag{80}$$

such that for every $\phi_s(x) \in \Phi$, the solution is well-defined at $\sigma = 1/2$.

Eqs. (57) and (58) are two vector representations of the same Hilbert space $\mathscr{H} = \mathscr{L}^{p=2}(1,\infty)$. From Eq. (19), it can be seen that

$$\hat{T} = -2i\hbar x\partial_x, \tag{81}$$

such that we define a multiplicative operator $\hat{T}_0$ in momentum space $(\hat{T}_0\Phi_s)(p) = \hat{T}_0(p)\Phi_s(p)$, where

$$\hat{T}_0(p) = 2\hat{x}\hat{p}. \tag{82}$$

Here, it should be pointed out that as $\hat{x} = i\hbar d/dp$, Eq. (82) reduces to

$$\hat{T}_0(p) = 2i\hbar, \tag{83}$$

and Eq. (19) is then rewritten in momentum space as $\hat{H}(p) = i\hbar$. The domain $\mathscr{D}_0$ of $\hat{T}_0$ is defined as the set of all functions $\Phi_s(p) \in \mathscr{H}$ such that $\hat{T}_0(p)\Phi_s(p) \in \mathscr{H}$. As such, $\hat{T}_0$ is definitively self-adjoint. From Eq. (22) we have defined the set $\mathscr{D}_1$ of functions in configuration space. From the Plancherel transform [32] of Eq. (22), we obtain the set $\mathscr{D}_1$ of functions in momentum space having the form

$$G(p,s) = P(p,s)\exp\left(-\frac{p^2}{2}\right), \tag{84}$$

where P is a polynomial of p and s. Eqs. (57) and (58) are true if $\phi_s(x) \in \mathscr{D}_1$ or $\Phi_s(p) \in \mathscr{D}_1$ and since $\Phi_s(p) \in \mathscr{D}_1 \to 0$ as $p \to \infty$ then $\mathscr{D}_1 \subseteq \mathscr{D}_0$. Moreover, for $\phi \in \mathscr{D}_1$, $\hat{T}_0$ coincides with Eq. (81) [31]. Using Eq. (57) and $\hat{H}(p) = i\hbar$, the eigenrelation

$$\hat{H}(p)\left|\Phi_s(p)\right\rangle = \lambda\left|\Phi_s(p)\right\rangle \tag{85}$$

is obtained. In order to find the expectation value for $\hat{H}$ we take the complex conjugate of Eq. (85), set $\hbar = 1$, multiply by the eigenfunction $\Phi_s(p)$, and then integrate over p to obtain

$$\int_{-\infty}^{\infty}\left(i\frac{e^{-\frac{1}{2}i\pi s}(\mathrm{sgn}(p)+1)\sin(\pi s)\Gamma(1-s)\left|p\right|^{s-1}}{2\pi^{1/2}}\right)^*\left(\frac{e^{-\frac{1}{2}i\pi s}(\mathrm{sgn}(p)+1)\sin(\pi s)\Gamma(1-s)\left|p\right|^{s-1}}{2\pi^{1/2}}\right)dp = \lambda^* \parallel \Phi_s \parallel_p, \tag{86}$$

where λ is the eigenvalue.

Theorem 3. *Let the complex valued eigenstate $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$ where $s = \sigma + it$, and let the measurable subset $\mathbb{E} \to (-\infty, -1] \cup [1, \infty)$. The following are equivalent for $\sigma, t \in \mathbb{R}$:*

1. *For each real number r, the set $\{x \in \mathbb{E} : \phi_\sigma(x) > r\}$ is measurable.*
2. *For each real number r, the set $\{x \in \mathbb{E} : \phi_t(x) > r\}$ is measurable.*
3. *For each real number r, the set $\{x \in \mathbb{E} : \phi_\sigma(x) \geq r\}$ is measurable.*
4. *For each real number r, the set $\{x \in \mathbb{E} : \phi_t(x) \geq r\}$ is measurable.*
5. *For each real number r, the set $\{x \in \mathbb{E} : \phi_\sigma(x) < r\}$ is measurable.*
6. *For each real number r, the set $\{x \in \mathbb{E} : \phi_t(x) < r\}$ is measurable.*
7. *For each real number r, the set $\{x \in \mathbb{E} : \phi_\sigma(x) \leq r\}$ is measurable.*
8. *For each real number r, the set $\{x \in \mathbb{E} : \phi_t(x) \leq r\}$ is measurable.*

Proof. Note that the intersection of sets,

$$\{x \in \mathbb{E} : \phi_\sigma(x) \geq r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_\sigma(x) > r - \frac{1}{n}\}, \tag{87}$$

$$\{x \in \mathbb{E} : \phi_t(x) \geq r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_t(x) > r - \frac{1}{n}\}, \tag{88}$$

$$\{x \in \mathbb{E} : \phi_\sigma(x) > r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_\sigma(x) \geq r + \frac{1}{n}\}, \tag{89}$$

$$\{x \in \mathbb{E} : \phi_t(x) > r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_t(x) \geq r + \frac{1}{n}\}, \tag{90}$$

where

$$\phi_\sigma(x) = (x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \cos\Big(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\Big), \tag{91}$$

and

$$\phi_t(x) = -(x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \sin\Big(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\Big). \tag{92}$$

□

Theorem 4. *Let $\mathbb{E} \to (-\infty, -1] \cup [1, \infty)$ be a measurable subset of the measure space $\mathbb{X}$. If the complex valued eigenstate $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$ where $s = \sigma + it$, and $\phi_\sigma(x)$,and ϕ_t are continuous a.e. on $\mathbb{E}$, then $\phi_s(x)$ is measurable for $\sigma, t \in \mathbb{R}$.*

Proof. Let $\mathcal{D}$ be the singleton $\{0\}$ owing to the singularity at $x = 0$ of $\phi_s(x) = x^{-s}$. Then $\mu(\mathcal{D}) = 0$ and all of its subsets are measurable. Let $r \in \mathbb{R}$ and note that

$$\{x \in \mathbb{E} : \phi_\sigma(x) > r\} = \{x \in \mathbb{E} - \mathcal{D} : \phi_\sigma(x) > r\} \cup \{x \in \mathcal{D} : \phi_\sigma(x) > r\}, \tag{93}$$

where

$$\phi_\sigma(x) = (x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \cos\Big(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\Big), \tag{94}$$

and

$$\phi_t(x) = -(x^2)^{-\sigma/2} \exp\Big(t \cdot \arg(x)\Big) \sin\Big(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\Big). \tag{95}$$

Letting

$$C_\sigma = \{x \in \mathbb{E} - \mathcal{D} :\ \phi_\sigma(x) > r\}, \tag{96}$$

for each $x \in C_\sigma$, as $\phi_\sigma(x)$ is continuous at x, we can find $\delta_x > 0$ such that if $y \in V_{\delta_x}(x)$ then $\phi_\sigma(y) > r$. It can be seen that $\phi_\sigma(x)$ is measurable, since

$$C_\sigma = (\mathbb{E} - \mathcal{D}) \bigcap_{x \in C_\sigma} V_{\delta_x}(x). \tag{97}$$

Similarly, noting that

$$\{x \in \mathbb{E} :\ \phi_t(x) > r\} = \{x \in \mathbb{E} - \mathcal{D} :\ \phi_t(x) > r\} \cup \{x \in \mathcal{D} :\ \phi_t(x) > r\}, \tag{98}$$

and letting

$$C_t = \{x \in \mathbb{E} - \mathcal{D} :\ \phi_t(x) > r\}, \tag{99}$$

for each $x \in C_t$, as $\phi_t(x)$ is continuous at x, we can find $\delta_x > 0$ such that if $y \in V_{\delta_x}(x)$ then $\phi_t(y) > r$. It can be seen that $\phi_t(x)$ is measurable since

$$C_t = (\mathbb{E} - \mathcal{D}) \bigcap_{x \in C_t} V_{\delta_x}(x). \tag{100}$$

□

Let $\{\phi_s\} = \{\phi_\sigma\} + i\{\phi_t\}$ be a sequence of functions defined on the measure space $\mathbb{X} \to \mathbb{C}$. Denoting

$$\sup_s \phi_s(x) = \sup\{\phi_s(x) :\ s \in \mathbb{C}\} \tag{101}$$

and

$$\limsup_s \phi_s(x) = \lim_s \Big(\sup_{k \geq s} \phi_k(x) \Big), \tag{102}$$

it can be seen that

$$\limsup_s \phi_s(x) = \inf_s \Big(\sup_{k \geq s} \phi_k(x) \Big). \tag{103}$$

Similarly, from

$$\inf_s \phi_s(x) = \inf\{\phi_s(x) :\ s \in \mathbb{C}\} \tag{104}$$

and

$$\liminf_s \phi_s(x) = \lim_s \Big(\inf_{k \geq s} \phi_k(x) \Big), \tag{105}$$

it can be seen that

$$\inf_s \phi_s(x) = -\sup_s \Big(-\phi_s(x) \Big), \tag{106}$$

and

$$\liminf_s \phi_s(x) = -\limsup_s \Big(-\phi_s(x) \Big). \tag{107}$$

Theorem 5. *Let the sequence of measurable eigenstates $\{\phi_s\} = \{\phi_\sigma\} + i\{\phi_t\}$ be defined on the measure space $\mathbb{X} \to \mathbb{C}$. For the sequence of measurable eigenstates $\{\phi_\sigma\} :\ \mathbb{E} \to (-\infty, -1] \cup [1, \infty)$*

$$g(x) = \sup_\sigma \phi_\sigma(x), \tag{108}$$

and

$$h(x) = \limsup_\sigma \phi_\sigma(x), \tag{109}$$

such that g and h are measurable for $x \in \mathbb{E}$.

Proof. For any $r \in \mathbb{R}$, we obtain

$$\{x \in \mathbb{E}:\ g(x) > r\} = \bigcup_{\sigma} \{x \in \mathbb{E}:\ \phi_\sigma(x) > r\}. \tag{110}$$

From Eqs. (103) and (106)-(107), this implies that h is also measurable. □

Corollary 2. *Let ϕ_σ be a sequence of measurable eigenstates defined on the measure space $\mathbb{X}$, and $\phi_\sigma :\ \mathbb{E} \to (-\infty, -1] \cup [1, \infty)$. Since $\{\phi_\sigma\}$ converges pointwise to ϕ_σ a.e. on $\mathbb{E}$, then ϕ_σ is measurable.*

Theorem 6. *Let the sequence of measurable eigenstates $\{\phi_s\} = \{\phi_\sigma\} + i\{\phi_t\}$ be defined on the measure space $\mathbb{X} \to \mathbb{C}$. For the sequence of measurable eigenstates $\{\phi_t\} :\ \mathbb{E} \to (-\infty, -1] \cup [1, \infty)$*

$$g(x) = \sup_t \phi_t(x), \tag{111}$$

and

$$h(x) = \limsup_t \phi_t(x), \tag{112}$$

such that g and h are measurable for $x \in \mathbb{E}$.

Proof. For any $r \in \mathbb{R}$, we obtain

$$\{x \in \mathbb{E}:\ g(x) > r\} = \bigcup_{t} \{x \in \mathbb{E}:\ \phi_t(x) > r\}. \tag{113}$$

From Eqs. (103) and (106)-(107), this implies that h is also measurable. □

Corollary 3. *Let ϕ_t be a sequence of measurable eigenstates defined on the measure space $\mathbb{X}$, and $\phi_t :\ \mathbb{E} \to (-\infty, -1] \cup [1, \infty)$. Since $\{\phi_t\}$ converges pointwise to ϕ_t a.e. on $\mathbb{E}$, then ϕ_t is measurable.*

Corollary 4. *Let $\phi_s = \phi_\sigma + i\phi_t$ be a sequence of measurable eigenstates defined on the measure space $\mathbb{X} \to \mathbb{C}$. Since $\{\phi_\sigma\}$ converges pointwise to ϕ_σ a.e. on $\mathbb{E} \to (-\infty, -1] \cup [1, \infty)$, and $\{\phi_t\}$ converges pointwise to ϕ_t a.e. on $\mathbb{E} \to (-\infty, -1] \cup [1, \infty)$, then ϕ_s is measurable.*

C. Expectation Value of the Observable

Definition 7. *The Riemann zeta Schrödinger equation is*

$$-\hbar\partial_s |\Psi_s(x)\rangle = i\left[\hat{\Delta}^{-1}\hat{x}\hat{p}\hat{\Delta} + \hat{\Delta}^{-1}\hat{p}\hat{x}\hat{\Delta}\right]|\Psi_s(x)\rangle, \tag{114}$$

where $\hat{\Delta} = 1 - \exp(-\partial_x)$, $\hat{x} = x$, $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, $x \in \mathbb{R}^+ \geq 1$ owing to the difference operator $\hat{\Delta}\,|\Psi_s(x)\rangle$, and $s \in \mathbb{C}$.

Upon inserting Eq. (13) into Eq. (114) for $x \in \mathbb{R}^+$, we obtain the symmetrized Riemann zeta Schrödinger equation, i.e.,

$$\begin{aligned} \partial_s |\phi_s(x)\rangle &= 1/2(\partial_\sigma - i\partial_t)\,|\phi_s(x)\rangle \\ &= -\frac{2}{\hbar}\sqrt{x}\partial_x\sqrt{x}\,|\phi_s(x)\rangle. \end{aligned} \tag{115}$$

Theorem 7. *Let the complex-valued eigenstate*

$$\phi_s(x) = \frac{\sqrt{\pi}2^{-\sigma-it-\frac{3}{2}}e^{-\frac{1}{2}\pi(t+3i\sigma)}\left(e^{2\pi t} - e^{2i\pi\sigma}\right)\left(t - x\sqrt{\frac{t^2}{x^2}}\right)\Gamma(-it-\sigma)\left(\frac{t^2}{x^2}\right)^{\frac{1}{2}(\sigma+it)}}{t}, \tag{116}$$

where $s = \sigma + it$ and $\sigma, t \in \mathbb{R}$, and let the measurable subset of the measure space $\mathbb{X}$ be $\mathbb{E} \to (-\infty, -1] \cup [1, \infty)$, for the Hamiltonian operator $\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}$.

Proof. Let $|\phi_s(x)\rangle$ be an eigenstate of $\hat{H}$ with eigenvalue t, i.e.,

$$\hat{H}\,|\phi_s(x)\rangle = t\,|\phi_s(x)\rangle\,. \tag{117}$$

In order to find the expectation value of $\hat{H}$ we multiply $\hat{H}$ by the eigenstate, take the complex conjugate, and then multiply the result by the eigenstate and integrate over $\mathbb{E}$ to obtain

$$\begin{aligned} 2i\int_{\mathbb{E}}\Big(\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\Big)^*\phi_s(x)dx &= t^*\int_{\mathbb{E}}\phi_s^*(x)\phi_s(x)dx \\ &= t^*\parallel\phi\parallel\,. \end{aligned} \tag{118}$$

Integrating by parts on the LHS then gives

$$-2i\,\Big(\parallel\phi\parallel+\int_{-\infty}^{-1}\phi_s^*(x)x\frac{d}{dx}\phi_s(x)dx+\int_1^{\infty}\phi_s^*(x)x\frac{d}{dx}\phi_s(x)dx\Big)=t^*\parallel\phi\parallel\,. \tag{119}$$

Carrying out the integration on the LHS we obtain

$$\int_{-2\pi n}^{0}\int_{-\infty}^{-1}\phi_s^*(x)x\frac{d}{dx}\phi_s(x)dxdt=\int_0^{2\pi n}\int_1^{\infty}\phi_s^*(x)x\frac{d}{dx}\phi_s(x)dxdt=0\;\forall\,n. \tag{120}$$

Hence it can be seen that

$$\int_{-2\pi n}^{0}\int_{-\infty}^{-1}\phi_s^*(x)\phi_s(x)dxdt=\int_0^{2\pi n}\int_1^{\infty}\phi_s^*(x)\phi_s(x)dxdt\;=0\;\forall\,n. \tag{121}$$

□

Theorem 8. *Let the complex-valued eigenstate* $\phi_s(x)=\phi_\sigma(x)+i\phi_t(x)=x^{-s}$ *where* $s=\sigma+it$ *and* $\sigma,t\in\mathbb{R}$*, and let the measurable subset of the measure space* $\mathbb{X}$ *be* $\mathbb{E}\to(-\infty,-1]\cup[1,\infty)$*. For the Hamiltonian operator* $\hat{H}=-2i\hbar\sqrt{x}\partial_x\sqrt{x}$*, all of the eigenvalues* t *occur at* $\mid\sigma\mid=1/2$ *with* $\hbar=1$*.*

Proof. Let $|\phi_s(x)\rangle$ be an eigenstate of $\hat{H}$ with eigenvalue t, i.e.,

$$\hat{H}\,|\phi_s(x)\rangle = t\,|\phi_s(x)\rangle\,. \tag{122}$$

In order to find the expectation value of $\hat{H}$ we multiply $\hat{H}$ by the eigenstate, take the complex conjugate, and then multiply the result by the eigenstate and integrate over $\mathbb{E}$ to obtain

$$\begin{aligned} 2i\int_{\mathbb{E}}\Big(\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\Big)^*\phi_s(x)dx &= t^*\int_{\mathbb{E}}\phi_s^*(x)\phi_s(x)dx \\ &= t^*\parallel\phi\parallel\,. \end{aligned} \tag{123}$$

Integrating by parts on the LHS then gives

$$-2i\,\Big(\parallel\phi\parallel+\int_{-\infty}^{-1}\phi_s^*(x)x\frac{d}{dx}\phi_s(x)dx+\int_1^{\infty}\phi_s^*(x)x\frac{d}{dx}\phi_s(x)dx\Big)=t^*\parallel\phi\parallel\,. \tag{124}$$

Carrying out the integration on the LHS we obtain

$$2i(-1)^{-2\sigma}\Big((-1)^{2\sigma}+1\Big)(\sigma+it)=(2\sigma-1)(t^*+2i)\parallel\phi\parallel\,. \tag{125}$$

Hence it can be seen that

$$\mid\sigma\mid=\frac{1}{2}\;\forall\,t. \tag{126}$$

□

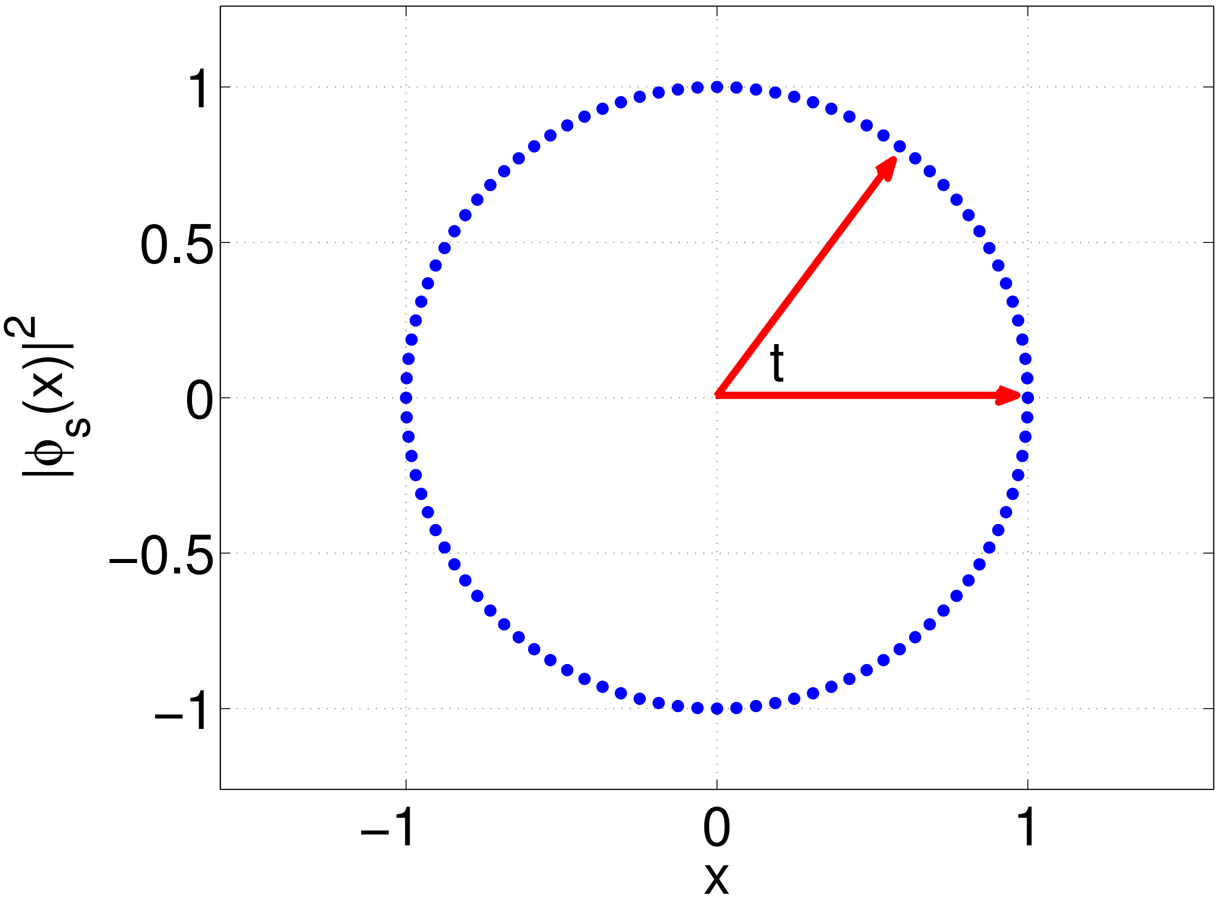

Figure 1: Plot of $s = |\sigma| \exp(it) = 1/2 - \log(x)/2$, Eq. (127). The density is normalized when $x \cos(t) = 1$ (color online).

D. Convergence

Theorem 9. *For the symmetrized Riemann zeta Schrödinger equation, i.e.,* $\hbar\partial_s |\phi_s(x)\rangle = -2\sqrt{x}\partial_x\sqrt{x}\,|\phi_s(x)\rangle$*, the complex-valued eigenstate* $|\phi_s(x)\rangle = x^{-s}$ *where* $s = |\sigma| \exp(it)$ *and* $\sigma, t \in \mathbb{R}$ *normalizes at* $x\cos(t) = 1$*, i.e., the density* $|\phi_s(x)|^2 = 1$*.*

Proof. In order to obtain convergent solutions to the unsymmetric Riemann zeta Schrödinger Eq. (114), it can be seen that upon inserting Eq. (13) into the symmetric Eq. (115), we obtain

$$\begin{aligned} s &= |\sigma| \exp(it) \\ &= \frac{1}{2} - \frac{\log(x)}{2}. \end{aligned} \tag{127}$$

Hence, at $x = \sec(t)$,

$$\sigma = \pm\frac{1}{2}e^{-it}\Big(\log(\sec(t)) - 1\Big), \tag{128}$$

such that at $|\sigma| = 1/2$ in agreement with Eq. (126) for

$$t = 2\pi n, \tag{129}$$

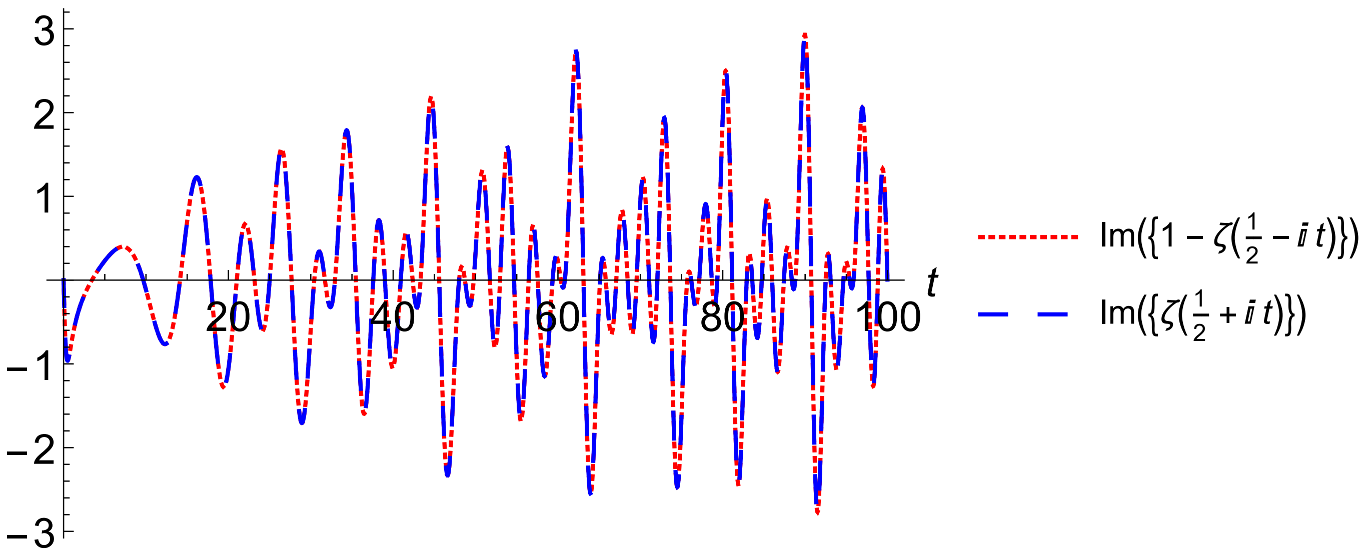

Figure 2: Plot of the imaginary components of Eq. (1). Results are compared with Eq. (137) (color online).

where $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. This condition is required such that the density is normalized in agreement with Eq. (75), i.e.,

$$\begin{aligned} \parallel \phi_s \parallel_2 &= \sum_m \sum_n \hat{b}_n(s)\hat{b}^\dagger_m(s) \langle \phi_m | \phi_n \rangle \\ &= \sum_n |\hat{b}_n(s)|^2 \\ &= 1. \end{aligned} \tag{130}$$

□

Theorem 10. *For the Bender-Brody-Müller equation [7], i.e.,*

$$\frac{1}{1-e^{-i\hat{p}}}(\hat{x}\hat{p}+\hat{p}\hat{x})(1-e^{-i\hat{p}})\,|\Psi_s(x)\rangle = t\,|\Psi_s(x)\rangle\,, \tag{131}$$

the nontrivial zeros of the Riemann zeta function can be obtained from the analytic continuation of the Riemann zeta function, i.e. $\zeta(s) = (1-2^{1-s})^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ *at the normalization constraint* $x = \sec(t = 2\pi n) = 1$, *such that* $|\sigma| = 1/2 \; \forall \, t \in \mathbb{R}$ *where* $s = \sigma + it$ *and* $\sigma, t \in \mathbb{R}$. *The nontrivial zeros of the Riemann zeta function are not observable at* $|\sigma| = 1/2$, $\forall \, n \in \mathbb{Z}$.

Proof. At $x = \sec(t = 2\pi n) = 1$, the normalization constraint Eq. (130) is satisfied, $\sigma = \frac{1}{2} - it$, and Eq. (10) can be written

$$\begin{aligned} \Psi_s(x=1) &= -\zeta(s=1/2,2) \\ &= -\Gamma(1/2)\frac{1}{2\pi i}\oint_C \frac{\sqrt{z}e^{2z}}{1-e^z}dz \\ &= 1-\zeta(\sigma=\frac{1}{2}-it). \end{aligned} \tag{132}$$

where the contour C is about $\mathbb{R}^-$. From the analytic continuation relations of Eq. (1)

$$\begin{aligned} \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} &= \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\exp\Big(-i\cdot t\ln(n)\Big)}{n^\sigma} \\ &= \frac{1}{1-2^{1-s}}\Big[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\Big(t\cdot\ln(n)\Big)}{n^\sigma} - i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\Big(t\cdot\ln(n)\Big)}{n^\sigma}\Big], \end{aligned} \tag{133}$$

$$1-\Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^* = 1-\frac{1}{1-2^{1-s^*}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\exp\Big(i\cdot t\ln(n)\Big)}{n^\sigma}$$
$$= 1-\frac{1}{1-2^{1-s^*}}\Big[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\Big(t\cdot\ln(n)\Big)}{n^\sigma}$$
$$+\ i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\Big(t\cdot\ln(n)\Big)}{n^\sigma}\Big]. \tag{134}$$

$$\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{2^{-\sigma+1}\cos\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}, \tag{135}$$

$$1-\Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^* = 1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{2^{-\sigma+1}\cos\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-2^{-\sigma+1}\sin\Big(t\log(2)\Big)\cos\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{2^{-\sigma+1}\cos\Big(t\log(2)\Big)\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}$$
$$+\ i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cdot\frac{-\sin\Big(t\ln(n)\Big)}{2^{-2\sigma+2}\sin^2\Big(t\log(2)\Big)+\Big[1-2^{-\sigma+1}\cos\Big(t\log(2)\Big)\Big]^2}, \tag{136}$$

such that Owing to the periodicity of $t = 2\pi n$ at $x = \sec(t)$, i.e. Eq. (129), it can be seen that

$$\Im\Big[\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big] = \Im\Big[1 - \Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^*\Big]. \tag{137}$$

Owing to Eq. (126), at $|\sigma| = 1/2$ we obtain

$$\Im\Big[\zeta(s)\Big] \;=\; i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\cdot\frac{\sin\Big(t\ln(n)\Big) - \sqrt{2}\sin\Big(t\log\left(\frac{n}{2}\right)\Big)}{2\sqrt{2}\cos\Big(t\log(2)\Big) - 3}. \tag{138}$$

However, since at $|\sigma| = 1/2$ the eigenvalues t are not observable, i.e., $\langle\hat{H}\rangle = t = 0$, we have

$$\Im\Big[\zeta(s)\Big] \;=\; i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\cdot\frac{\sin\Big(\cancelto{0}{2\pi n}\ln(n)\Big) - \sqrt{2}\sin\Big(\cancelto{0}{2\pi n}\log\left(\frac{n}{2}\right)\Big)}{2\sqrt{2}\cos\Big(\cancelto{0}{2\pi n}\log(2)\Big) - 3} = 0\;\forall\, n\in\mathbb{Z}. \tag{139}$$

□

Remark 2. *It has been noted that there is a uniquely defined relation between prime numbers and the imaginary parts of the nontrivial Riemann zeros, independent of their real part [36].*

E. Second Quantization

Theorem 11. *By representing the complex-valued eigenstate* $|\phi_s(x)\rangle = |\phi_\sigma(x)\rangle + i\,|\phi_t(x)\rangle = x^{-s}$ *where* $s = |\sigma|\exp(it)$ *and* $\sigma, t \in \mathbb{R}$ *as a linear combination of basis states, then the eigenspectrum of the Hamiltonian operator* $-2i\hbar\sqrt{x}\partial_x\sqrt{x}$ *is not observable, i.e. zero, on the measure space* $\mathbb{E} \to (-\infty, -1] \cup [1, \infty)$ *when* $|\sigma| = 1/2$ *and* $\hbar = 1$.

Proof. A standard way to introduce topology into the algebra of observables is to make them operators on a Hilbert space. In order to perform a second quantization [35], we can express the complex-valued eigenstate as a linear combination of basis states

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}}\hat{b}_n(s)\,|\phi_n(x)\rangle\,, \tag{140}$$

where $s = |\sigma|\exp(it) \in \mathbb{C}$, and $\sigma, t \in \mathbb{R}$. As such, using Eq. (16) we can rewrite Eq. (140) as

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}}\hat{b}_n(s)x^{-n}. \tag{141}$$

From using this second quantization in Eq. (115), we find

$$\hbar\frac{d}{ds}\hat{b}_n(s) \;=\; -t_n\hat{b}_n(s). \tag{142}$$

We now find a Hamiltonian that yields Eq. (142) as the equation of motion, hence, we take

$$\langle\phi_{s'}(x)|\,\hat{H}\,|\phi_s(x)\rangle \;=\; -2\int_1^{\infty}\langle\phi_{s'}(x)|\,\sqrt{x}\partial_x\sqrt{x}\,|\phi_s(x)\rangle\;dx - 2\int_{-\infty}^{-1}\langle\phi_{s'}(x)|\,\sqrt{x}\partial_x\sqrt{x}\,|\phi_s(x)\rangle\;dx, \tag{143}$$

as the expectation value. Upon substituting Eq. (141) into Eq. (143), we obtain the harmonic oscillator

$$\begin{aligned}\langle\phi_m(x)|\,\hat{H}\,|\phi_n(x)\rangle \;&=\; -2\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_1^{\infty}\frac{1}{x^{\frac{1}{2}-im}}\sqrt{x}\partial_x\sqrt{x}\;\frac{1}{x^{\frac{1}{2}+in}}dx - 2\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{-1}\frac{1}{x^{\frac{1}{2}-im}}\sqrt{x}\partial_x\sqrt{x}\;\frac{1}{x^{\frac{1}{2}+in}}dx \\ &=\; \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\hat{b}^\dagger_m(s)\hat{b}_n(s)\,\langle m|\Big(\frac{2n(\exp(\pi(n-m))-1)}{m-n}\Big)|n\rangle\,,\end{aligned} \tag{144}$$

for $|m\rangle, |n\rangle = 1, 2, 3, \ldots, \infty$. Hence at $m = n$, $\langle n|n\rangle = \delta_{nn} = 1$ and

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle \;=\; \sum_{n\in\mathbb{Z}} |\hat{b}_n(s)|^2\Big(-2\pi n\Big). \tag{145}$$

In accordance with Eq. (126) and Eq. (130), at $|\sigma| = 1/2$ and the zero periodicity of the eigenvalues t,

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle \;=\; 0. \tag{146}$$

□

Taking $\hat{b}_n(s)$ as an operator, and $\hat{b}^\dagger_n(s)$ as the adjoint, we obtain the usual properties:

$$\begin{aligned} [\hat{b}_n(s), \hat{b}_m(s)] &= [\hat{b}^\dagger_n(s), \hat{b}^\dagger_m(s)] = 0, \\ [\hat{b}_n(s), \hat{b}^\dagger_m(s)] &= \delta_{nm}. \end{aligned} \tag{147}$$

From the analogous Heisenberg equations of motion,

$$\begin{aligned} -\hbar\frac{d}{ds}\sum_{n\in\mathbb{Z}}\hat{b}_n(s) &= [\hat{b}_n(s), \hat{H}]_- \\ &= \sum_{m\in\mathbb{Z}} E_m\Big(\hat{b}_n(s)\hat{b}^\dagger_m(s)\hat{b}_m(s) - \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\ &= \sum_{m\in\mathbb{Z}} E_m\Big(\delta_{nm}\hat{b}_m(s) - \hat{b}^\dagger_m(s)\hat{b}_n(s)\hat{b}_m(s) - \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\ &= \sum_{m\in\mathbb{Z}} E_m\Big(\delta_{nm}\hat{b}_m(s) + \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s) - \hat{b}^\dagger_m(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\ &= \sum_{n\in\mathbb{Z}} \hat{b}^\dagger_m(s)\hat{b}_n(s)t_n. \end{aligned} \tag{148}$$

The eigenvalues of $\hat{H}$ are then unobservable, i.e.,

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle \;=\; 0. \tag{149}$$

From Eq. (148) it can be seen that

$$\begin{aligned} -\hbar\frac{d}{ds}\hat{b}_n &= 0, \\ -\hbar\frac{d}{ds}\hat{b}^\dagger_m &= -0. \end{aligned} \tag{150}$$

Remark 3. *Theorem 11 implies the Riemann hypothesis, as the spectrum of a Hermitian operator consists of real numbers as seen in Theorem 7, and* 0 *is a real number.*

F. Holomorphicity

Theorem 12. *The densely defined Hamiltonian operator $\hat{H} = -2\sqrt{x}\partial_x\sqrt{x}$ on the Hilbert space $\mathscr{H} = L^2[1,\infty)$ is symmetric (Hermitian) [34], for the complex-valued eigenstate $|\phi_s(x)\rangle = |\phi_\sigma(x)\rangle + i\,|\phi_t(x)\rangle = x^{-s}$ where $s = |\sigma|\exp(it)$ and $\sigma, t \in \mathbb{R}$ when $|\sigma| = 1/2$ and $\hbar = 1$.*

Proof. By expressing the complex-valued eigenstate as a linear combination of basis states such that

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}}\hat{b}_n(s)\,|\phi_n(x)\rangle\,, \tag{151}$$

where $s = |\sigma|\exp(it) \in \mathbb{C}$, and $\sigma, t \in \mathbb{R}$, it can be seen that by using Eq. (16) we can rewrite Eq. (151) as

$$|\phi_s(x)\rangle = \sum_{n\in\mathbb{Z}}\hat{b}_n(s)x^{-n}. \tag{152}$$

By taking the inner product

$$(\hat{H}\phi_n^*, \phi_m) = -2\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_1^{\infty}\frac{1}{x^{\frac{1}{2}+im}}\sqrt{x}\partial_x\sqrt{x}\,\frac{1}{x^{\frac{1}{2}-in}}dx - 2\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{-1}\frac{1}{x^{\frac{1}{2}+im}}\sqrt{x}\partial_x\sqrt{x}\,\frac{1}{x^{\frac{1}{2}-in}}dx$$
$$= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\hat{b}_m^{\dagger}(s)\hat{b}_n(s)\langle m|\Big(\frac{2n(\exp(\pi(n-m))-1)}{m-n}\Big)|n\rangle, \tag{153}$$

for $|m\rangle, |n\rangle = 1, 2, 3, \ldots, \infty$. Hence at $m = n$, $\langle n|n\rangle = \delta_{nn} = 1$ and

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle = \sum_{n\in\mathbb{Z}}|\hat{b}_n(s)|^2\Big(2\pi n\Big). \tag{154}$$

In accordance with Eq. (126) and Eq. (130), at $|\sigma| = 1/2$ and the zero periodicity of the eignenvalues t,

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle = 0. \tag{155}$$

Furthermore, by taking the inner product

$$(\phi_m^*, \hat{H}\phi_n) = -2\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_1^{\infty}\frac{1}{x^{\frac{1}{2}-im}}\sqrt{x}\partial_x\sqrt{x}\,\frac{1}{x^{\frac{1}{2}+in}}dx - 2\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{-1}\frac{1}{x^{\frac{1}{2}-im}}\sqrt{x}\partial_x\sqrt{x}\,\frac{1}{x^{\frac{1}{2}+in}}dx$$
$$= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\hat{b}_m^{\dagger}(s)\hat{b}_n(s)\langle m|\Big(\frac{2n(\exp(\pi(n-m))-1)}{m-n}\Big)|n\rangle, \tag{156}$$

for $|m\rangle, |n\rangle = 1, 2, 3, \ldots, \infty$. Hence at $m = n$, $\langle n|n\rangle = \delta_{nn} = 1$ and

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle = \sum_{n\in\mathbb{Z}}|\hat{b}_n(s)|^2\Big(-2\pi n\Big). \tag{157}$$

In accordance with Eq. (126) and Eq. (130), at $|\sigma| = 1/2$,

$$\langle\phi_n(x)|\,\hat{H}\,|\phi_n(x)\rangle = 0. \tag{158}$$

Finally,

$$(\hat{H}\phi_n^*, \phi_m) = (\phi_m^*, \hat{H}\phi_n) = 2\pi n = 0\ \forall\ n \in \mathbb{Z}. \tag{159}$$

□

Remark 4. *The Riemann Hypothesis states that the real part of all of the nontrivial zeros of the Riemann zeta function are located at $\sigma = 1/2$ [8].*

III. SIMILARITY SOLUTIONS

Since Eq. (115), the Riemann zeta Schrödinger equation (RZSE) possesses symmetry about the origin $x = 0$, we then seek a similarity solution [38] of the form:

$$\phi_s(x) = x^{\alpha} f(\eta), \tag{160}$$

where $\eta = s/x^{\beta}$, and the RZSE becomes an ordinary differential equation (ODE) for f. As such, we consider Eq. (115), and introduce the transformation $\xi = \epsilon^a x$, and $\tau = \epsilon^b s$, so that

$$w(\xi, \tau) = \epsilon^c\phi(\epsilon^{-a}\xi, \epsilon^{-b}\tau), \tag{161}$$

where $\epsilon \in \mathbb{R}$, and $\tau \in \mathbb{C}$.

From performing this change of variable we obtain

$$\frac{\partial}{\partial s}\phi = \epsilon^{-c}\frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial s} = \epsilon^{b-c}\frac{\partial w}{\partial \tau}, \tag{162}$$

and

$$\begin{aligned} -2\sqrt{x}\frac{\partial}{\partial x}\sqrt{x}\phi &= -2\sqrt{x}\Big(\frac{\partial\sqrt{x}}{\partial x}\phi + \sqrt{x}\frac{\partial \phi}{\partial x}\Big) \\ &= -2\sqrt{x}\frac{1}{2\sqrt{x}}\phi - 2\sqrt{x}\sqrt{x}\frac{\partial \phi}{\partial x} \\ &= -\phi - 2x\frac{\partial \phi}{\partial x}, \end{aligned} \tag{163}$$

where

$$\frac{\partial \phi}{\partial x} = \epsilon^{-c}\frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} = \epsilon^{a-c}\frac{\partial w}{\partial \xi}. \tag{164}$$

By using Eqs. (162)-(164) in Eq. (115), the RZSE is then written

$$\epsilon^{-c}\Big[\epsilon^{b}\frac{\partial w}{\partial \tau} + w + 2\xi\frac{\partial w}{\partial \xi}\Big] = 0, \tag{165}$$

and is invariant under the transformation $\forall\, \epsilon$ if $\epsilon^b = 2$, i.e.,

$$\epsilon^{-c}\Big[\frac{\epsilon^b}{2}\Big(\frac{\partial w}{\partial \tau_{\Re}} - i\frac{\partial w}{\partial \tau_{\Im}}\Big) + w + 2\xi\frac{\partial w}{\partial \xi}\Big] = 0, \tag{166}$$

and

$$b = \frac{\log(2) + 2i\pi n}{\log(\epsilon)},\ \forall\, n \in \mathbb{Z}. \tag{167}$$

Therefore, it can be seen that since ϕ solves the RZSE for x and s, then $w = \epsilon^{-c}\phi$ solves the RZSE at $x = \epsilon^{-a}\xi$, and $s = \epsilon^{-b}\tau$. We now construct a group of independent variables such that

$$\frac{\xi}{\tau^{a/b}} = \frac{\epsilon^a x}{(\epsilon^b s)^{a/b}} = \frac{x}{s^{a/b}} = \eta(x,s), \tag{168}$$

and the similarity variable is then

$$\eta(x,s) = xs^{-\frac{a\log(\epsilon)}{\log(2)+2i\pi n}}. \tag{169}$$

Also,

$$\frac{w}{\tau^{c/b}} = \frac{\epsilon^c \phi}{(\epsilon^b s)^{c/b}} = \frac{\phi}{s^{c/b}} = \nu(\eta), \tag{170}$$

suggesting that we seek a solution of the RZSE with the form

$$\phi_s(x) = s^{\frac{c\log(\epsilon)}{\log(2)+2i\pi n}}\nu(\eta). \tag{171}$$

Since the RZSE is invariant under the transformation, it is to be expected that the solution will also be invariant under the variable transformation. Taking $a = c = \log^{-1}(\epsilon)$, the partial derivatives transform like

$$\begin{aligned} \frac{\partial}{\partial s}\phi_s(x) &= \frac{\partial}{\partial s}\Big(s^{\frac{1}{\log(2)+2i\pi n}}\Big)\nu(\eta) + \Big(s^{\frac{1}{\log(2)+2i\pi n}}\Big)\nu'(\eta)\frac{\partial \eta}{\partial s} \\ &= \frac{s^{-1+\frac{1}{\log(2)+2i\pi n}}}{\log(2)+2i\pi n}\Big[\nu(\eta) - \nu'(\eta)\Big], \end{aligned} \tag{172}$$

and

$$\begin{aligned} \frac{\partial}{\partial x}\phi_s(x) &= \Big(s^{\frac{1}{\log(2)+2i\pi n}}\Big)\nu'(\eta)\frac{\partial \eta}{\partial x} \\ &= \nu'(\eta), \end{aligned} \tag{173}$$

where

$$\frac{\partial \eta}{\partial s} = -\frac{s^{-1}}{2i\pi n + \log(2)}, \tag{174}$$

and

$$\frac{\partial \eta}{\partial x} = s^{-\frac{1}{2i\pi n+\log(2)}}. \tag{175}$$

The RZSE then reduces to the ODE

$$\Big[s^{-1} + \log(2) + 2i\pi n\Big]\nu(\eta) + \Big[-s^{-1} + 2\log(2)\eta + 4i\pi n\eta\Big]\nu'(\eta) = 0,\ \forall\, n \in \mathbb{Z}. \tag{176}$$

A. General Solution

The homogenous linear differential Eq. (176) is separable [39], viz.,

$$\frac{d\nu}{\nu} = \frac{2i\pi n + s^{-1} + \log(2)}{s^{-1} - 4i\pi n\eta - \eta\log(4)}d\eta. \tag{177}$$

Integrating on both sides, we obtain

$$\ln|\nu| = c_1 - \frac{\Big(2i\pi n + s^{-1} + \log(2)\Big)\log\Big(s^{-1} - 4i\pi n\eta - \eta\log(4)\Big)}{4i\pi n + \log(4)}. \tag{178}$$

Exponentiating both sides,

$$|\nu| = \exp(c_1)\Big(s^{-1} - 4i\pi n\eta - \eta\log(4)\Big)^{-\frac{2i\pi n + s^{-1} + \log(2)}{4i\pi n + \log(4)}}. \tag{179}$$

Renaming the constant $\exp(c_1) = C$ and dropping the absolute value recovers the lost solution $\nu(\eta) = 0$, giving the general solution to Eq. (176)

$$\nu_n(\eta) = C\Big(s^{-1} - 4i\pi n\eta - \eta\log(4)\Big)^{-\frac{2i\pi n + s^{-1} + \log(2)}{4i\pi n + \log(4)}},\ \forall\, n \in \mathbb{Z},\ \forall\, C \in \mathbb{R}. \tag{180}$$

By setting $C = 1$, and using Eqs. (169) and (171) in Eq. (180), we obtain the general solution to the RZSE Eq. (115), written

$$\phi_s(x) = s^{\frac{1}{\log(2)+2i\pi n}}\left[\frac{1}{s} + s^{-\frac{1}{\log(2)+2i\pi n}}\Big(-x\log(4) - 4i\pi n x\Big)\right]^{-\frac{2\pi n s + is\log(2) + i}{4\pi n s - is\log(4)}},\ \forall\, n \in \mathbb{Z}. \tag{181}$$

IV. CONCLUSION

In this study, we have discussed the convergence of the real part of every nontrivial zero of the analytic continuation of the Riemann zeta function. This was accomplished by developing a Riemann zeta Schrödinger equation and comparing it with the Bender-Brody-Müller conjecture in both configuration space and momentum space. A symmetrization procedure was implemented to study the convergence of the system, and the expectation values were calculated from the resulting system to study the nontrivial zeros of the analytic continuation of the Riemann zeta function. It was found using Green's functions that the expectation value of the Hamiltonian operator for the eigenstates along the critical line $\sigma = 1/2$ is also zero such that the nontrivial zeros of the Riemann zeta function are not observable. A Gelfand triplet was implemented to ensure that the eigenvalues are well defined. Moreover, a second quantization procedure was performed for the Riemann zeta Schrödinger equation to obtain the equations of motion and an analytical expression for the eigenvalues. It was also demonstrated that the eigenvalues are holomorphic across the measurable subspace of the measure space. A normalized convergent expression for the analytic continuation of the nontrivial zeros of the Riemann zeta function was obtained, and a convergence test for the expression was performed demonstrating that the real part of every nontrivial zero of the Riemann zeta function exists at $\sigma = 1/2$. Finally, a general solution to the Riemann zeta Schrödinger equation was found from performing an invariant similarity transformation.

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